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TRUTH-TELLING EQUILIBRIA FOR BAYESIAN GAMES ARISING FROM SEQUENCING SITUATIONS*

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Abstract This paper considers one-machine sequencing situations with linear costs in which the urgency of players is private information. To study strategic behavior of players based on neighbor switches we associate with such a situation a Bayesian game where the utility functions are based on gain split rules and study whether the truth-telling strategy profile is an equilibrium of the game. The existence of such truth-telling equilibria turns out to be exceptional.

Keywords: Sequencing situations, bayesian games, truth-telling, queueing situations

1. Introduction

This paper deals with one-machine sequencing situations with linear costs under incomplete information. In such sequencing situations some players in need for specific service are standing in a queue in front of a server in the order of their arrival and have costs proportionally with the time spent in the queue. A basic question addressed in the sequencing literature is that of reordering the queue such that the aggregate cost savings are maximized. To cope with this question one needs to know the relevant characteristics of players, specifically the individual cost per unit of time and the processing (or service) time. The ratio between the individual cost per unit of time and the processing time, called the player's urgency, plays a key role. It is often assumed that these parameters are deterministic and common knowledge. For deterministic sequencing situations Smith [12] proved that ordering the players according to the decreasing order of their urgency, results in maximum (total) cost savings. A natural way to reach the optimal order according to Smith is via neighbor switches. To make such switches are attractive, transfer payoffs and division of neighbor gains between players are tackled via various division rules. The class of (neighbor) gain split rules where each switch of positions for neighbors is coupled with a split of the gain generated by the neighbor switch has received much attention in the (deterministic) sequencing situations literature. Curiel et al. [1] consider interaction in sequencing situations based on the equal gain split rule (EGS-rule) where the gain generated by each neighbor switch is divided equally between the two switching players. To motivate their interest in the EGS-rule they have constructed a cooperative game related with a deterministic sequencing situation and proved that the EGS-rule leads to a core element of the corresponding sequencing game. Hamers et al. [7] have considered gain split rules where the gain generated by each neigh-

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bor switch is divided between the switching players according to a quota and this splitting fraction may be different in each neighbor switch. Such split rules generate the split core of a related cooperative sequencing game. In our paper we consider gain split rules where the splitting fraction ε is the same for all neighbor switches, which we refer to as GS^{ε}-rules. Such rules are appealing for players in a sequencing situation because no player loses from any neighbor switch made, and, consequently, all players are (weakly) better off. For other sharing rules for sequencing situations with complete information we refer the reader to Fernandez et al. [5]. Some other properties of players in sequencing situations with complete information, like due dates and ready times, have been considered by Hamers et al. [6].

Sequencing situations under incomplete information have been also studied in the Operations Research literature and in the Game Theory literature. In such situations, each player's characteristics are private information and have to be reported by the players. In most literature on sequencing situations under incomplete information, dominant strategy incentive compatibility has been investigated. We refer here to the papers by Suijs [13], Mitra [9][10] dealing with public decision problems arising from one-machine sequencing situations with linear costs under incomplete information. Suijs [13] has shown that the only incentive compatible payoff transfer schemes for such a public decision problem are budget balanced Groves schemes which are not individually rational. Mitra [9] has proved that it is possible to find dominant strategy incentive mechanisms satisfying efficiency, budget balancedness and individual rationality. A more detailed comparison between our model and those of Suijs and Mitra can be found at the end of Section 5. Looking for mechanisms that will also imply an incentive for the individual players to participate taking into account their rights due to their initial place in the queue, we relate in our paper a sequencing situation under incomplete information with a Bayesian game where switches and transfer payoffs between players are based on the class of GS^{ε}-rules, and investigate under which GS^{ε}-rules the truth-telling strategy profile is an equilibrium, i.e. a truth-telling equilibrium exists. We mainly deal with *queueing situations*, which are special sequencing situations where the time of service is fixed and equal for all players. For such situations we find that with two possible types regarding the cost per unit of time, a truth-telling equilibrium of the related Bayesian game exists under any GS^{ε}-rule. Further, with three possible types, the subclass of GS^{ε}-rules for which a truth-telling equilibrium exists is characterized. For queueing situations with more than three types and for any sequencing situation where each player's processing time is private information, we show that no GS^{ε}-rule exists that assures the existence of a truth-telling equilibrium for the related Bayesian game.

The paper is organized as follows. In Section 2 the sequencing situation where the players' characteristics are private information is formalized, a related Bayesian game whose transfer payment functions are based on the class of GS^{ε}-rules is introduced and the notion of truth-telling equilibrium is defined. Special attention is paid to Bayesian games arising from *queueing* situations and their truth-telling equilibria. Section 3 presents our possibility results on the existence of truth-telling equilibria of Bayesian games arising from queueing situations. In Section 4 we present our impossibility results on the existence of truth-telling equilibria under GS^{ε}-rules. The concluding Section 5 is followed by an Appendix dealing with a graphical representation of the domain of GS^{ε}-rules for the case of three possible types from Section 3.

2. Models of Discrete Sequencing Situations

We consider one-machine sequencing situations with linear costs and n players where each player's relevant characteristics, namely the cost per unit of time and the processing (or service) time, are private information. It is common knowledge that all players are risk-neutral and their relevant characteristics are independently drawn from general populations C and S of possible values (or types) where $C = \{c_1, c_2, \dots, c_k\}$ is the finite set of all possible values for the cost-per-unit-of-time parameter, and $S = \{s_1, s_2, \dots, s_m\}$ is the finite set of all possible values for the service-time parameter. The probability distributions of possible values across each population, which we denote here by $p^C \in \Re_+^{|C|}$ and $p^S \in \Re_+^{|S|}$ with $\sum_{c\in C} p^C(c) = 1$, $\sum_{s\in S} p^S(s) = 1$ respectively, are commonly known by all players. We assume that these distributions are independent and the same for all players. A sequencing situation under incomplete information can be formalized as a tuple $\langle N, T, p\rangle$ where:

- N is the finite set of players of the form $N = \{1, 2, \dots, n\}$, with a player $i \in N$ having the i-th position in the (initial) queue;
- $T = C \times S$ is the finite set describing all the possible values of the players' relevant characteristics;
- p is a probability distribution on T such that $p(c, s) = p^C(c)p^S(s)$ for each $(c, s) \in T$.

