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NUMERICAL EXPLORATION OF DYNAMIC BEHAVIOR OF ORNSTEIN-UHLENBECK PROCESSES VIA EHRENFEST PROCESS APPROXIMATION

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Abstract Recently Ornstein-Uhlenbeck (O-U) processes have been drawing much attention in financial engineering for modeling stochastic behavior of spot interest rates. While transition probabilities of the O-U processes are readily accessible, it is numerically cumbersome to quantify their dynamic behavior much needed in certain applications, e.g., computing the prices of barrier options and the like in financial engineering. The purpose of this paper is to develop numerical procedures for evaluating distributions of first passage times and the historical maximums of the O-U processes via the Ehrenfest process approximation. Using the fact that a sequence of Ehrenfest processes with appropriate scaling and shifting converges in law to an O-U process, it is shown that first passage times and the historical maximum of the Ehrenfest processes converge in law to those of the O-U process. Through analysis of the spectral structure of the Ehrenfest process, efficient numerical algorithms are developed, thereby providing effective approximation tools for capturing the dynamic behavior of the O-U process. The proposed numerical algorithms are systematic in that the needed computations can be done repeatedly for different values of the underlying parameters with little alterations. Some numerical results are also exhibited, demonstrating speed and accuracy of the algorithms.

Keywords: Markov process, Ornstein-Uhlenbeck (O-U) process, Ehrenfest process, dynamic behavior, convergence in law, first passage times, historical maximum, numerical approximation.

1. Introduction

We consider a Markov diffusion process on the real continuum $-\infty < x < \infty$ characterized by the forward diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t) + \frac{\partial}{\partial x}\left[x\, f(x,t)\right], \tag{1.1}$$

where $f(x,t)$ is the probability density function defined by $f(x,t) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[X_{\mathrm{OU}}(t) \le x\right]$. This process is called an Ornstein-Uhlenbeck (O-U) process denoted by $\{X_{\mathrm{OU}}(t) \,:\, t \ge 0\}$. A basic function describing this stochastic process is the conditional transition density $g(x_0, x, \tau) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[X_{\mathrm{OU}}(t+\tau) \le x | X_{\mathrm{OU}}(t) = x_0\right]$ given by

$$g(x_0, x, \tau) = \frac{1}{\sqrt{2\pi}\sqrt{1-\mathrm{e}^{-2\tau}}} \exp\left\{-\frac{\left(x - x_0\,\mathrm{e}^{-\tau}\right)^2}{2\left(1-\mathrm{e}^{-2\tau}\right)}\right\}, \quad -\infty < x < \infty. \tag{1.2}$$

Its stationary or ergodic density is obtained as

$$f_\infty(x) \stackrel{\text{def.}}{=} \lim_{t\to\infty} f(x,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}, \quad -\infty < x < \infty. \tag{1.3}$$

O-U processes generated by $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ through appropriate shifting and scaling find many applications to statistics, including the studies of "goodness of fit" of a set of observations to a distribution function, see e.g. Anderson and Darling [1] and the studies of stopping time for sample sequences, see e.g. Armitage, McPherson and Rowe [2]. During the past three decades, the usefulness of the O-U processes has been reinforced in the area of financial engineering where spot interest rates are often represented by such O-U processes, see e.g. Vasicek [27]. As we saw in (1.2), because of the underlying simplicity associated with the Gaussian transition structure, the transition probabilities of the O-U processes are readily accessible. However, quantifying their dynamic behaviors is numerically cumbersome. Furthermore, when the O-U processes are modified with various boundaries, the corresponding dynamic behaviors become analytically intractable.

