Journal of the Operations Research
Society of Japan
2006, Vol. 49, No. 3, 222-237

OPTIMAL INSPECTION SCHEDULE IN AN IMPERFECT EMQ MODEL WITH FREE REPAIR WARRANTY POLICY

Bibhas Chandra Giri
Jadavpur University

Tadashi Dohi
Hiroshima University

(Received November 16, 2004; Revised October 31, 2005)

Abstract The paper considers a sequential inspection policy in an imperfect production process which shifts randomly from an 'in-control' state to an 'out-of-control' state following a general probability distribution. Two different inspection policies are adopted in the proposed model: (i) no action is taken in the intermediate of a production run unless the process is found in an 'out-of-control' state by inspection and (ii) preventive repair action is undertaken once the 'in-control' state of the process is detected by inspection. The manufacturer is in a contractual agreement with the customer to provide free minimal repair service until a certain (warranty) period from the time of initial purchase. The objective is to determine the optimal number of inspections and inspection time sequence during a production run which minimize the manufacturer's future expected costs in present term or average cost in distant future. The proposed model is formulated under discounted as well as long-run average cost criteria and some structural properties on the optimal inspection policy are derived analytically. For a numerical example, the optimal inspection policy is determined and several managerial insights are investigated.

Keywords: Maintenance, inspection, EMQ model, imperfect repair, free repair warranty

1. Introduction

Economic manufacturing quantity (EMQ) models for batch production systems have been studied extensively in the literature and in most of the cases, manufacturer's optimal decisions have been derived based on the assumption that the outputs are of perfect quality. But the output quality often deteriorates due to variability in manufacturing processes. So production systems need to be maintained through adequate maintenance program to reduce product defects and cost, and to provide both quality products and better service to customer. Smart manufacturers are nowadays increasing their focus even on the post-sale maintenance (warranty service) to build good relationship with the customer and enhance brand reputation. During the past few decades, production and maintenance have so far been treated as two separate issues. Recently many researchers have turned their attention into the joint optimization of these two aspects of manufacturer's concern.

At about the same time, Rosenblatt and Lee [13] and Porteus [9] developed economic manufacturing quantity (EMQ) models which characterize the impact of process deterioration on lot sizing decisions. Assuming that the process shift from an 'in-control' state to an 'out-of-control' state follows an exponential distribution, Rosenblatt and Lee [13] found that the corresponding optimal EMQ is smaller than that of the classical one. Porteus [9] assumed that the process goes to an 'out-of-control' state with a given probability at each time when it produces an item, and observed the similar results to Rosenblatt and Lee [13]. After these seminal works, Lee and Rosenblatt [5], Porteus [10] and Lee and Park [6], among others, addressed EMQ problems where the process is monitored through inspections as it may randomly go to 'out-of-control'. For a deteriorating production process with increasing failure rate (IFR), Banerjee and Rahim [1] determined jointly the optimal design parameters on an $\overline{x}$ control chart and preventive replacement time. Rahim [11] and Rahim and Ben-Daya [12] investigated the effect of EMQ on the economic design of $\overline{x}$ control chart for deteriorating production processes where the 'in-control' period follows a general probability distribution with IFR. The process was inspected using a sampling frequency that increases with the age of the system. Tseng [14] introduced a preventive maintenance policy to enhance the system reliability instead of inspection policy into an imperfect EMQ model. Makis [7] considered the joint determination of the lot size and the inspection schedule, minimizing the long-run expected cost per unit time when in-control periods are generally distributed and inspections are imperfect.

To remain in the fray of the competitive market, manufacturers or distributors frequently offer a warranty service with the sale of their products which in turn results in additional costs due to servicing of any item fails during the warranty period. If any product does not work satisfactorily within a limited period after the sale, then it is manufacturer's responsibility to repair or replace the product with no cost to the customer or refund a fraction or whole of the sale price. Djamaludin *et al.* [3] were the first who studied the effect of product warranty on the optimal lot sizing policy. They assumed that the production system can shift to an 'out-of-control' state with a given probability at each time when an item is produced. Describing the production process as a two-state discrete time Markov chain, they derived the expected cost per unit item under free repair warranty (FRW) and use it as a criterion of optimality. Yeh *et al.* [18] reformulated Djamaludin *et al.*'s model [3] by considering exponentially distributed process shift distribution. Wang [16] and Wang and Sheu [17] further extended Yeh *et al.*'s model [18] by incorporating continuous and discrete time general shift distributions, respectively. However, inspection and maintenance in the intermediate of a production run are not allowed in Wang's models [15, 16]. Only at the end of a production lot, the system is inspected once to detect the state of the production process. If the system is found in an 'out-of-control' state then it is restored to 'in-control' state with some additional cost, otherwise, preventive maintenance is carried out to return back the system to 'as good as new' condition.

Inspection of production process is common in modern industries, especially in deteriorating production systems. It reduces the defective item cost and the post-sale warranty servicing cost but at the same time increases the maintenance cost. So there must be a trade off relationship between these costs to keep the manufacturer's expected cost at the minimum level. In this paper, we consider an imperfect production process where process shift follows a general probability distribution, sequential inspections are performed during a production run and produced items are sold with a free minimal repair warranty. The objective of this study is to determine the optimal number of inspections and the inspection sequence which minimize the manufacturer's expected cost. We develop the stochastic model under two different cost criteria: discounted and long-run average costs.

The rest of the paper is organized as follows. The next section explains the assumptions and notation of the proposed model and provides the model description under two types of inspection policies I and II adopted in this paper. Sections 3 and 4 deal with the formulation and analysis of the stochastic model under inspection policy I and II, respectively. Some characteristics of the model are studied analytically in these sections. Section 5 presents the numerical solution of the developed model and gives managerial insights on optimal decisions. Finally, in Section 6, we conclude the paper and suggest some future research directions.