We suppose that players have the possibility to switch positions with their neighbors, and payoff transfers according to a commonly agreed upon rule are allowed. The payoff transferred between players may depend on their announced urgencies. Recall that the urgency of a player is the ratio between his cost per unit of time and the service time, and efficiency of a queue can be achieved by rearranging the players via neighbor switches in a decreasing order of their urgencies. Efficiency of the queue means here minimizing the expected aggregate costs of time spent in the system. Each player is asked independently and simultaneously to announce his relevant characteristics and each player's objective is to maximize his expected payoff. Given a rule under which payoff transfers are made, the question is whether it can be advantageous for a player to misrepresent his true characteristics, and, consequently, his true urgency. Specifically, we want to find out whether there are sequencing situations under incomplete information and rules such that truth-telling is an equilibrium in the sense that unilateral deviation from truth-telling does not pay. We would like the rules under consideration to have the property that players' participation is voluntary (their participation cannot decrease their expected payoff).

To approach these questions we focus on the class of gain split rules GS^ε with $\varepsilon \in [0, 1]$. We say that a group applies a GS^ε-rule if for each neighbor switch the follower (completely) compensates the predecessor for the position's loss and additionally gives the predecessor the fraction ε of the net gain generated by the switch.

One can easily see that (in case of truth-telling) each switching player is weakly better off since the follower reduces his costs by (switching and) getting a better position, and keeps the fraction $(1 - \varepsilon)$ of the achieved gain, whereas the predecessor earns the splitting fraction ε of the achieved gain.

With each sequencing situation under incomplete information $\langle N, T, p\rangle$ and each GS^ε-rule we associate a Bayesian game (cf. Myerson [11]), which we denote by G^ε, of the form $\langle N, \Omega, p, (A_i, u_i)_{i\in N}, \varepsilon\rangle$ where:

- N is the set of players waiting for service in the initial queue;

- $\Omega = T^N$ is the set of possible "states of nature", each of which is a description of all the players' relevant characteristics; the profile of the players' types is noted by $t \in \Omega$ with $t = (t_1, t_2, \ldots, t_n)$, and where each player's type is of the form $t_i = (t_i^C, t_i^S)$ with $t_i^C \in C$ and $t_i^S \in S$; hereafter, we also use the notation $t_{-i} = (t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$;
- p is the probability distribution on T as described above;
- ε here specifies a GS^ε-rule as described above;

and for each i in N:

- $A_i = T$ is the action set for i; we denote by A the set $\prod_{i \in N} A_i$; the profile of the players' announced types (or actions) is denoted by $a \in A$ with $a = (a_1, a_2, \ldots, a_n)$, and where each player's announced type is of the form $a_i = (a_i^C, a_i^S)$ with $a_i^C \in C$ and $a_i^S \in S$;
- u_i is the payoff function for player i, $u_i{:}A \times \Omega \to \Re$.

Let (a, t) be a realized play of the game G^ε, with $a \in A$ and $t \in \Omega$. Then $u_i(a, t)$ is the payoff of i. We denote by $u_i(t, t)$ i's payoff in case all players report their true characteristics and by $u_i((t_{-i}, a_i), t)$ i's payoff if player i unilaterally deviates from truth-telling by announcing $a_i \neq t_i$.

Each agent i follows a strategy function of the form $\chi_i : T \to T$, $\chi_i(t_i) = a_i$. We denote by χ_i^t the truth-telling strategy for player i and by χ^t the truth-telling strategy profile for all players; then (χ_{-i}^t, a_i) represents a strategy profile where player i chooses to announce $a_i \neq t_i$ while all other players follow their truth-telling strategy.

Given his own realized type and some assumption about other players' strategies, player i can consider his expected payoff $U_i((\chi_{-i}, a_i), t_i)$. In this paper we do not try to find all Bayesian equilibria, but rather check whether the truth-telling strategy profile is an equilibrium of the game G^ε. We therefore concentrate on the expected payoff $U_i((\chi_{-i}^t, a_i), t_i)$ where a player i assumes that all the other players follow their truth-telling strategy.

The truth-telling strategy profile is an equilibrium of the game G^ε, which we call in the following a *truth-telling equilibrium* (*TT-equilibrium*), if for each $i \in N$ and for any realization t, it holds that

$$U_i((\chi_{-i}^t, t_i), t_i) \geq U_i((\chi_{-i}^t, a_i), t_i) \text{ for all } a_i \in T, a_i \neq t_i. \tag{1}$$

The notions of a Bayesian game and a Bayesian equilibrium were introduced by Harsanyi [8]; see also Myerson [11].

In the following we concentrate on GS^ε-rules and give explicit expressions for the payoff functions and the expected payoffs of players. Each player i, $i \in N$ with true type $t_i = (t_i^C, t_i^S)$ has to announce a type $a_i = (a_i^C, a_i^S)$. A neighbor switch will happen for two players i and j such that $i < j$ if and only if $a_i^C/a_i^S < a_j^C/a_j^S$(i.e. player j's urgency is higher than player i's urgency). In such a case, player j will move in front of player i, and the payment made by j to i will be $a_i^C \cdot a_j^S + \varepsilon(a_j^C \cdot a_i^S - a_i^C \cdot a_j^S)$ where the first part is a compensation for the reported loss due to i's move back, and the second part is the share ε of the net gain according to the reported values.