For the case of absorbing boundaries, Keilson and Ross [14] find the Laplace transform of the first passage time density to any absorbing state. The inversion process is then numerically established by locating the singular points of the Laplace transform on the complex plane and using the residue theorem. Park and Schuurmann [18] establish an integral equation to be satisfied by the first passage time density with a moving boundary. This integral equation is solved by discretizing the time axis with equal distance. In a subsequent paper by Park and Schuurmann [19], a non-equal distance discretization procedure is proposed for dealing with a large time interval or a moving boundary near the origin at time zero. Dinardo, Nobile, Pirozzi and Ricciardi [3] show that the first passage time density satisfies a Volterra integral equation of second-kind and develop numerical procedures to solve it.

While the above procedures may enable one to evaluate the moments and distributions of the first passage times of the O-U processes, they may not be necessarily appropriate to automate the required computations when such measures have to be evaluated repeatedly under different parameter values. In the approach of Keilson and Ross [14], for example, it is necessary to locate the singular points on the complex plane for each parameter value repeatedly, which is laborious and cumbersome. In certain applications, however, repeated computations of this sort become crucial. In financial engineering, for example, let us consider the Hull-White model [5] which is a one factor term structure model characterized by a stochastic differential equation of the form

$$\mathrm{d}R(t) = (\phi(t) - \alpha(t)R(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \tag{1.4}$$

where $R(t)$ is a random short rate, $W(t)$ is the standard Wiener process, $\phi(t)$ is a market fitting function, $\alpha(t) > 0$ is a reversion function and $\sigma(t) > 0$ is a volatility function. When $\phi(t)$, $\alpha(t)$ and $\sigma(t)$ are constant, the Hull-White model is reduced to the Vasicek model [27] specified by

$$\mathrm{d}\widehat{X}_{\mathrm{OU}}(t) = (\phi - \alpha\widehat{X}_{\mathrm{OU}}(t))\mathrm{d}t + \sigma\mathrm{d}W(t). \tag{1.5}$$

Let $X_{\mathrm{OU}}(t)$ be the limiting O-U process of $X_V(t)$, to be introduced later in (1.15), as $V \to \infty$ and define $\widetilde{X}_{\mathrm{OU}}(t) = \dfrac{\sigma}{\sqrt{2\alpha}}X_{\mathrm{OU}}(\alpha t)$. After a little algebra, one finds that

$$\widehat{X}_{\mathrm{OU}}(t) = \widetilde{X}_{\mathrm{OU}}(t) + \theta(t), \tag{1.6}$$

where $\widetilde{X}_{\mathrm{OU}}(0) = 0$ and

$$\theta(t) \stackrel{\mathrm{def}}{=} \frac{\phi}{\alpha}(1 - \mathrm{e}^{-\alpha t}) + \widehat{X}_{\mathrm{OU}}(0)\mathrm{e}^{-\alpha t}. \tag{1.7}$$

We now consider an up-and-out call option maturing at time τ with strike price K_{S}, written on a discount bond of maturity at time T where the upper limit is given by r_{b}. Let

$D(\tau|T)$ be the price of the discount bond at time τ. Then the price of the up-and-out call option at time τ, denoted by $\pi_{\mathrm{KO}}(\tau|T)$, can be expressed in terms of the first passage time $T_{\hat{x}_0 r_{\mathrm{b}}} = \inf\{t : \widehat{X}_{\mathrm{OU}}(t) = r_{\mathrm{b}} | \widehat{X}_{\mathrm{OU}}(0) = \hat{x}_0\}$ as

$$\pi_{\mathrm{KO}}(\tau|T) = \mathrm{E}[\{D(\tau|T) - K_{\mathrm{S}}\}^+ 1_{\{T_{\hat{x}_0 r_{\mathrm{b}}} > \tau\}}], \tag{1.8}$$

where $\{a\}^+ = \max\{a, 0\}$ and

$$1_{\{A\}} = \begin{cases} 1, & A \textit{ is true}, \\ 0, & A \textit{ is false}. \end{cases} \tag{1.9}$$

Evaluating $\pi_{\mathrm{KO}}(\tau|T)$ requires the joint distribution of $\mathrm{P}[\widehat{X}_{\mathrm{OU}}(t) \le x, T_{\hat{x}_0 r_{\mathrm{b}}} > \tau | \widehat{X}_{\mathrm{OU}}(0) = \hat{x}_0]$. In addition, the joint distribution has to be computed repeatedly with speed and accuracy for different values of the underlying parameters. To the authors' best knowledge, there exist no systematic algorithms to overcome this difficulty in the literature. The computational algorithms proposed in this paper provide a powerful numerical vehicle for filling this gap. Achieving this goal in financial engineering, however, requires independent and additional efforts and will be reported in a sequel to this paper.