2. Model Description

2.1. Notation

Throughout the paper we use the following notation:

$f(\cdot)$, $F(\cdot)$: probability density function, probability distribution function of the time to process shift
$\overline{F}(\cdot)$: the survivor function *i.e.*, $\overline{F}(\cdot) = 1 - F(\cdot)$
$r(\cdot)$: failure rate of $F(\cdot)$
D (> 0): constant demand rate
P $(> D)$: constant production rate
c_s (> 0): setup cost
c_m (> 0): manufacturing cost per unit product
c_h (> 0): inventory holding cost per unit product per unit time
c_r (> 0): cost of each minimal repair
v_0 (> 0): inspection cost
v_1 (> 0): preventive maintenance cost
W (> 0): free repair warranty period
n (≥ 1): number of inspections undertaken during each production run
T_i: elapsed time from the beginning of the production run until the i-th inspection takes place, $\forall\ i = 1, 2, \cdots, n$
$\{T_1, T_2, \cdots, T_n\}$: inspection time sequence
T $(= T_n)$: production run length
t_i: the i-th inspection interval, *i.e.*, $t_i = T_i - T_{i-1},\ \forall\ i = 1, 2, \cdots, n;\ T_0 = 0$
ρ (> 0): restoration cost per unit detection delay time
θ_1: probability that the produced item in the 'in-control' state is non-conforming
θ_2 $(> \theta_1)$: probability that the produced item in the 'out-of-control' state is non-conforming
$F_1(t), F_2(t)$: lifetime distributions of conforming and non-conforming items
$r_1(t)$: hazard rate of a conforming item
$r_2(t)$: hazard rate of a non-conforming item, where $r_2(t) >> r_1(t),\ 0 < t < \infty$
δ (> 0): discount factor.

2.2. Basic assumptions

(1) The production process always starts in an 'in-control' state but it may shift to an 'out-of-control' state at any random time.
(2) The duration of 'in-control' period follows an arbitrary probability distribution.
(3) Each inspection is perfect and takes only a negligible time.
(4) Restoration cost is proportional to the detection delay time. After restoration, the system can be returned back to the original 'in-control' state.
(5) The product defects are not readily detectable; the discovery occurs only through time testing.
(6) The produced items are repairable and are sold with a free minimal repair warranty.

2.3. Inspection scheme

We consider the production of a single item on a single machine which starts in an 'in-control' state to produce items of acceptable quality at a rate P (> 0) to meet a demand rate D $(< P)$ for the product. The inventory builds up with a rate $(P - D)$ during the production run and reaches the maximum level $(P - D)T$ at time T. Afterwards it decreases monotonically with a rate D and reaches ultimately to the zero level at time PT/D. However, in the production phase, if the process shifts from an 'in-control' state to an 'out-of-control' state then some

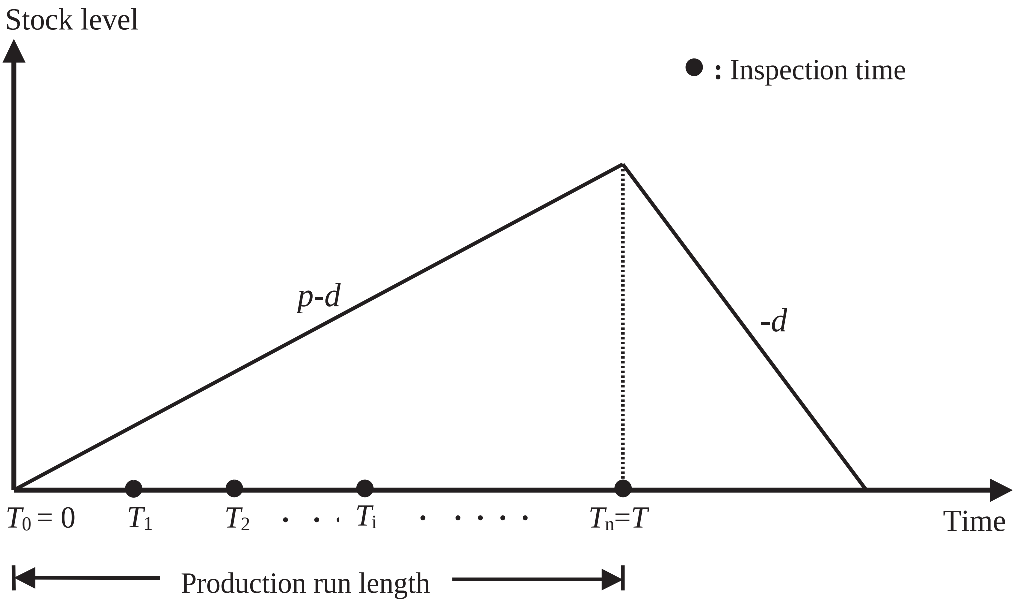

Figure 1: Configuration of the EMQ model with inspections.

percentage of produced items are of substandard quality, although a few substandard items could have been produced when the process was in 'in-control' state. We assume that the time to process shift follows a general probability distribution $F(t)$ with density function $f(t)$. During the production run, inspections are carried out at times $T_1, T_2, \cdots, T_n = T$ to assess its state. See Figure 1 for the configuration of the EMQ process with inspections.

The following two inspection policies are adopted for the proposed model:

Policy I: At each inspection, if the process is found in an 'out-of-control' state then restoration is done, otherwise, no action is taken with the exception of the last inspection at time T_n where preventive maintenance (PM) is done to ensure that the system will be in 'in-control' state at the beginning of the next production run.

Policy II: At each inspection, if the process is found in an 'out-of-control' state then restoration is done, otherwise, PM is performed to enhance the system reliability.

3. Model Formulation and Analysis under Inspection Policy I

In the following, we derive the manufacturer's expected total discounted cost (pre-sale + post-sale) over an infinite time horizon under inspection policy I and use it as the criterion of optimality. We define the inventory holding time of one production lot together with the warranty contract time as one cycle. Then the expected total discounted cost for holding inventory per cycle is given by

$$c_h \left[\int_0^T (P-D) z e^{-\delta z} dz + \int_T^{PT/D} (PT - Dz) e^{-\delta z} dz \right] = \frac{c_h}{\delta^2} \Big[(P-D)(1-e^{-\delta T}) + D\left(e^{-\delta PT/D} - e^{-\delta T} \right) \Big].$$

Let P_j be the probability that the production process needs to be restored at time T_j, $\forall\, j = 1, 2, \cdots, n$. Then, we have

$$\begin{aligned}
P_1 &= P_0 \int_0^{T_1} dF(t) = P_0[F(T_1) - F(0)] = P_0[F(T_1 - T_0) - F(T_0 - T_0)], \\
P_2 &= P_0 \int_{T_1}^{T_2} dF(t) + P_1 \int_0^{T_2 - T_1} dF(t) \\
&= P_0[F(T_2 - T_0) - F(T_1 - T_0)] + P_1[F(T_2 - T_1) - F(T_1 - T_1)], \\
P_3 &= P_0 \int_{T_2}^{T_3} dF(t) + P_1 \int_{T_2 - T_1}^{T_3 - T_1} dF(t) + P_2 \int_0^{T_3 - T_2} dF(t) \\
&= P_0[F(T_3 - T_0) - F(T_2 - T_0)] + P_1[F(T_3 - T_1) - F(T_2 - T_1)] \\
&\quad + P_2[F(T_3 - T_2) - F(T_2 - T_2)],
\end{aligned}$$

and in general,

$$P_j = \sum_{i=0}^{j-1} P_i \{F(T_j - T_i) - F(T_{j-1} - T_i)\}, \quad j = 1, 2, \cdots, n;$$

where $P_0 = 1$.