Consequently, for each $i \in N$, i's payoff function is given by

$$u_i(a, t) = \sum_{j<i} (sgn(a_i^C/a_i^S - a_j^C/a_j^S)) \cdot f_i^{j,i}(a, t) + \sum_{j>i} (sgn(a_j^C/a_j^S - a_i^C/a_i^S)) \cdot f_i^{i,j}(a, t). \tag{2}$$

where the function sgn, defined by:

$$sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & otherwise \end{cases}, \tag{3}$$

is used to indicate whether a switch of i with a player j occurs; $f_i^{j,i}(a,t)$ is the payoff for i generated by a switch with a predecessor j(in case $i > j$, $a_i^C \cdot a_j^S > a_j^C \cdot a_i^S$), and $f_i^{i,j}(a,t)$ is the payoff for i generated by a switch with a follower j (in case $i < j$, $a_i^C \cdot a_j^S < a_j^C \cdot a_i^S$) as follows:

$$f_i^{j,i}(a,t) = t_i^C \cdot t_j^S - a_j^C \cdot a_i^S - \varepsilon(a_i^C \cdot a_j^S - a_j^C \cdot a_i^S); \tag{4}$$
$$f_i^{i,j}(a,t) = -t_i^C \cdot t_j^S + a_i^C \cdot a_j^S + \varepsilon(a_j^C \cdot a_i^S - a_i^C \cdot a_j^S). \tag{5}$$

Specifically, in case of truth-telling, the payoff and the expected payoff are given by:

$$u_i(t,t) = \sum_{j<i}(sgn(t_i^C \cdot t_j^S - t_j^C \cdot t_i^S)) \cdot (t_i^C \cdot t_j^S - t_j^C \cdot t_i^S)(1-\varepsilon) + \sum_{j>i}(sgn(t_j^C \cdot t_i^S - t_i^C \cdot t_j^S)) \cdot \varepsilon(t_j^C \cdot t_i^S - t_i^C \cdot t_j^S); \tag{6}$$

$$U_i((\chi_{-i}^t, t_i), t_i) = (i-1) \sum_{(c_r,s_l)|\frac{c_r}{s_l}<\frac{t_i^C}{t_i^S}} p^C(c_r)p^S(s_l) \cdot (t_i^C \cdot s_l - c_r \cdot t_i^S)(1-\varepsilon) + (n-i) \sum_{(c_r,s_l)|\frac{c_r}{s_l}>\frac{t_i^C}{t_i^S}} p^C(c_r)p^S(s_l) \cdot \varepsilon(c_r \cdot t_i^S - t_i^C \cdot s_l); \tag{7}$$

where $(c_r, s_l) \in T$ is any possible type for any player $j \in N \setminus \{i\}$.

When player i deviates unilaterally from truth-telling, we have:

$$u_i((t_{-i}, a_i), t) = \sum_{j<i}(sgn(a_i^C \cdot t_j^S - t_j^C \cdot a_i^S)) \cdot \lfloor t_i^C \cdot t_j^S - t_j^C \cdot a_i^S - \varepsilon(a_i^C \cdot t_j^S - t_j^C \cdot a_i^S) \rfloor + \sum_{j>i}(sgn(t_j^C \cdot a_i^S - a_i^C \cdot t_j^S)) \cdot \lfloor -t_i^C \cdot t_j^S + a_i^C \cdot t_j^S + \varepsilon(t_j^C \cdot a_i^S - a_i^C \cdot t_j^S) \rfloor; \tag{8}$$

$$U_i((\chi_{-i}^t, a_i), t_i) = (i-1) \sum_{(c_r,s_l)|\frac{c_r}{s_l}<\frac{a_i^C}{a_i^S}} p^C(c_r)p^S(s_l) \cdot \lfloor t_i^C \cdot s_l - c_r \cdot a_i^S - \varepsilon(a_i^C \cdot s_l - c_r \cdot a_i^S) \rfloor + (n-i) \sum_{(c_r,s_l)|\frac{c_r}{s_l}>\frac{a_i^C}{a_i^S}} p^C(c_r)p^S(s_l) \cdot \lfloor -t_i^C \cdot s_l + a_i^C \cdot s_l + \varepsilon(c_r \cdot a_i^S - a_i^C \cdot s_l) \rfloor. \tag{9}$$

Special attention is paid in the following sections to *queueing situations* with linear costs, i.e. sequencing situations with linear costs where all players have the same service time, say one unit of time, and this is common knowledge. When referring to *queueing situations* we hereafter simply replace t_i^S, a_i^S, s_l by 1, and use $t_i, a_i \in C$ instead of t_i^C, a_i^C to denote player i's cost per unit of time and his announced value of this parameter, respectively.

A *queueing situation under incomplete information* is a tuple $\langle N, C, p \rangle$, where C is the finite set of all possible values for the cost per unit of time for players in N, and $p = p^C$ is a probability distribution on C. Each player's cost per unit of time is private information, while C and p are common knowledge.

The Bayesian game arising from such a queueing situation where a GS^{ε}-rule is used for payoff transfers is a simplified version of G^{ε}. Agents are asked to announce a value: a_i. When a GS^{ε}-rule is used, then a switch will happen for two players such that $i < j$ and $a_i < a_j$. In such a case, the payment made by j to i will be $a_i + \varepsilon(a_j - a_i)$.

For each $i \in N$, the payoff function is given by

$$u_i(a,t) = \sum_{j<i}(sgn(a_i - a_j)) \cdot f_i^{j,i}(a,t) + \sum_{j>i}(sgn(a_j - a_i)) \cdot f_i^{i,j}(a,t) \tag{10}$$

where the function sgn is as before, and where $f_i^{j,i}(a,t)$ refers to a switch of i with a predecessor j (in case $i > j$, $a_i > a_j$), and $f_i^{i,j}(a,t)$ refers to a switch of i with a follower j (in case $i < j$, $a_i < a_j$) as follows:

$$f_i^{j,i}(a,t) = t_i - a_j - \varepsilon(a_i - a_j) \quad i > j, a_i > a_j; \tag{11}$$

$$f_i^{i,j}(a,t) = -t_i + a_i + \varepsilon(a_j - a_i) \quad i < j, a_i < a_j. \tag{12}$$

Specifically, in case of truth-telling, the payoff and the expected payoff are given by:

$$u_i(t,t) = \sum_{j<i}(sgn(t_i - t_j) \cdot (t_i - t_j)(1-\varepsilon) + \sum_{j>i}(sgn(t_j - t_i)) \cdot \varepsilon(t_j - t_i); \tag{13}$$

$$U_i((\chi_{-i}^t, t_i), t_i) = (i-1)\sum_{c_r<t_i} p(c_r) \cdot (t_i - c_r)(1-\varepsilon) + (n-i)\sum_{c_r>t_i} p(c_r) \cdot \varepsilon(c_r - t_i). \tag{14}$$