The O-U processes have been employed extensively in the area of diffusion approximation. In this approach, a practical system of interest is often modeled as a birth-death process. Then, under certain conditions, a sequence of such birth-death processes is shown to converge in law to a Gaussian diffusion process for each time $t \ge 0$. Various performance indicators of the Markov model is then approximated by those of the Gaussian diffusion process. In this context, the convergence theorem involving the O-U process dates back to Stone [20, 21], where a necessary and sufficient condition is established for a sequence of birth-death processes $\{N_{\mathrm{BD}:K}(t) : t \ge 0\}$ to converge in law to an O-U process $\{X_{\mathrm{OU}}(t) : t \ge 0\}$ with appropriate shifting and scaling as $K \to \infty$. Convergence in law involving stochastic processes represents certain subtlety. Stone [20, 21] introduces the concept of "weak convergence in the Markov sense of a sequence of birth-death processes," which essentially requires the convergence in law of $f(N_{\mathrm{BD}:K}(\cdot))$ to $f(X_{\mathrm{OU}}(\cdot))$ for all functionals f defined on the space of path-functions of $\{N_{\mathrm{BD}:K}(t) : t \ge 0\}$ and $\{X_{\mathrm{OU}}(t) : t \ge 0\}$ in the topology of uniform convergence in compact time intervals almost everywhere with respect to the measure corresponding to $\{X_{\mathrm{OU}}(t) : t \ge 0\}$.

Since then, many authors have elaborated on the convergence theorem of Stone [20, 21] by constructing specific birth-death processes for specific applications. Iglehart [6], for example, applies the theorem to establish diffusion approximation of a class of queueing models. Using the theorem of Stone [20, 21], McNeil and Schach [17] identify central limit analogues involving O-U processes. With reference to Iglehart [6], Kulkarni and Rolski [15] prove diffusion approximation of a class of prevalent models in communication networks generated by O-U processes. For these applications, the central idea is to approximate the performance indices of the underlying birth-death process by those of the limiting O-U process. The thrust of this paper can be found in that this typical approach in diffusion approximation is reversed. More specifically, a simple sequence of birth-death processes, which converges in law in the weak Markov sense to an O-U process, is chosen so as to approximate the dynamic behavior of the O-U process, possibly with various boundaries, by that of the simple birth-death process. This approach enables one to mechanize the computational procedures for evaluating the dynamic behavior of the O-U processes under different parameter values repeatedly.

A class of simple discrete time processes for approximating O-U processes is a set of Ehrenfest urn models, see e.g. Karlin and Taylor [11]. For capturing the dynamic behavior of an O-U process, however, this approach is rather cumbersome since both the state space and the time axis are discretized. In this paper, we propose to utilize the continuous time Ehrenfest process for approximating the O-U process where only the state space is discretized. The underlying spectral structure enables one to develop efficient numerical procedures for computing the distributions of the first passage times and the historical maximum of the O-U process, which are of considerable importance in certain applications, as we saw.