Let N_i denote the conditional expectation of the number of non-conforming items produced in the time interval $(T_{i-1}, T_i]$, $i = 1, 2, \cdots, n$ and τ_k denote the random elapsed time of the first shift from the 'in-control' state to the 'out-of-control' state given that the previous restoration was held at time T_k, $k = 0, 1, 2, \cdots, i-1$. Then we have

$$N_1 = \begin{cases} P\theta_1 \tau_0 + P\theta_2 (T_1 - \tau_0), & \text{if } T_0 < \tau_0 \le T_1; \\ P\theta_1 T_1, & \text{otherwise.} \end{cases}$$

Therefore, the expected number of non-conforming items produced in $(0, T_1]$ is

$$E(N_1) = \int_0^{T_1} \{P\theta_1 \tau_0 + P\theta_2 (T_1 - \tau_0)\}\, dF(\tau_0) + \int_{T_1}^{\infty} P\theta_1 T_1\, dF(\tau_0). \tag{3.1}$$

The conditional expectation of the number of non-conforming items produced in the second period $(T_1, T_2]$ is

$$N_2 = \begin{cases} P\theta_1 (\tau_j - T_{1-j}) + P\theta_2 (T_2 - T_j - \tau_j), & \text{if } T_1 - T_j < \tau_j \le T_2 - T_j; \\ P\theta_1 (T_2 - T_1), & \text{if } \tau_j > T_2 - T_j, \quad \forall\, j = 0, 1. \end{cases}$$

Therefore, the expected number of non-conforming items produced in the time interval $(T_1, T_2]$ is given by

$$\begin{aligned}
E(N_2) &= \sum_{j=0}^{1} P_j \Big[\int_{T_1 - T_j}^{T_2 - T_j} \{P\theta_1 (\tau_j - T_{1-j}) + P\theta_2 (T_2 - T_j - \tau_j)\}\, dF(\tau_j) \\
&\quad + \int_{T_2 - T_j}^{\infty} P\theta_1 (T_2 - T_1)\, dF(\tau_j) \Big].
\end{aligned} \tag{3.2}$$

By the same argument, the expected number of non-conforming items produced in the i-th period $(T_{i-1}, T_i]$ is given by

$$E(N_i) = \sum_{j=0}^{i-1} P_j \Big[\int_{T_{i-1}-T_j}^{T_i-T_j} \{P\theta_1(\tau_j - T_{i-1-j}) + P\theta_2(T_i - T_j - \tau_j)\}\, dF(\tau_j) + P\theta_1(T_i - T_{i-1})\overline{F}(T_i - T_j)\Big]. \quad (3.3)$$

Therefore, the fraction of non-conforming items produced in a production lot is

$$\Delta_{12} = \frac{1}{PT}\sum_{i=1}^{n}\sum_{j=0}^{i-1} P_j \Big[\int_{T_{i-1}-T_j}^{T_i-T_j} \{P\theta_1(\tau_j - T_{i-1-j}) + P\theta_2(T_i - T_j - \tau_j)\}\, dF(\tau_j) + P\theta_1(T_i - T_{i-1})\overline{F}(T_i - T_j)\Big] \quad (3.4)$$

and hence the fraction of conforming items produced in a lot is $\Delta_{11} = 1 - \Delta_{12}$.

The present value of the expected restoration cost in the time interval $(0, T_1]$ is

$$\rho \int_0^{T_1} \int_0^{T_1-\tau_0} e^{-\delta(\tau_0+y)} dy\; dF(\tau_0).$$

Similarly, the present value of the expected restoration cost in the time interval $(T_1, T_2]$ is

$$\rho \sum_{j=0}^{1} P_j \int_{T_1-T_j}^{T_2-T_j} \int_0^{T_2-T_j-\tau_j} e^{-\delta(T_j+\tau_j+y)} dy\; dF(\tau_j). \quad (3.5)$$

In general, since the present value of the expected restoration cost in the i-th period $(T_{i-1}, T_i]$ is given by

$$\rho \sum_{j=0}^{i-1} P_j \int_{T_{i-1}-T_j}^{T_i-T_j} \int_0^{T_i-T_j-\tau_j} e^{-\delta(T_j+\tau_j+y)} dy\; dF(\tau_j), \quad (3.6)$$

the net present value of the expected pre-sale cost per cycle is derived as

$$\begin{aligned} A(n, T_1, T_2, \cdots, T_n) &= c_s + c_m P T_n e^{-\delta T_n} + \sum_{i=1}^{n} v_0 e^{-\delta T_i} + v_1 \sum_{j=0}^{n-1} P_j \overline{F}(T_n - T_j) e^{-\delta T_n} \\ &\quad + \frac{c_h}{\delta^2}\left[(P - D)(1 - e^{-\delta T_n}) + D\left(e^{-\delta P T_n/D} - e^{-\delta T_n}\right)\right] \\ &\quad + \rho \sum_{i=1}^{n}\sum_{j=0}^{i-1} P_j \int_{T_{i-1}-T_j}^{T_i-T_j} \int_0^{T_i-T_j-\tau_j} e^{-\delta(T_j+\tau_j+y)} dy\; dF(\tau_j). \quad (3.7) \end{aligned}$$

Evidently the failure rates of conforming and non-conforming items are different in the post-sale period. So, the present value of the expected warranty cost per cycle depends on the individual number of conforming and non-conforming items sold in each time unit of the stock holding period. Since it is difficult to estimate these numbers, we assume for simplicity that the fractions of conforming and non-conforming items sold in each time unit are the same as those produced in a production lot. Further, we assume that the minimal repair time of an item is short enough relative to the FRW period W. Under these assumptions, the present value of the warranty servicing cost is given by

$$B(T_1, T_2, \cdots, T_n) = c_r \int_0^{PT_n/D} \left\{ \int_0^W D\Delta_{11} e^{-\delta(z+t)} r_1(t)dt + \int_0^W D\Delta_{12} e^{-\delta(z+t)} r_2(t)dt \right\} dz,$$

where $r_1(t)$ and $r_2(t)$ are the failure rates of the associated lifetime distributions of conforming and non-conforming items, respectively. Hence, the expected total discounted cost (pre-sale + post-sale) per cycle is given by