When player i deviates unilaterally from truth-telling, we have:

$$u_i((t_{-i}, a_i), t) = \sum_{j<i}(sgn(a_i - t_j)) \cdot \lfloor t_i - t_j - \varepsilon(a_i - t_j)\rfloor + \sum_{j>i}(sgn(t_j - a_i)) \cdot \lfloor -t_i + a_i + \varepsilon(t_j - a_i)\rfloor; \tag{15}$$

$$U_i((\chi_{-i}^t, a_i), t_i) = (i-1)\sum_{c_r<a_i} p(c_r) \cdot [t_i - c_r - \varepsilon(a_i - c_r)] + (n-i)\sum_{c_r>a_i} p(c_r) \cdot [-t_i + a_i + \varepsilon(c_r - a_i). \tag{16}$$

This model of a queueing situation under incomplete information is considered in Sections 3,4 and in Appendix 1 for the cases $|C| = 2$, $|C| = 3$ and $|C| > 3$ to analyze the existence of a truth-telling equilibrium in G^{ε} with respect to the cost per unit of time.

3. Existence of a Truth-Telling Equilibrium

In this section we consider the model $\langle N, C, p\rangle$ of queueing situations with incomplete information where $|C| \leq 3$.

We first analyze the case with only two possible values for cost per unit of time and prove that for any GS^{ε}-rule with $\varepsilon \in [0,1]$ supports a TT-equilibrium of G^{ε}.

Theorem 3.1 *Let $\langle N, C, p\rangle$ be a queueing situation with $|C| = 2$, and let G^{ε} be the corresponding Bayesian game. Then for each $\varepsilon \in [0,1]$ the GS^{ε}-rule assures the existence of a truth-telling equilibrium of the game G^{ε}.*

proof We show that for each $\varepsilon \in [0,1]$ and for each $i \in N$ we have $U_i((\chi^t_{-i}, c_1), c_1) \geq U_i((\chi^t_{-i}, c_2), c_1)$ and $U_i((\chi^t_{-i}, c_2), c_2) \geq U_i((\chi^t_{-i}, c_1), c_2)$. Take $\varepsilon \in [0,1]$ and $i \in N$, and suppose that all players in $N \setminus \{i\}$ reveal their true type. We consider two cases:

(i) Let i be of type c_1. If i reveals his true type c_1 then there are no switches of i with his $i-1$ predecessors. Further, switches with followers of i of type c_2 take place. The expected number of followers of type c_2 is $(n-i)p(c_2)$ and i gains a payoff of $\varepsilon(c_2 - c_1)$ from each of these switches. We obtain that the expected payoff for player i if he reveals his true type c_1 is given by

$$U_i((\chi^t_{-i}, c_1), c_1) = (i-1) \cdot 0 + (n-i)p(c_2)\varepsilon(c_2 - c_1) \geq 0. \tag{17}$$

If player i announces c_2 as his type then there are only switches with predecessors of type c_1. The expected number of predecessors of type c_1 is $(i-1)p(c_1)$. Each of these switches will lead to a loss of $\varepsilon(c_2 - c_1)$, since player i will pay the switching player $c_1 + \varepsilon(c_2 - c_1)$, but he will only gain c_1 from moving forward. We obtain that

$$U_i((\chi^t_{-i}, c_2), c_1) = (n-i) \cdot 0 - (i-1)p(c_1)\varepsilon(c_2 - c_1) \leq 0. \tag{18}$$

Hence, we find that $U_i((\chi^t_{-i}, c_1), c_1) \geq U_i((\chi^t_{-i}, c_2), c_1)$.

(ii) Let i be of type c_2. We follow similar considerations as above. Specifically, note that if i chooses to misrepresent his type by announcing c_1, then he will make switches with followers of type c_2 from whom he will receive a payoff of $c_1 + \varepsilon(c_2 - c_1)$. This payoff will not be sufficient to compensate for the loss of the amount c_2 from moving backward. So, we have $U_i((\chi^t_{-i}, c_2), c_2) \geq U_i((\chi^t_{-i}, c_1), c_2)$, because

$$U_i((\chi^t_{-i}, c_2), c_2) = (i-1)p(c_1)(1-\varepsilon)(c_2 - c_1) \geq 0 \tag{19}$$

$$U_i((\chi^t_{-i}, c_1), c_2) = -(n-i)p(c_2)(1-\varepsilon)(c_2 - c_1) \leq 0. \blacksquare \tag{20}$$

Remark 3.1 : The existence of TT-equilibria of a Bayesian game arising from a queueing situation with linear costs is also guaranteed for each gain split rule in case there are only two types - the same for all players - on which the players may have different a priori probability distributions.

We now focus on queueing situations $\langle N, C, p \rangle$ with $|C| = 3$ and characterize a subclass of queueing situations for which the truth-telling strategy profile is an equilibrium of the game G^ε. We denote by $E(p)$ the set of GS^ε-rules for which a TT-equilibrium exists, given the probability distribution p.

Theorem 3.2 *Let $\langle N, C, p \rangle$ be a queueing situation with $|C| = 3$, and let G^ε be the corresponding Bayesian game. In case $p > 0$, $E(p)$ is nonempty if and only if the probability distribution p is such that $p(c_2) \geq \sqrt{p(c_1) \cdot p(c_3)}$; specifically, for each $\varepsilon \in \left[\frac{p(c_3)}{p(c_3)+p(c_2)}, \frac{p(c_2)}{p(c_1)+p(c_2)}\right]$ the GS^ε-rule assures the existence of a truth-telling equilibrium of the game G^ε. In case $p(c_1) = 1$, $E(p) = \{GS^0\text{-rule}\}$, and in case $p(c_3) = 1$, $E(p) = \{GS^1\text{-rule}\}$.*

proof We consider player $i \in N$ and, assuming that all other players report their true types, analyze if it is a best strategy for player i to report his true type or not. We look for necessary and sufficient conditions for ε and p such that inequality (1) holds for each $i \in N$, $t_i, a_i \in T$.

(i) Suppose first that player i is of type c_1.