A finite Markov chain in continuous time of practical importance arises from the sum of K independent identical chains $\{J_j(t) : t \geq 0\}$, $j = 1, ..., K$, each on state space $\{0, 1\}$ governed by transition rates $\nu_{01} = \nu_{10} = \frac{1}{2}$. The Markov chain of interest

$$\left\{ N_K(t) \,:\, N_K(t) = \sum_{j=1}^{K} J_j(t),\ t \geq 0 \right\} \tag{1.10}$$

on state space $\{0, 1, ..., K\}$ is called an Ehrenfest process and has transition rates

$$\nu_{n,n+1} = \frac{1}{2}(K - n),\ 0 \leq n \leq K - 1,\ \text{and}\ \nu_{n,n-1} = \frac{1}{2}n,\ 1 \leq n \leq K. \tag{1.11}$$

Consequently the local growth rate of the variance is given by

$$\nu_{n,n+1} + \nu_{n,n-1} = \frac{K}{2}, \tag{1.12}$$

which is independent of n, and the local velocity is given by

$$\nu_{n,n+1} - \nu_{n,n-1} = \frac{K}{2} - n. \tag{1.13}$$

For the associated stationary chain $\{N_{K:\mathrm{S}}(t) : t \geq 0\}$, one has

$$\mathrm{cov}\left[\, N_{K:\mathrm{S}}(t),\ N_{K:\mathrm{S}}(t + \tau) \,\right] = \frac{K}{4}\, \mathrm{e}^{-\tau}, \tag{1.14}$$

and asymptotic normality.

O-U processes are characterized by its Markov property, normal distribution, and exponential covariance function. Because of the properties of the Ehrenfest process specified in (1.11) through (1.14) together with its asymptotic normality, one then expects that a sequence of processes $\{X_V(t) : t \geq 0\}, V = 1, 2, 3, ...,$ defined by

$$X_V(t) = \sqrt{\frac{2}{V}}\, N_{2V}(t) - \sqrt{2V} \tag{1.15}$$

converges in law to an O-U process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ as $V \to \infty$. Indeed, it can be shown that $\{N_{2V}(t) : t \geq 0\}$ with shifting and scaling as specified in (1.15) satisfies the sufficient condition of Stone [20, 21] so that the convergence of $\{X_V(t) : t \geq 0\}$ in the weak Markov sense to the O-U process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ as $V \to \infty$ is assured. In this paper, because of the simplicity of $\{X_V(t) : t \geq 0\}$, we provide an independent proof for the convergence of $X_V(t)$ to an O-U process $X_{\mathrm{OU}}(t)$ for each $t \geq 0$. In addition, we analyze the spectral structure of $\{N_{2V}(t) : t \geq 0\}$ for quantifying its dynamic behavior numerically, which in turn provides a numerical foundation for capturing the dynamic behavior of the O-U processes. The proposed approach enables one to mechanize the computational procedures with speed and accuracy, which is essential for certain applications where such computations are required repeatedly under different parameter values as we saw.

The structure of this paper is as follows. In Section 2, a succinct summary is given concerning the spectral representation and the first passage time structure of birth-death processes based on Karlin and McGregor [7–10] and Keilson [13]. Based on these results, in Section 3, the Ehrenfest process is studied in detail, deriving new results to establish a numerical foundation for evaluating the distributions of the first passage times and the historical maximum. In Section 4, an independent proof is given for the convergence in law of $X_V(t)$ to $X_{\mathrm{OU}}(t)$ as $V \to \infty$ for all $t \geq 0$ and some related results are also obtained. Section 5 is devoted to development of numerical algorithms for evaluating transition probabilities, first passage times, and the historical maximum of the O-U process via the Ehrenfest process approximation. Numerical results are also exhibited, demonstrating speed and accuracy. Some additional results concerning the first passage time structure of the Ehrenfest process are given in Appendix, which are of interest in their own right.