$$\begin{aligned} C_1(n, T_1, T_2, \cdots, T_n) &= A(n, T_1, T_2, \cdots, T_n) + B(T_1, T_2, \cdots, T_n) \\ &= c_s + c_m P T_n e^{-\delta T_n} + \sum_{i=1}^{n} v_0 e^{-\delta T_i} + v_1 \sum_{j=0}^{n-1} P_j \overline{F}(T_n - T_j) e^{-\delta T_n} \\ &\quad + \frac{c_h}{\delta^2} \left[(P - D)(1 - e^{-\delta T_n}) + D \left(e^{-\delta P T_n / D} - e^{-\delta T_n} \right) \right] \\ &\quad + \rho \sum_{i=1}^{n} \sum_{j=0}^{i-1} P_j \int_{T_{i-1}-T_j}^{T_i - T_j} \frac{e^{-\delta(T_j + \tau_j)} - e^{-\delta T_i}}{\delta} dF(\tau_j) \\ &\quad + c_r D \int_0^{PT_n/D} \left\{ \int_0^W \Delta_{11} e^{-\delta t} r_1(t) dt + \int_0^W \Delta_{12} e^{-\delta t} r_2(t) dt \right\} e^{-\delta z} dz. \end{aligned} \tag{3.8}$$

The expected length of a cycle is clearly given by $PT_n/D + W$. Therefore, the present value of one unit cost after one cycle becomes $\phi(T_n) = \exp\{-\delta(PT_n/D + W)\}$. Hence, the manufacturer's expected total discounted cost (TC_1) over the time horizon $[0, \infty)$, when the initial time point is taken to be the starting point of the production lot, is

$$\begin{aligned} TC_1(n, T_1, T_2, \cdots, T_n) &= \sum_{l=0}^{\infty} C_1(n, T_1, T_2, \cdots, T_n) \{\phi(T_n)\}^l \\ &= \frac{C_1(n, T_1, T_2, \cdots, T_n)}{1 - e^{-\delta(PT_n/D + W)}}. \end{aligned} \tag{3.9}$$

Our objective is to seek the optimal number of inspections n and the inspection sequence $\{T_1, T_2, \cdots, T_n\}$ which minimize $TC_1(n, T_1, T_2, \cdots, T_n)$.

Unfortunately, it is quite hard to determine the optimal values of $n, T_1, T_2, \cdots, T_n$ simultaneously. So, we proceed in the following to derive a sub-optimal inspection policy. We choose approximately the lengths of inspection intervals in such way that the integrated hazard over each interval is same for all intervals *i.e.*,

$$\int_{T_j}^{T_{j+1}} r(t)dt = \int_0^{T_1} r(t)dt, \quad j = 0, 1, 2, \cdots, n-1. \tag{3.10}$$

This approximate method is proposed by Munford [8]. For the application to the EMQ problem, see Rahim [11] and its references. If the time of production process staying in the in-control state follows a Weibull distribution whose probability density function is given by

$$f(t) = \lambda\beta(\lambda t)^{\beta-1} e^{-(\lambda t)^\beta}, \quad t > 0, \quad \beta \geq 1, \quad \lambda > 0,$$

then, from equation (3.10), the approximate inspection times can be determined recursively as

$$T_j = (T_{j-1}^{\beta} + T_1^{\beta})^{1/\beta}, \quad j = 1, 2, \cdots, n. \tag{3.11}$$

This reduces the cost function $TC_1(n, T_1, T_2, \cdots, T_n)$ to a function of two independent variables n and T_1 only. That is, we have

$$TC_1(n, T_1) = \frac{C_1(n, T_1)}{1 - e^{-\delta(PT_n/D+W)}}.$$

For any given n, the optimal value of T_1 can be obtained numerically by any one-dimensional search technique. Consequently, a line search on n would determine a local optimal solution. The assumption of constant hazard policy leads to the following fundamental result without the proof.

Proposition 3.1 *For any arbitrary number of inspections* n *during a production run, there exists a sub-optimal inspection time sequence* $\{T_1^0, T_2^0, \cdots, T_n^0\}$ *such that*

(i) $t_1^0 \geq t_2^0 \geq t_3^0 \geq \cdots \geq t_n^0$, *where* $t_j^0 = T_j^0 - T_{j-1}^0, \quad \forall\, j = 1, 2, \cdots, n,$

(ii) $\lim_{n\to\infty} \sum_{j=1}^{n} t_j^0 = \infty,$

(iii) $t_j^0 = t_1^0, \quad \forall\, j = 2, 3, \cdots, n$ *when* $\beta = 1$ *(exponential failure case), i.e., the periodic inspection policy is optimal.*

Remark 3.1 *A non-gradient based approach (see e.g. Hooke and Jeeves [4]) can be applied to minimize the cost function (3.9) directly by developing a pattern search algorithm.*

Remark 3.2 *The long-run average cost* $AC_1(n, T_1, T_2, \cdots, T_n)$ *can be easily obtained as the limiting value of the annualized total discounted cost as* $\delta \to 0^+$, *although we omit to show the result.*

4. Model Formulation and Analysis under Inspection Policy II

Under inspection policy II, either restoration or preventive maintenance is performed at each inspection. So, the process is in the original 'in-control' state immediately after each inspection. In this case, the fraction of non-conforming items produced in a production lot is given by

$$\Delta_{22} = \frac{1}{PT}\sum_{i=1}^{n}\Big[\int_0^{t_i}\{P\theta_1 t + P\theta_2(t_i - t)\}\, dF(t) + \int_{t_i}^{\infty} P\theta_1 t_i dF(t)\Big]. \tag{4.1}$$

Therefore, the fraction of conforming items produced in a production lot is $\Delta_{21} = 1 - \Delta_{22}$. The expected total discounted cost over an infinite time horizon is given by

$$TC_2(n, t_1, t_2, \cdots, t_n) = \frac{C_2(n, t_1, t_2, \cdots, t_n)}{1 - e^{-\delta(PT/D+W)}}, \tag{4.2}$$

where $T = t_1 + t_2 + \cdots + t_n$ and

$$\begin{aligned} C_2(n, t_1, t_2, \cdots, t_n) = \Big[& c_s + c_m PTe^{-\delta T} + \sum_{i=1}^{n} e^{-\delta\sum_{k=1}^{i} t_k}(v_0 + v_1\overline{F}(t_i)) \\ & + \frac{c_h}{\delta^2}\left\{(P - D)(1 - e^{-\delta T}) + D\left(e^{-\delta PT/D} - e^{-\delta T}\right)\right\} \\ & + (\rho/\delta)\sum_{i=1}^{n}\int_0^{t_i}\left\{e^{-\delta(\sum_{k=1}^{i-1} t_k + t)} - e^{-\delta\sum_{k=1}^{i} t_k}\right\}\, dF(t) \\ & + c_r D\int_0^{PT/D} e^{-\delta z}\left\{\int_0^W \Delta_{21} e^{-\delta t} r_1(t)dt + \int_0^W \Delta_{22} e^{-\delta t} r_2(t)dt\right\} dz\Big]. \end{aligned}$$