If all players, including i are truthfully reporting their cost per unit of time in the system, then i will change places with all followers of type c_2 or c_3, and will be paid by them according to a GS^ε-rule.

- If player i decides to report c_3 instead of c_1, then he gains nothing and loses the possible gains associated with the switches with followers of type c_2 and c_3 that will not take place due to his false report. In addition, he will make switches with predecessors of type c_1 and c_2, and pay them more than these switches are worth for him.
- If player i decides to report c_2 instead of c_1, then he loses the gains associated with the switches with followers of type c_2; he will switch with predecessors of type c_1 paying them more than such a switch is worth for him. However, there will be an increase in the payment player i will receive from followers of type c_3. Therefore

$$\begin{aligned} &U_i((\chi^t_{-i}, c_2), c_1) - U_i((\chi^t_{-i}, c_1), c_1) = (n-i)\cdot p(c_3)[c_2 + \varepsilon(c_3 - c_2) - c_1 - \varepsilon(c_3 - c_1)] \\ &\qquad - (n-i)\cdot p(c_2)[c_1 + \varepsilon(c_2 - c_1) - c_1] - (i-1)\cdot p(c_1)[c_1 + \varepsilon(c_2 - c_1) - c_1] \\ &= (n-i)\cdot[p(c_3)\cdot(c_2 - c_1)(1-\varepsilon) - p(c_2)\cdot\varepsilon(c_2 - c_1)] - (i-1)\cdot p(c_1)\cdot\varepsilon(c_2 - c_1) \end{aligned} \tag{21}$$

(ii) Suppose now that player i is of type c_2.

For a player i of type c_2, assuming that all other players are truthfully reporting their urgency types, there is no incentive to deviate from TT-equilibrium, as we show in the following:

- If player i reports c_3 instead of c_2, then he will lose his chance to gain from switching with followers of type c_3. On the other hand, he will have to pay more to predecessors of type c_1 with whom he switches, and he will have to make additional switches with predecessors of type c_2, paying them more than he gains from the switches.
- Similarly, player i will lose from reporting c_1 instead of c_2 since he will have to switch with followers of type c_2 (who will not pay enough for compensating him); he will still switch with followers of type c_3, but he will receive smaller payments from them (relative to the payments if he announced c_2); and he will lose the gains from switches with predecessors of type c_1.

(iii) Finally, suppose that player i is of type c_3.

- If player i reports c_1 instead of c_3, he will lose the gains from switches with predecessors of type c_1 and c_2. In addition he will lose from switches he will make with followers of type c_2 and c_3 (as he will not be paid sufficiently).
- If player i reports c_2 instead of c_3, he will lose the gains from switches with predecessors of type c_2, and will also lose from switches he will make with followers of type c_3. However, he will gain from switches with predecessors of type c_1, to whom he will have to pay less. Hence,

$$\begin{aligned} &U_i((\chi^t_{-i}, c_2), c_3) - U_i((\chi^t_{-i}, c_3), c_3) = (i-1)\cdot p(c_1)[c_1 + \varepsilon(c_3 - c_1) - c_1 - \varepsilon(c_2 - c_1)] \\ &\qquad - (i-1)\cdot p(c_2)[c_3 - c_2 - \varepsilon(c_3 - c_2)] - (n-i)\cdot p(c_3)[c_3 - c_2 - \varepsilon(c_3 - c_2)] \\ &= (i-1)\cdot[p(c_1)\cdot\varepsilon(c_3 - c_2) - p(c_2)\cdot(c_3 - c_2)(1-\varepsilon)] - (n-i)\cdot p(c_3)(c_3 - c_2)(1-\varepsilon) \end{aligned} \tag{22}$$

Following the above analysis, all the conditions come down to two effective conditions that need to be satisfied for every player i to ensure the existence of the TT-equilibrium in G^ε, which are

$$U_i((\chi^t_{-i}, c_2), c_1) - U_i((\chi^t_{-i}, c_1), c_1) \le 0 \text{ and } U_i((\chi^t_{-i}, c_2), c_3) - U_i((\chi^t_{-i}, c_3), c_3) \le 0. \tag{23}$$

Now, consider (21), and notice that for player 1 we have

$$U_1((\chi^t_{-1}, c_2), c_1) - U_1((\chi^t_{-1}, c_1), c_1) = (n-1)[p(c_3)\cdot(c_2 - c_1)(1-\varepsilon) - p(c_2)\cdot\varepsilon(c_2 - c_1)] \le 0, \tag{24}$$

implying that

$$p(c_3) \cdot (a_2 - a_1)(1 - \varepsilon) - p(c_2) \cdot \varepsilon(a_2 - a_1) \leq 0. \tag{25}$$

One can easily verify that when (25) holds, then $U_i((\chi^t_{-i}, c_2), c_1) - U_i((\chi^t_{-i}, c_1), c_1) \leq 0$ for each $i \in N$, since the last term in (21) is always non-positive.

Similarly, for $U_i((\chi^t_{-i}, c_2), c_3) - U_i((\chi^t_{-i}, c_3), c_3) \leq 0$ to hold for each $i \in N$, we can find from (22) (by considering player n) that it is necessary and sufficient that

$$p(c_1) \cdot \varepsilon(c_3 - c_2) - p(c_2) \cdot (c_3 - c_2)(1 - \varepsilon) \leq 0. \tag{26}$$

In case $p(c_1) \neq 1$ and $p(c_3) \neq 1$, one obtains from (25) and (26) .

$\varepsilon \geq \frac{p(c_3)}{p(c_3)+p(c_2)}$ and $\varepsilon \leq \frac{p(c_2)}{p(c_1)+p(c_2)}$, respectively, and it is easy to further verify that these conditions may only hold if $p(c_2) \geq \sqrt{p(c_1) \cdot p(c_3)}$.

Now, to analyze the case $p(c_1) = 1$, and the case $p(c_3) = 1$, we use straightforwardly inequalities (25) and (26). We obtain that $E(p) = \{GS^0\text{-rule}\}$ in case $p(c_1) = 1$, and $E(p) = \{GS^1\text{-rule}\}$ in case $p(c_3) = 1$. ■

See Appendix 1 for a graphical representation of the subclass of distributions described in this section.