2. Review of Spectral Representation and First Passage Time Structure of Birth-Death Processes

In this section, we briefly review key results concerning the spectral representation and the first passage time structure of birth-death processes. Based on these results, the Ehrenfest process $\{N_K(t) : t \geq 0\}$ given in (1.10) with $K = 2V$ will be analyzed in detail later. In a series of papers [7–10], Karlin and McGregor analyze the spectral representation of the transition probability matrix $\underline{\underline{P}}(t) = [p_{mn}(t)]$ for birth-death processes and use the results to evaluate various probabilistic quantities. More specifically, for a general birth-death process $\{N_{\mathrm{BD}}(t) : t \geq 0\}$ on $\mathcal{N}_{\mathrm{BD}} = \{0, 1, 2, \cdots\}$ governed by upward transition rates $\lambda_n,\ n \geq 0$, and downward transition rates $\mu_n,\ n \geq 1$, let $\underline{\underline{Q}}_{\mathrm{BD}}$ be the infinitesimal generator associated with $\underline{\underline{P}}(t)$ satisfying the Kolmogorov forward equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\underline{P}}(t) = \underline{\underline{P}}(t)\underline{\underline{Q}}_{\mathrm{BD}}, \tag{2.1}$$

where

$$\underline{\underline{Q}}_{\mathrm{BD}} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \dots & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} & \dots \\ 0 & 0 & 0 & \dots & 0 & \mu_n & -(\lambda_n+\mu_n) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

One then sees that $\underline{\underline{Q}}_{\mathrm{BD}}$ has a vector eigenfunction $\underline{y}(x) = [y_n(x)]_{n \in \mathcal{N}_{\mathrm{BD}}}$ with eigenvalue $-x$, i.e.

$$\underline{\underline{Q}}_{\mathrm{BD}}\, \underline{y}(x) = -x\, \underline{y}(x). \tag{2.2}$$

From (2.1), this then leads to

$$\begin{cases} -\lambda_0\, y_0(x) + \lambda_0\, y_1(x) = -x\, y_0(x), \\ \mu_n\, y_{n-1}(x) - (\lambda_n + \mu_n)\, y_n(x) + \lambda_n\, y_{n+1}(x) = -x\, y_n(x), \quad n \geq 1, \end{cases} \tag{2.3}$$

starting with $y_0(x) = 1$. It then follows that $y_n(x)$ is a polynomial of degree n with leading coefficients $\left(\prod_{j=0}^{n-1} \lambda_j\right)^{-1}$.

Let $\underline{f}(x,t)$ be a vector function defined by

$$\underline{f}(x,t) = \underline{\underline{P}}(t)\,\underline{y}(x). \tag{2.4}$$

From (2.1) and (2.2), one then finds that

$$\frac{\partial}{\partial t}\underline{f}(x,t) = \frac{\partial}{\partial t}\underline{\underline{P}}(t)\,\underline{y}(x) = \underline{\underline{P}}(t)\,\underline{\underline{Q}}_{\mathrm{BD}}\,\underline{y}(x) = -x\,\underline{\underline{P}}(t)\,\underline{y}(x)$$

so that from (2.4)

$$\frac{\partial}{\partial t}\underline{f}(x,t) = -x\,\underline{f}(x,t). \tag{2.5}$$

Since $\underline{f}(x,0+) = \underline{y}(x)$, the vector partial differential equation (2.5) has the unique solution

$$\underline{f}(x,t) = \mathrm{e}^{-xt}\underline{y}(x). \tag{2.6}$$

Combining with (2.4), Equation (2.6) then implies that

$$\sum_{n\in\mathcal{N}_{\mathrm{BD}}} p_{mn}(t)\,y_n(x) = \mathrm{e}^{-xt}y_m(x), \quad m \in \mathcal{N}_{\mathrm{BD}}. \tag{2.7}$$