Remark 4.1 *When $n = 1$, i.e., the process is inspected only once at the end of each production run, (from equations (3.9) and (4.2)) we have $TC_1(1, T) = TC_2(1, T)$.*

Proposition 4.1 *Suppose that $v_1 f(t) \le (\rho + \delta v_1)F(t) \le v_1 f(t) + \delta(v_0 + v_1)$ for $0 \le t \le T$. For any given n and T, if $\theta_1 \to 0$ (i.e., no non-conforming item is produced in 'in-control' state) then there exists a unique inspection interval sequence $\{t_1^*, t_2^*, \cdots t_n^*\}$ which is a minimizer of $TC_2(n, t_1, t_2, \cdots, t_n)$ provided that the associated Hessian matrix, $\mathcal{H}$, is positive definite at $(t_1^*, t_2^*, \cdots t_n^*)$.*

Proof. For any given n and T, the optimal inspection schedule can be obtained by solving the following nonlinear programming (NP) problem:

$$\min_{t_1, t_2, \cdots, t_n} C_2(n, t_1, t_2, \cdots, t_n)$$

$$\text{subject to} \quad \sum_{i=1}^{n} t_i = T, \quad t_i > 0 \ \ \forall\, i = 1, 2, \cdots, n. \tag{4.3}$$

The Lagrangian function for the NP problem (4.3) is given by

$$L(t_1, t_2, \cdots, t_n) = C_2(n, t_1, t_2, \cdots, t_n) + \mu \left(T - \sum_{i=1}^{n} t_i \right),$$

where μ is the Lagrange multiplier. It is immediate to get the first order necessary conditions of optimality as

$$\begin{aligned} \frac{\partial L}{\partial t_i} &= 0, \quad \text{for} \quad i = 1, 2, \cdots, n; \\ \frac{\partial L}{\partial \mu} &= 0, \end{aligned} \tag{4.4}$$

which yield

$$\begin{aligned} G_i(t_1, t_2, \cdots, t_n) \equiv\ & -\delta \sum_{j=i}^{n} e^{-\delta \sum_{k=1}^{j} t_k} \left[v_0 + v_1 \overline{F}(t_j) + \rho \int_0^{t_j} \frac{e^{-\delta(t - t_j)} - 1}{\delta} dF(t) \right] \\ & - e^{-\delta \sum_{k=1}^{i} t_k} \left[v_1 f(t_i) - \rho \int_0^{t_i} e^{-\delta(t - t_i)} dF(t) \right] \\ & + \frac{c_r D}{T} \{\theta_1 + (\theta_2 - \theta_1) F(t_i)\} \int_0^{PT/D} \left\{ \int_0^{W} e^{-\delta(z+t)} r_2(t) dt \right. \\ & \left. - \int_0^{W} e^{-\delta(z+t)} r_1(t) dt \right\} dz - \mu = 0, \qquad i = 1, 2, \cdots, n \end{aligned} \tag{4.5}$$

and

$$\sum_{i=1}^{n} t_i = T. \tag{4.6}$$

Again, differentiating $G_i(t_1, t_2, \cdots, t_n)$ with respect to t_i we get

$$\frac{\partial G_i}{\partial t_i} = \delta^2 \sum_{j=i}^{n} e^{-\delta \sum_{k=1}^{j} t_k} \left[v_0 + v_1 \overline{F}(t_j) + \rho \int_0^{t_j} \frac{e^{-\delta(t - t_j)} - 1}{\delta} dF(t) \right]$$

$$
\begin{aligned}
&+e^{-\delta\sum_{k=1}^{i}t_k}\Big[(\rho+2\delta v_1)f(t_i)-\rho\delta\int_0^{t_i}e^{-\delta(t-t_i)}dF(t)-v_1\frac{\partial f(t_i)}{\partial t_i}\Big]\\
&+\frac{c_rD}{T}(\theta_2-\theta_1)f(t_i)\int_0^{PT/D}e^{-\delta z}\Big\{\int_0^W e^{-\delta t}r_2(t)dt-\int_0^W e^{-\delta t}r_1(t)dt\Big\}dz\\
&\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\cdots,n\\
>&\ \ 0, \qquad (4.7)
\end{aligned}
$$

by assumption $v_1f(t)\le(\rho+\delta v_1)F(t)\le v_1f(t)+\delta(v_0+v_1)$. A closer look on equations (4.5) reveals that a sequential backward search procedure can be applied to determine the optimal values of $t_n,t_{n-1},\cdots,t_1$ satisfying equations (4.5) and (4.6). The sequential approach for each of $i=n,n-1,n-2,\cdots,1$ in fact finds G_i as a function of single variable only. Also note that when $\theta_1\to0^+$, we have $\lim_{t_n\to0^+}G_n(t_n)<0$ and $\lim_{t_n\to\infty}G_n(t_n)>0$. Since $G_n(t_n)$ is increasing in $(0,\infty)$, therefore, there exists a unique positive root t_n^* of $G_n(t_n)=0$. Replacing $\sum_{k=1}^{n-1}t_k$ by $(T-t_n^*)$ in equation $G_{n-1}(t_{n-1})=0$ the optimal value of t_{n-1} can be determined in the next step. Proceeding in this way, the optimal inspection interval sequence $\{t_1^*,t_2^*,\cdots t_n^*\}$ can be obtained. This inspection sequence will be a minimizer of $TC_2(n,t_1,t_2,\cdots,t_n)$ provided that the associated Hessian matrix $\mathcal{H}$ is positive definite at the point $(t_1^*,t_2^*,\cdots t_n^*)$. This completes the proof of the proposition.

The following result follows immediately from the proof of Proposition 4.1.