4. Impossibility Results

First, we consider queueing situations $\langle N, C, p\rangle$ with $|C| > 3$ and $p > 0$.

Theorem 4.1 *Let $\langle N, C, p\rangle$ be a queueing situation with $|C| > 3$ and $p > 0$, and let G^ε be the corresponding Bayesian game. Then $E(p) = \phi$.*

proof For a TT-equilibrium to hold in the case of more than three urgency types there are many non-trivial conditions to be satisfied (e.g. when $|C| = 4$, for each player there are six inequalities instead of two). However, to prove the above statement, it is enough to look at two conditions. Specifically, we consider $U_1((\chi^t_{-i}, c_2), c_1) - U_1((\chi^t_{-i}, c_1), c_1) \leq 0$ and $U_n((\chi^t_{-i}, c_{k-1}), c_k) - U_n((\chi^t_{-i}, c_k), c_k) \leq 0$.

For a TT-equilibrium of G^ε to exist, a GS^ε-rule needs to satisfy (among other conditions):

$$U_1((\chi^t_{-i}, c_2), c_1) - U_1((\chi^t_{-i}, c_1), c_1) = \\ [(1-\varepsilon)(c_2 - c_1) \cdot \sum_{r=3}^{k} p(c_r) - p(c_2) \cdot \varepsilon(c_2 - c_1)](n-1) \leq 0, \tag{27}$$

$$U_n((\chi^t_{-i}, c_{k-1}), c_k) - U_n((\chi^t_{-i}, c_k), c_k) = \\ [\varepsilon(c_k - c_{k-1}) \cdot \sum_{r=1}^{k-2} p(c_r) - p(c_{k-1}) \cdot (1-\varepsilon)(c_k - c_{k-1})](n-1) \leq 0. \tag{28}$$

Since $p > 0$, from (27) and (28) we obtain $\varepsilon \geq \dfrac{\sum_{r=3}^{k} p(c_r)}{\sum_{r=2}^{k} p(c_r)}$ and $\varepsilon \leq \dfrac{p(c_{k-1})}{\sum_{r=1}^{k-1} p(c_r)}$.

Hence, for such a GS^ε-rule to exist, we need $\dfrac{\sum_{r=3}^{k} p(c_r)}{\sum_{r=2}^{k} p(c_r)} \leq \dfrac{p(c_{k-1})}{\sum_{r=1}^{k-1} p(c_r)}$ to hold or, equivalently,

$$\sum_{r=1}^{k-2} p(c_r) \cdot \sum_{r=3}^{k} p(c_r) \leq p(c_2) \cdot p(c_{k-1}). \tag{29}$$

For $|C| \geq 4$ and $p > 0$ one can verify that $\sum_{r=1}^{k-2} p(c_r) \cdot \sum_{r=3}^{k} p(c_r) > p(c_2) \cdot p(c_{k-1})$. Hence, we can conclude that for queueing situations $\langle N, C, p\rangle$ with $|C| > 3$ and $p > 0$ there is no GS^{ε}-rule for which a TT-equilibrium of the Bayesian game G^{ε} exists. ■

Remark 4.1 : Note that in case $|C| = 3$ inequality (29) holds for a subclass of queueing situations and we obtain again the condition $p(c_2) \geq \sqrt{p(c_1) \cdot p(c_3)}$ that we have established before in Section 3.

Remark 4.2 : For queueing situations $\langle N, C, p\rangle$ with $|C| > 3$ and a degenerate probability distribution p, there may still exist GS^{ε}-rules leading to truth-telling. For example, if $|C| = 4$ in case $p(c_1) = 1$, $E(p) = \{GS^0\}$, while $E(p) = \{GS^1\}$ in case $p(c_4) = 1$.

Remark 4.3 : At least partial efficiency and TT-equilibrium can be reached in some queueing situations $\langle N, C, p\rangle$ with $|C| > 3$ and $p > 0$ by clustering the possible values for cost-per-unit-of-time parameter as to obtain three type-ranges: a low-cost group, a middle-cost group consisting of a single value, and a high-cost group. A representative value for each type-range is defined (which serves for calculating the payments according to a GS^{ε}-rule), and players are asked to report to which type-range they belong. If the representative value is defined as the highest value in the low-cost group , the lowest value for the high-cost group, and the unique value for the middle-cost group, then a truth-telling equilibrium can be achieved under a condition similar to that given in Theorem 4.1.

Until now we have considered queueing situations with incomplete information $\langle N, C, p\rangle$, where players face a revelation problem only with respect to their cost per unit of time. In the following we focus on the model $\langle N, T, p\rangle$ with $|S| \geq 2$. We show that for such situations there is no GS^{ε}-rule such that a TT-equilibrium of the related game G^{ε} exists if players are asked to report their processing time (either in addition to reporting the cost per unit of time, or in the case all players have the same cost per unit of time).

Theorem 4.2 *Let $\langle N, T, p\rangle$ be a sequencing situation with incomplete information, with $|S| \geq 2$ and $p > 0$, and let G^{ε} be the corresponding Bayesian game. Then $E(p) = \phi$.*

proof We look at player 1, and assume that his true type is $t_1 = (c_k, s_1)$. Assume also that all other players are reporting their true types. If player 1 reports his true type, then his urgency (c_k/s_1) is the highest possible, and he will not change places with any other player.

We now consider the situation where player 1 deviates from truth-telling by reporting $a_1 = (c_k, s_m)$. Then, with positive probability, there will be followers i of player 1 whose type is $t_i = (t_i^C, t_i^S)$, such that $\frac{t_i^C}{t_i^S} > \frac{c_k}{s_m}$, with whom he will switch places (assuming they announce their true value). As a result of each such switch player 1 will lose $c_k \cdot t_i^S$ due to moving back, but he will receive from player i the amount $c_k \cdot t_i^S + \varepsilon(t_i^C \cdot s_m - c_k \cdot t_i^S)$. This shows that for any $\varepsilon > 0$ player 1 is (strictly) better off by deviating from the truth-telling strategy and reporting $a_1 = (c_k, s_m)$.