There exists a measure $\psi(x)$ on $[0,\infty)$ such that $\{y_n(x)\}_{n\in\mathcal{N}_{\mathrm{BD}}}$ becomes a set of orthogonal polynomials with respect to $\psi(x)$, see Karlin and McGregor [7]. Accordingly one has

$$\int_0^\infty y_m(x)\,y_n(x)\,\mathrm{d}\psi(x) = \frac{\delta_{mn}}{\pi_n}, \quad m,n \in \mathcal{N}_{\mathrm{BD}}, \tag{2.8}$$

where $\delta_{mn} = 1$ if $m = n$, $\delta_{mn} = 0$ if $m \neq n$, and $\pi_n = \prod_{j=0}^{n-1}\lambda_j / \prod_{j=1}^{n}\mu_j,\ n \geq 1$, with $\pi_0 = 1$. The integral in (2.8) is a Lebesgue-Stieltjes integral. Indeed, it should be noted from (2.2) that the support of ψ is the set of the eigenvalues of $\underline{\underline{Q}}_{\mathrm{BD}}$ multiplied by -1, and therefore discrete. From (2.4) and (2.8), $\{p_{mn}(t)\}_{m\in\mathcal{N}_{\mathrm{BD}}}$ may be recognized as the generalized Fourier coefficients of the m-th component $f_m(x,t)$ of $\underline{f}(x,t)$ associated with $\{y_n(x)\}_{n\in\mathcal{N}_{\mathrm{BD}}}$ and $\psi(x)$ for each $m \in \mathcal{N}_{\mathrm{BD}}$. Accordingly, one finds from (2.7) that

$$p_{mn}(t) = \pi_n \int_0^\infty \mathrm{e}^{-xt}y_m(x)\,y_n(x)\,\mathrm{d}\psi(x), \quad m,n \in \mathcal{N}_{\mathrm{BD}}. \tag{2.9}$$

The first passage time structure of $\{N_{\mathrm{BD}}(t) : t \geq 0\}$ is discussed extensively in Keilson [13], from which some relevant results are extracted here. Let T_{mn} be the first passage time of $\{N_{\mathrm{BD}}(t) : t \geq 0\}$ from state m to state n. Formally, we define

$$T_{mn} = \inf\{\, t : N_{\mathrm{BD}}(t) = n \,|\, N_{\mathrm{BD}}(0) = m \,\}. \tag{2.10}$$

Let $s_{mn}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}\left[T_{mn} \leq \tau\right]$ and define the Laplace transform $\sigma_{mn}(s) = \mathrm{E}\left[\mathrm{e}^{-s\,T_{mn}}\right]$. For notational convenience, we denote $T_{m,m+1}$ by T_m^+, and $s_m^+(\tau)$ and $\sigma_m^+(s)$ are defined similarly. From the consistency relations, one has

$$\sigma_n^+(s) = \frac{\nu_n}{s+\nu_n}\left[\frac{\lambda_n}{\nu_n} + \frac{\mu_n}{\nu_n}\,\sigma_{n-1}^+(s)\,\sigma_n^+(s)\right], \quad n \geq 1,$$

where $\nu_n = \lambda_n + \mu_n$. This then yields

$$\sigma_n^+(s) = \lambda_n \left[s + \nu_n - \mu_n \, \sigma_{n-1}^+(s) \right]^{-1}, \quad n \geq 1; \quad \sigma_0^+(s) = \frac{\lambda_0}{s + \lambda_0}. \tag{2.11}$$

It is clear that

$$\sigma_{0n}(s) = \sigma_{0\,n-1}(s) \, \sigma_{n-1}^+(s), \quad n \geq 1. \tag{2.12}$$

From (2.11), it can be readily seen by induction that

$$\sigma_{0n}(s) = \frac{1}{g_n(s)}, \quad n \geq 1; \quad g_0(s) = 1, \tag{2.13}$$

where $g_n(s)$ is a polynomial of order n. It then follows from (2.11) that

$$g_{n+1}(s) = \frac{1}{\lambda_n} \left[(s + \nu_n) \, g_n(s) - \mu_n \, g_{n-1}(s) \right], \; n \geq 0, \tag{2.14}$$

where $g_{-1}(s) = 0$ and $g_0(s) = 1$. From (2.3), one sees that

$$y_{n+1}(x) = \frac{1}{\lambda_n} \left[(-x + \nu_n) \, y_n(x) - \mu_n \, y_{n-1}(x) \right], \; n \geq 0, \tag{2.15}$$