Proposition 4.2 *When $\delta\to0$, for any given n and T, there exists a unique optimal periodic inspection schedule for which $t_1^*=t_2^*=\cdots=t_n^*=T/n$.*

By taking the limit as $\delta\to0$ and applying L'Hospital's rule, the long-run average cost $AC_2(n,t_1,t_2,\cdots,t_n)$ can be obtained from equation (4.2) as

$$
\begin{aligned}
AC_2(n,t_1,t_2,\cdots,t_n)&=\lim_{\delta\to0}\delta\cdot TC_2(n,t_1,t_2,\cdots,t_n)\\
&=\Big[c_s+c_mPT+\sum_{i=1}^{n}(v_0+v_1\overline{F}(t_i)+\frac{c_h(P-D)PT^2}{2D}\\
&\quad+\rho\sum_{i=1}^{n}\int_0^{t_i}(t_i-t)\,dF(t)+c_rD\int_0^{PT/D}\Big\{\int_0^W\Delta_{21}r_1(t)dt\\
&\quad+\int_0^W\Delta_{22}r_2(t)dt\Big\}\,dz\Big]/(PT/D+W). \qquad (4.8)
\end{aligned}
$$

Under the periodic inspection policy, the above equation takes the form:

$$
\begin{aligned}
AC_2(n,T)&=\Big[c_s+c_mPT+n(v_0+v_1\overline{F}(T/n)+\frac{c_h(P-D)PT^2}{2D}\\
&\quad+\rho n\int_0^{T/n}(T/n-t)\,dF(t)+c_rPT\Big\{\Delta_{31}\int_0^W r_1(t)dt\\
&\quad+\Delta_{32}\int_0^W r_2(t)dt\Big\}\Big]/(PT/D+W), \qquad (4.9)
\end{aligned}
$$

where

$$\Delta_{32}=\theta_1+(n/T)(\theta_2-\theta_1)\int_0^{T/n}F(t)dt\quad\text{and}\quad\Delta_{31}=1-\Delta_{32}.$$

To derive the optimal number of inspections for a known production time T, we define

$$\begin{aligned} R_W &= c_r P(\theta_2-\theta_1)\left\{\int_0^W r_2(t)dt - \int_0^W r_1(t)dt\right\}, \\ \psi(t) &= (\rho+R_W)F(t) - v_1 f(t). \end{aligned}$$

Proposition 4.3 *For any given T, if*

$$n\int_{T/(n+1)}^{T/n}\psi(t)dt - (n+1)\int_0^{T/n}\psi(t)dt < v_0+v_1, \quad \forall n \geq 1, \tag{4.10}$$

then the optimal number of inspections which minimizes $AC_2(n,T)$ is given by $n^*=1$, otherwise, n^* is the smallest positive integer greater than 1 which satisfies the following inequality:

$$(n^*+2)\int_{T/(n^*+2)}^{T/(n^*+1)}\psi(t)dt < (n^*+1)\int_0^{T/n^*}\psi(t)dt + v_0 + v_1. \tag{4.11}$$

Proof. Taking the difference of $AC_2(T,n)$ with respect to n we get

$$\begin{aligned} q(n) &= AC_2(n+1,T) - AC_2(n,T) \\ &= v_0+v_1+(n+1)\left\{(\rho+R_W)\int_0^{T/(n+1)}F(t)dt - v_1F(T/(n+1))\right\} \\ &\quad -n\left\{(\rho+R_W)\int_0^{T/n}F(t)dt - v_1F(T/n)\right\}. \end{aligned}$$

If the condition (4.10) holds then $AC_2(n,T)$ is a strictly increasing function of n and consequently, the optimal number of inspections is $n^*=1$, *i.e.*, the only one inspection is to be performed at the end of the production run. On the other hand, there exists a unique number of inspections $n>1$ provided that $AC_2(n,T)$ is convex with respect to n. Taking the difference of $q(n)$ with respect to n, after some algebraic manipulations, we get

$$q(n+1)-q(n) = n\int_{T/(n+1)}^{T/n}\psi(t)dt - (n+2)\int_{T/(n+2)}^{T/(n+1)}\psi(t)dt. \tag{4.12}$$

From equation (4.12) and the condition given in equation (4.10), it follows that the optimal number of inspections n^* is the smallest integer greater than 1 which satisfies the inequality (4.11). This completes the proof of the proposition.

Proposition 4.4 *Suppose that both $F_1(t)$ and $F_2(t)$ have IFR (increasing failure rate) property. Then, for any given n and T, there exists a unique warranty period W^* $(0<W^*<\infty)$ which minimizes the expected total cost per unit time in the steady state and the corresponding minimum average cost is $c_rPT\{\Delta_{31}r_1(W^*)+\Delta_{32}r_1(W^*)\}$.*

Proof. Considering W as a decision variable in equation (4.9), the first order condition of optimality gives

$$\begin{aligned} &c_r(PT/D+W)PT\{\Delta_{31}r_1(W)+\Delta_{32}r_2(W)\} - \Bigg[c_s + c_mPT + n(v_0+v_1\overline{F}(T/n) \\ &+\frac{c_h(P-D)PT^2}{2D} + \rho n\int_0^{T/n}(T/n-t)\,dF(t) + c_rPT\left\{\Delta_{31}\int_0^W r_1(t)dt\right. \\ &\left.+\Delta_{32}\int_0^W r_2(t)dt\right\}\Bigg] = 0. \end{aligned} \tag{4.13}$$

Table 1: Influence of λ on the expected total discounted cost when $n = 4$

λ	Inspection policy I[a]	Inspection policy II				
	TC_1^0	t_1^*	t_2^*	t_3^*	t_4^*	TC_2^*
0.1	**7254.17**	0.32754	0.28315	0.23042	0.16089	7353.36
0.2	**7267.87**	0.27012	0.25716	0.24358	0.22923	7356.14
0.3	**7290.19**	0.25906	0.25310	0.24703	0.24083	7360.58
0.4	**7320.39**	0.25517	0.25176	0.24832	0.24485	7366.75
0.5	**7357.54**	0.25336	0.25115	0.24893	0.24670	7374.67
0.6	7400.57	0.25235	0.25080	0.24924	0.24769	**7384.30**
0.7	7448.35	0.25175	0.25059	0.24943	0.24828	**7395.63**
0.8	7499.78	0.25142	0.25052	0.24963	0.24873	**7408.69**
0.9	7553.82	0.25108	0.25036	0.24965	0.24893	**7423.32**

[a] Sub-optimal inspection schedule $\{T_1^0, T_2^0, T_3^0, T_4^0\}$={0.5, 0.70711, 0.86602, 1.0}, where superscript 0 indicates sub-optimal results.