To complete the proof, we need only to consider the case $\varepsilon = 0$. Consider that player n is of type $t_n = (c_1, s_m)$. It can be easily checked that for every $\varepsilon < 1$, it is strictly profitable for player n to deviate from the truth-telling strategy by reporting $a_n = (c_1, s_1)$. ■

5. Concluding Remarks

This paper deals with sequencing situations with incomplete information where the cost per unit of time and/or processing time of players are private information. Interaction among players to rearrange the queue optimally (i.e., in decreasing order of their urgencies) is only possible if players announce their characteristics. The question we address here is: Can we expect the truthful revelation of players' private information?

To analyze players' incentives to misreport their characteristics we use here sharing rules of type GS^{ε} and characterize those GS^{ε}-rules for which the related Bayesian game has a truth-telling equilibrium. While seeking efficiency of the queue (i.e., minimizing the expected aggregate costs of time spent in the system) , one would like to consider the rights of the players, granted by their original place in the queue. When devising rules for reordering the queue, we would like the players to have an incentive to willingly participate in the process. This leads us to considering only rules that lead to core elements of cooperative sequencing games related to sequencing situations with complete information. The class of GS^{ε}-rules has such a property, and has received much attention in the analysis of cooperative sequencing games arising from sequencing situations with complete information. Intuition might suggest that using other rules that lead to core elements of the cooperative sequencing game will not generate qualitatively different results for the existence of TT-equilibrium of the corresponding Bayesian games. Future research, however, might try to locate other rules for which a TT-equilibrium exists for the case of 4 or more possible values for the cost-per-unit-of-time parameter.

We have shown that in the case with only two possible values for cost per unit of time and processing time 1, for each $\varepsilon \in [0,1]$ the corresponding GS^{ε}-rule supports a truth-telling equilibrium. No player has any incentive to deviate from truth-telling strategy. Indeed, it is not difficult to verify, that for this model, truth-telling is actually a weakly dominant strategy for any GS^{ε}-rule with $\varepsilon \in [0,1]$. The incentives for deviating from the truth-telling strategy come into life when there are at least three possible values for the cost per unit of time. For sequencing situations with three possible values for cost per unit of time and processing time 1 we find and characterize a subclass of GS^{ε}-rules that support truthful revelation of private information for a subclass of probability distributions. Notice that this subclass of GS^{ε}-rules does not depend on the differences in the valuation of the parameter cost per unit of time, but only on the probabilities with which these values may occur. Specifically, the EGS-rule is the only rule that supports the truth-telling equilibrium when the probabilities for the three different urgency types are the same, and it also supports truth-telling in the widest range of sequencing situations with incomplete information.

However, for nondegenerate probability distributions beyond the class characterized in Theorem 3.2, no GS^{ε}-rule which supports a truth-telling mechanism can be found. So, we can conclude that there is no "magic rule" of type GS^{ε} that supports truth-telling in all queueing situations $\langle N, C, p\rangle$ with $|C| = 3$. Moreover, we show that for nondegenerate queueing situations $\langle N, C, p\rangle$ with $|C| > 3$, as well as for sequencing situations where players have to announce their processing time, there is no GS^{ε}-rule which supports a truth-telling equilibrium of the related Bayesian game.

We end this section by comparing our work on incentive compatibility in the framework of one-machine sequencing situations with linear costs under incomplete information with the models by Suijs [13] and Mitra [9][10]. First, we note that in our model such situations are considered as completely decentralized decision making problems whereas Suijs and Mitra consider them as (examples of centralized) public decision making problems. As a result, based on the announced types by the individual players, in the models by Suijs and Mitra the social decision maker has to create a queue for service, while in our model the initial queue has to be rearranged by (the decentralized pairwise interaction of) the players themselves. Secondly, the study of incentive compatibility in Suijs and Mitra focuses on Groves schemes whereas we base our study on gain split rules which consider explicitly players' original rights in the queue. Thirdly, Suijs and Mitra deal with the first best implementability of a sequencing (queueing) problem, whereas we deal with the existence of TT-equilibria for the related Bayesian game. Specifically, in the models by Suijs and Mitra, the social decision maker aims to design a mechanism (whose transfers are based on Groves schemes) that satisfies efficiency (i.e. minimizes the total cost), budget balancedness (i.e. the sum of all transfers equals zero) and dominant strategy incentive compatibility (i.e. announcing the true type is always weakly better for any agent whatever the other agents announce). We remark that in their possibility and impossibility results for dominant strategy incentive compatibility, the number of agents plays a role. Specifically, an impossibility result for the first best implementability is obtained for the case of two agents (see Theorem 4.1 in Suijs [13]) and a basic possibility result is proved by Mitra if and only if there are at least three agents (see Theorem 3.1 in Mitra [10]). In our model, the pairwise interaction among agents via gain split rules corresponds to a mechanism (implicitly designed by the whole group of agents) based on binary transfers in neighbor switches. This mechanism also satisfies efficiency and budget balancedness. In our possibility and impossibility results concerning the existence of TT-equilibria, the number of types of the players plays a role, while the number of players is unrestricted. We obtain a possibility result for the case of two types, find conditions to be satisfied by the common a priori probability distribution of players on the set of types for the case of three types to assure the existence of TT-equilibria, and obtain impossibility results otherwise. Finally, both Suijs and Mitra address the question of individual rationality of a Groves mechanism (i.e. individual incentives for agents to participate in the process) and answer it either negatively for sequencing situations (see Suijs [13]) or show that if the benefit derived for each player from the service is sufficiently high, then a first best implementable queueing problem satisfies individual rationality (see Section 6 in Mitra [9]). In our model individual rationality is satisfied always, even in a Bayesian Equilibrium which is not a TT-equilibrium.

The above comparative study leads us to the conclusion that our work and that of Suijs and Mitra on sequencing (queueing) situations with linear costs and incomplete information could be considered rather complementary.

Appendix 1: Regions of Feasible Probability Vectors

In the following, we visualize graphically the region(s) in the 3-type probability space for which a truth-telling equilibrium may be achieved.

Let $\Delta = \{(x_1, x_2, x_3) \in \Re^3_+ | x_1 + x_2 + x_3 = 1\}$ be the probability simplex. Clearly $(p(c_1), p(c_2), p(c_3)) \in \Delta$.