where $y_{-1}(x) = 0$ and $y_0(x) = 1$. By comparing (2.14) with (2.15), it can be seen that the orthogonal polynomials $y_n(x)$, defined for x where $-x$ is an eigenvalue of $\underline{\underline{Q}}_{\mathrm{BD}}$, are related to $g_n(s)$ by

$$y_n(x) = g_n(-x), \quad n \geq 0. \tag{2.16}$$

Accordingly, $\{g_n(s)\}_{n \in \mathcal{N}_{\mathrm{BD}}}$ are orthogonal polynomials so that the zeros of $g_n(s)$ are distinct, the zeros of any two successive polynomials interleave, and the zeros are all negative, see e.g. Szegö [26]. It should be noted from (2.12) and (2.13) that

$$\sigma_n^+(s) = \frac{g_n(s)}{g_{n+1}(s)}, \quad n \geq 0. \tag{2.17}$$

Consequently, $\sigma_n^+(s)$ can be written as

$$\sigma_n^+(s) = \sum_{j=0}^{n} r_{n+1,j} \, \frac{\alpha_{n+1,j}}{s + \alpha_{n+1,j}}, \tag{2.18}$$

where $-\alpha_{n+1,j}$ are the zeros of $g_{n+1}(s)$, $r_{n+1,j} = \lim_{s \to -\alpha_{n+1,j}} \frac{s+\alpha_{n+1,j}}{\alpha_{n+1,j}} \frac{g_n(s)}{g_{n+1}(s)} \geq 0$ and $\sum_{j=0}^{n} r_{n+1,j} = \sigma_n^+(0+) = 1$. This implies that $s_n^+(t)$ is a mixture of exponential densities and is completely monotone. The downward first passage times $T_{n,n-1}^- = T_n^-$ and T_{n0} can be treated similarly.

We next turn our attention to the historical maximum of $\{N_{\mathrm{BD}}(t) : t \geq 0\}$ in the time interval $[0, \theta]$ given that $N_{\mathrm{BD}}(0) = n_0$. More specifically, let $M(n_0, \theta)$ be defined as

$$M(n_0, \theta) = \max_{0 \leq t \leq \theta} \{ N_{\mathrm{BD}}(t) \,|\, N_{\mathrm{BD}}(0) = n_0 \}. \tag{2.19}$$

Then the following dual relation holds between $T_{n_0\,n+1}$ $(n_0 \leq n)$ and $M(n_0, \theta)$.

$$F_{n_0\theta}(n) \stackrel{\text{def.}}{=} \mathrm{P}\left[M(n_0, \theta) \leq n \right] = \mathrm{P}\left[T_{n_0\,n+1} > \theta \right] \stackrel{\text{def.}}{=} \overline{S}_{n_0\,n+1}(\theta). \tag{2.20}$$

Hence one has

$$F_{n_0\theta}(n) = \begin{cases} 0 & n < n_0 \\ \overline{S}_{n_0\,n+1}(\theta) & n \geq n_0. \end{cases} \tag{2.21}$$

For the corresponding stationary process $\{ N_{\mathrm{BD:S}}(t) : t \geq 0 \}$, the distribution function $F_\theta(n)$ of the historical maximum is then given by

$$F_\theta(n) = \sum_{m \leq n} e_m \, \overline{S}_{m\,n+1}(\theta), \tag{2.22}$$

where $\underline{e}^\top = [\, e_m \,]$ is the ergodic distribution of $\{ N_{\mathrm{BD}}(t) : t \geq 0 \}$.