It is easy to see that the function on the left hand side of equation (4.13) is increasing in W for known values of n and T. Moreover, its limits as $W \to 0$ and $W \to \infty$ are negative and positive, respectively. So the equation (4.13) has a finite unique positive root W^* (> 0). Further, the second order derivative of the right hand side of equation (4.9) with respect to W at $W = W^*$ leads to $c_r PT \{\Delta_{31} r_1'(W^*) + \Delta_{32} r_2'(W^*)\} / (PT/D + W^*)^2 > 0$, where prime denotes differentiation with respect to W. Hence the proposition follows.

5. Numerical Examples

Suppose that the process shift from an 'in-control' state to an 'out-of-control' state follows the Weibull distribution:

$$F(t) = 1 - e^{-(\lambda t)^\beta}, \quad t > 0,\ \lambda > 0,\ \beta \geq 1.$$

The lifetime distributions of conforming and non-conforming items are also Weibull distributions with failure rates $r_1(t) = t/50$ and $r_2(t) = t/25$, respectively. The parameter values of the proposed model are as follows: $D = 90$ (units/week), $P = 150$ (units/week), $c_s = 250$ (\$), $c_h = 0.1$ (\$/unit/week), $c_m = 5$ (\$/unit), $c_r = 3$ (\$), $v_0 = 10$ (\$), $v_1 = 15$ (\$), $\rho = 20$ (\$), $\theta_1 = 0$, $\theta_2 = 1$, $\beta = 2$, $W = 24$ (weeks), $T = 1$ (week) and $\delta = 0.02$. Table 1 presents the influence of λ on the expected total discounted cost over an infinite time horizon when the process is inspected 4 times during a production run.

As expected, the expected total discounted cost increases with the failure parameter λ. Table 1 further shows that when the failure rate is low ($\lambda = 0.1 \sim 0.5$), inspection policy I performs better than inspection policy II. On the other hand, inspection policy II results in lower cost than inspection policy I for higher failure rates ($\lambda = 0.6 \sim 0.9$). When the failure rate is high, one would expect that preventive maintenance at the time of inspection when the system is in 'in-control' state could reduce the defective item cost and expected restoration cost. However, the overall cost performance strongly depends on each of the cost components. It is also to be observed from Table 1 that the first inspection interval in each inspection sequence is the longest one as the process is in 'in-control' state at the beginning. The subsequent intervals decrease as the cumulative hazards increase with time. The results in Table 2 show a high impact of δ on the expected total discounted cost. For

Table 2: Dependence of the expected total discounted cost on δ when $n = 4$ and $\lambda = 0.5$

	Inspection policy I[a]	Inspection policy II				
δ	TC_1^0	t_1^*	t_2^*	t_3^*	t_4^*	TC_2^*
0.02	**7357.54**	0.25331	0.25110	0.24888	0.24666	7374.67
0.03	**4961.51**	0.25583	0.25197	0.24807	0.24414	4983.54
0.04	**3772.31**	0.25928	0.25330	0.24724	0.24108	3797.17
0.05	**3065.90**	0.26341	0.25478	0.24597	0.23694	3092.50
0.06	**2600.85**	0.26843	0.25655	0.24428	0.23155	2629.12
0.07	**2273.61**	0.27465	0.25885	0.24231	0.22484	2303.06
0.08	**2032.33**	0.28230	0.26186	0.24012	0.21661	2062.76
0.09	**1848.18**	0.29127	0.26543	0.23741	0.20612	1879.40
0.10	**1703.83**	0.30210	0.27011	0.23464	0.19331	1735.76

[a] Sub-optimal inspection schedule $\{T_1^0, T_2^0, T_3^0, T_4^0\}$={0.5, 0.70711, 0.86602, 1.0}

example, when δ is increased from 2% to 3% the total discounted cost under inspection policy I reduces by 32.57%.

The pre-specified number of inspections during a production run may be restrictive in some manufacturing systems as it does not allow to control the production of defective items in an optimal way, whereas a fixed production run time or lot size in a production cycle is quite common in practice. We keep the production time duration $T = 1$ week as fixed and determine the optimal inspection policy. Table 3 presents the optimal/sub-optimal number of inspections, inspection sequence and the associated total discounted cost for different values of the failure parameter λ. From Tables 1 and 3, it could be observed that given a production run time T, a flexible number of inspections provide lower expected total discounted cost than that for inflexible one.

Table 4 exhibits the corresponding results of Tables 1 and 3 under long-run average cost criterion. It can be observed from Tables 1, 3 and 4 that the characteristics of the optimal/sub-optimal inspection policy are quite similar to those under discounted cost criterion. Table 5 explores that the zero warranty policy ($W = 0$) is not the true optimum. In fact, there exists a unique non-zero warranty period W which minimizes the manufacturer's expected cost under discounted or long-run average cost criterion. This signifies the importance of considering warranty issue in the integrated production and maintenance model under consideration.

6. Conclusions and Future Research Directions

The primary goal of this paper is to incorporate a sequential inspection policy during a production run in an imperfect production process. When the manufacturer involves in a contractual deal with the customer to provide post-sale free repair service until a certain period, such a sequential inspection policy not only reduces the number of defective items in a production lot but also lowers the warranty servicing cost. In this paper, we have developed an imperfect EMQ model under discounted as well as long-run average cost criteria assuming that the process shift time from an 'in-control' state to an 'out-of-control' state follows a general probability distribution and the process is subject to inspection policy I or II during a production run. Inspection policies I and II designate respectively no action and preventive repair action taken at each inspection in the intermediate of a

Table 3: Dependence of optimal/sub-optimal inspection policies on λ under discounted cost criterion when n is not fixed

	Inspection policy I			Inspection policy II		
λ	n^*	$\{T_1^0, T_2^0, \cdots, T_n^0\}$	TC_1^0	n^*	$\{t_1^*, t_2^*, \cdots, t_n^*\}$	TC_2^*
0.1	1	$\{1.0\}$	**7207.01**	2	$\{0.51344, 0.48666\}$	7233.06
0.2	2	$\{0.70760, 1.0\}$	**7240.14**	2	$\{0.50341, 0.49664\}$	7244.20
0.3	3	$\{0.57731, 0.81644, 1.0\}$	7279.51	2	$\{0.50155, 0.49850\}$	**7262.65**
0.4	3	$\{0.57731, 0.81644, 1.0\}$	7318.40	2	$\{0.49916, 0.50091\}$	**7288.26**
0.5	4	$\{0.5, 0.70711, 0.86602, 1.0\}$	7357.54	2	$\{0.50057, 0.49942\}$	**7320.79**
0.6	5	$\{0.44726, 0.63252, 0.77468, 0.89452, 1.0\}$	7396.67	3	$\{0.33457, 0.33338, 0.33220\}$	**7348.47**
0.7	6	$\{0.40837, 0.577522, 0.707318, 0.81674, 0.913143, 1.0\}$	7435.19	3	$\{0.33423, 0.33334, 0.33246\}$	**7368.82**
0.8	6	$\{0.40837, 0.577522, 0.707318, 0.81674, 0.913143, 1.0\}$	7471.45	3	$\{0.33399, 0.33330, 0.33261\}$	**7392.21**
0.9	7	$\{0.37800, 0.53457, 0.65472, 0.75600, 0.84523, 0.92591, 1.0\}$	7506.09	3	$\{0.33386, 0.33330, 0.33274\}$	**7418.42**