Then, for each $\varepsilon \in [0,1], R(\varepsilon) = \{x \in \Delta | (1-\varepsilon)x_3 - \varepsilon \cdot x_2 \leq 0, (1-\varepsilon)x_2 - \varepsilon \cdot x_1 \geq 0\}$ is the region of probability vectors for which TT-equilibrium exists for G^ε if a GS^ε-rule is used. The region $R(\varepsilon)$ for $\varepsilon = 0.25$ is depicted in Figure 1.

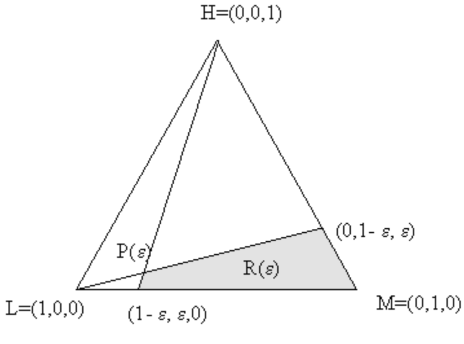

Figure 1: $R(\varepsilon)$

Note that for every $\varepsilon \in (0,1)$, $R(\varepsilon)$ is a quadrangle with extreme points $M = (0,1,0)$,

$(0,1-\varepsilon,\varepsilon)$, $P(\varepsilon)=(1-\varepsilon+\varepsilon^2)^{-1}(1-2\varepsilon+\varepsilon^2,\varepsilon-\varepsilon^2,\varepsilon^2)$, $(1-\varepsilon,\varepsilon,0)$. $R(0)$ is the line segment with L and M as endpoints; $R(1)$ is the line segment with H and M as endpoints. For $\varepsilon=0.5$, $P(\varepsilon)$ coincides with the barycenter $B=(1/3,1/3,1/3)$ of the triangle $\langle L,M,H\rangle$. So, TT-equilibrium exists for $p=(1/3,1/3,1/3)$ if the EGS-rule is considered. It is not difficult to see that for $p=(1/3,1/3,1/3)$ no other GS^{ε}-rule supports TT-equilibrium. Note further that for each $\varepsilon\in[0,1]$, the vector $M=(0,1,0)\in R(\varepsilon)$. Interesting is the region $R=\cup\{R(\varepsilon)|\varepsilon\in[0,1]\}$ since it contains the points with the property that there exists at least one $\varepsilon\in[0,1]$ that satisfies the conditions for the existence of a truth-telling equilibrium. The region R whose shape is shown in Figure 2, is bounded by the segment lines LM, MH and the curve LBH, where B is the barycenter of the triangle $\langle L,M,H\rangle$. The region $\Delta\setminus R$ is convex. The smooth curve LBH consists of the points $(x_1,x_2,x_3)\in\Delta$ with $x_2^2=x_1\cdot x_3$. The points on the segment curve LB are of the form $\left(\frac{1}{2}(1-x_2)+\frac{1}{2}\sqrt{1-2x_2-3x_2^2},x_2,\frac{1}{2}(1-x_2)-\frac{1}{2}\sqrt{1-2x_2-3x_2^2}\right)$ where $x_2\in\left[0,\frac{1}{3}\right]$, and the points on the segment curve BH are of the form
$\left(\frac{1}{2}(1-x_2)-\frac{1}{2}\sqrt{1-2x_2-3x_2^2},x_2,\frac{1}{2}(1-x_2)+\frac{1}{2}\sqrt{1-2x_2-3x_2^2}\right)$ where $x_2\in\left[0,\frac{1}{3}\right]$.

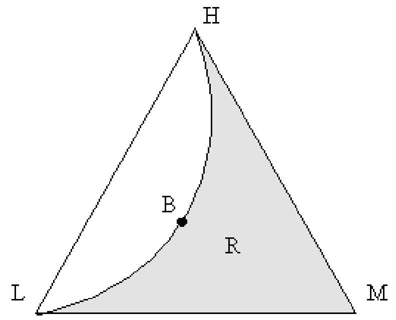

Figure 2: $R=\cup\{R(\varepsilon)|\varepsilon\in[0,1]\}$

For each $x_2\in\left[0,\frac{1}{3}\right]$ the above points are the unique points with $x_2^2=x_1\cdot x_3$. Note that $f:\left[0,\frac{1}{3}\right]\rightarrow\Re$ with $f(x_2)=\frac{1}{2}(1-x_2)+\frac{1}{2}\sqrt{1-2x_2-3x_2^2}$ is a concave function.

We next show that the surface of $R(\varepsilon)$ is directly proportional to z_2, the distance of $P(\varepsilon)$ from LH (see Figure 3). We look at the triangle $D(\varepsilon)=\langle L,M,P(\varepsilon)\rangle$. Its area equals the area of $R(\varepsilon)$ since the triangles $\langle L,M,(0,1-\varepsilon,\varepsilon)\rangle$ and $\langle H,L,(1-\varepsilon,\varepsilon,0)\rangle$ are similar and both contain the smaller triangle $\langle L,P(\varepsilon),(1-\varepsilon,\varepsilon,0)\rangle$. Since the area of $R(\varepsilon)$ grows as z_2 grows, it is maximized when $\varepsilon=0.5$, and $P(\varepsilon)=B$. Then the triangle $D(0.5)$ coincides with $\langle L,H,B\rangle$, and its area is exactly 1/3 of the area of Δ. This means that the EGS rule supports truth-telling equilibria for the widest range of possible discrete 3-point distributions, specifically, 1/3 of the possible distributions of this type.

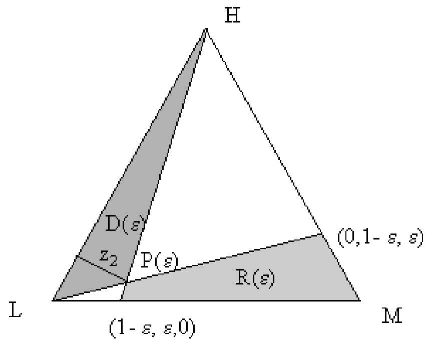

Figure 3: Surface of $R(\varepsilon)$

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