production run if the process is found in an 'in-control' state. Since it is hard to derive the optimal solution of the model analytically under inspection policy I, we have derived a sub-optimal inspection policy. Numerical results show that given the production run length T, the manager has to choose the right inspection policy from I and II, depending on the system failure information. The free minimal repair warranty which he or she has to offer the customer is non-zero and unique. The model developed in this paper can be extended in several ways. We have assumed that all non-conforming items are saleable. However, it would be more realistic if it is assumed that a fraction of non-conforming items are to be scrapped before sale. Another direction may be to study the model under pro-rata warranty policy (Blischke and Murthy [2]) for which the manufacturer agrees to refund a fraction of the purchase price when the item fails before the warranty period W. Analysis of the model under an extended warranty policy for which some options are available to the customers at the expiry time W of FRW would also be a worthful contribution in future.

Acknowledgement

This work was done when the first author has visited the Hiroshima University, Japan, as a JSPS Post-Doctoral Research Fellow. The authors would like to thank the financial support by a Grant-in-Aid for Scientific Research from the Ministry of Education, Sports, Science and Culture of Japan under Grant No.02296.

Table 4: Dependence of optimal/sub-optimal inspection policies on λ under the long-run average cost criterion

	$n = 4,\ T = 1$		$T = 1$	
λ	AC_1^0	AC_2^*	(n^0, AC_1^0)	(n^*, AC_2^*)
0.1	**142.514**	144.059	(1, 141.832)	(1, **141.449**)
0.2	**142.813**	144.118	(2, 142.468)	(2, **142.417**)
0.3	**143.301**	144.216	(3, 143.203)	(2, **142.832**)
0.4	**143.961**	144.354	(4, 143.961)	(2, **143.386**)
0.5	144.772	**144.530**	(5, 144.727)	(3, **143.951**)
0.6	145.712	**144.745**	(6, 145.467)	(3, **144.337**)
0.7	146.755	**144.998**	(7, 146.170)	(3, **144.789**)
0.8	147.877	**145.288**	(7, 146.860)	(4, **145.288**)
0.9	149.056	**145.615**	(8, 147.519)	(4, **145.615**)

Table 5: Impact of FRW on the expected total discounted cost/long-run average cost when $\lambda = 0.5$

	Discounted cost criterion				Long-run average cost criterion			
W	n^0	TC_1^0	n^*	TC_2^*	n^0	AC_1^0	n^*	AC_2^*
6	1	8297.80	1	**8240.04**	1	162.04	1	**156.88**
12	2	6674.28	2	**6660.93**	2	126.75	2	**125.48**
18	3	6777.75	2	**6744.27**	4	129.916	2	**129.22**
24	4	7357.54	2	**7320.79**	5	144.73	3	**143.95**
36	6	8932.84	4	**8898.31**	6	185.88	3	**184.90**
48	7	10602.80	5	**10566.50**	9	233.72	4	**232.07**

References

[1] P.K. Banerjee and M.A. Rahim: Economic design of $\overline{x}$ control charts under Weibull shock model. *Technometrics*, **30** (1988), 407–414.

[2] W.R. Blischke and D.N.P. Murthy: *Product Warranty Handbook*. Marcel Dekker, Inc., New York, 1996.

[3] I. Djamaludin, D.N.P. Murthy and R.J. Wilson: Quality control through lot sizing for items sold with warranty. *International Journal of Production Economics*, **33** (1994), 97–107.

[4] R. Hooke and T.A. Jeeves: Direct search solution of numerical statistical problems. *Journal of Association for Computing Machinery*, **8** (1961), 212–229.

[5] H.L. Lee and M.J. Rosenblatt: Simultaneous determination of production cycle and inspection schedules in a production system. *Management Science*, **33** (1987), 1125–1136.

[6] J.S. Lee and K.S. Park: Joint determination of production cycle and inspection intervals in a deteriorating production system. *Journal of the Operational Research Society*, **42** (1991), 775–783.

[7] V. Makis: Optimal lotsizing and inspection policy for an EMQ model with imperfect inspections. *Naval Research Logistics*, **45** (1998), 165–186.

[8] A.G. Munford: Comparison among certain inspection policies. *Management Science*, **27** (1981), 261–267.

[9] E.L. Porteus: Optimal lot sizing process quality improvement and setup cost reduction. *Operations Research*, **34** (1986), 137–144.

[10] E.L. Porteus: The impact of inspection delay on process and inspection lot sizing. *Management Science*, **36** (1990), 999–1007.

[11] M.A. Rahim: Joint determination of production quantity inspection schedule and control chart design. *IIE Transactions*, **26** (1994), 2–11.

[12] M.A. Rahim and M. Ben-Daya: A generalized economic model for the joint determination of production run, inspection schedules and control chart design. *International Journal of Production Research*, **36** (1996), 277–289.

[13] M.J. Rosenblatt and H.J. Lee: Economic production cycles with imperfect production processes. *IIE Transactions*, **18** (1986), 48–55.

[14] S.H. Tseng: Optimal preventive maintenance policy for deteriorating production systems. *IIE Transactions*, **28** (1996), 687–694.

[15] C.H. Wang: The impact of free-repair warranty policy on EMQ model for imperfect production systems. *Computers and Operations Research*, **31** (2004), 2021–2035.

[16] C.H. Wang: Integrated production and production inspection policy for a deteriorating production system. *International Journal of Production Economics*, **95** (2005), 123–134.

[17] C.H. Wang and S.H. Sheu: Optimal lot sizing for products sold under free-repair warranty. *European Journal of Operational Research*, **149** (2003), 131–141.

[18] R.H. Yeh, W.T. Ho and S.T. Tseng: Optimal production run length for products sold with warranty. *European Journal of Operational Research*, **120** (2000), 575–582.

Tadashi Dohi
Department of Information Engineering
Hiroshima University
4-1 Kagamiyama 1 Chome
Higashi-Hiroshima 739-8527, Japan
E-mail: `dohi@rel.hiroshima-u.ac.jp`