3. Spectral Representation and First Passage Time Structure of Ehrenfest Processes

We consider $2V$ independent and identical Markov chains $\{J_j(t) : t \geq 0\}$, $j = 1, ..., 2V$, in continuous time on $\{0, 1\}$ governed by the transition rate matrix

$$\underline{\underline{\nu}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{3.1}$$

The corresponding infinitesimal generator $\underline{\underline{Q}}$ is then given by

$$\underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}}; \quad \underline{\underline{\nu}}_D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{3.2}$$

Let $\underline{\underline{q}}(t) = [q_{ij}(t)]$, $0 \leq i, j \leq 1$, be the transition probability matrix of $\{ J_j(t) : t \geq 0 \}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\underline{\underline{q}}(t) = \underline{\underline{Q}}\,\underline{\underline{q}}(t)$. Since $\underline{\underline{q}}(0) = \underline{\underline{I}}$ which denotes the identity matrix, taking the Laplace transform of this matrix differential equation yields $s\,\hat{\underline{\underline{q}}}(s) - \underline{\underline{I}} = \underline{\underline{Q}}\,\hat{\underline{\underline{q}}}(s)$ or $\hat{\underline{\underline{q}}}(s) = \left[s\underline{\underline{I}} - \underline{\underline{Q}}\right]^{-1}$ where $\hat{\underline{\underline{q}}}(s) = \int_0^\infty \mathrm{e}^{-st}\underline{\underline{q}}(t)\,\mathrm{d}t$. From (3.2), one then finds that

$$\underline{\underline{q}}(t) = \begin{pmatrix} q_{00}(t) & q_{01}(t) \\ q_{10}(t) & q_{11}(t) \end{pmatrix} = \begin{pmatrix} f(t) & g(t) \\ g(t) & f(t) \end{pmatrix}, \tag{3.3}$$

where

$$f(t) = \frac{1}{2}\left(1 + \mathrm{e}^{-t}\right); \quad g(t) = \frac{1}{2}\left(1 - \mathrm{e}^{-t}\right). \tag{3.4}$$

For analytical convenience, we introduce two generating functions :

$$\alpha_0(t, u) \stackrel{\text{def.}}{=} q_{00}(t) + q_{01}(t)\,u = f(t) + g(t)\,u \tag{3.5}$$

and

$$\alpha_1(t, u) \stackrel{\text{def.}}{=} q_{10}(t) + q_{11}(t)\,u = g(t) + f(t)\,u. \tag{3.6}$$

Let $\{ N_{2V}(t) : t \geq 0 \}$ be defined by

$$N_{2V}(t) \stackrel{\text{def.}}{=} \sum_{j=1}^{2V} J_j(t). \tag{3.7}$$

Then $\{ N_{2V}(t) : t \geq 0 \}$ is a birth-death process on $\mathcal{N} = \{0, 1, ..., 2V\}$ governed by the upward transition rates λ_n and the downward transition rates μ_n, where

$$\lambda_n = \frac{1}{2}\,(2V - n)\,; \quad \mu_n = \frac{n}{2}, \quad n \in \mathcal{N}. \tag{3.8}$$

We note that

$$\nu_n \stackrel{\text{def.}}{=} \lambda_n + \mu_n = V, \quad n \in \mathcal{N}, \tag{3.9}$$

which is independent of state n. This birth-death process is called an Ehrenfest process, see e.g. Feller [4]. Let $\underline{\underline{P}}_{2V}(t) = [p_{2V:mn}(t)]\ \ m, n \in \mathcal{N}$ be the transition probability matrix of $\{\, N_{2V}(t) : t \geq 0\,\}$. As in (3.5) and (3.6), we introduce the following generating functions :

$$\beta_m(t,u) = \sum_{k=0}^{2V} p_{2V:mk}(t)\, u^k, \quad m \in \mathcal{N}. \tag{3.10}$$

From the independence of $\{J_j(t) : t \geq 0\}$, one then has

$$\beta_m(t,u) = \alpha_0(t,u)^{2V-m}\, \alpha_1(t,u)^m = \{f(t) + g(t)\,u\}^{2V-m}\{g(t) + f(t)\,u\}^m. \tag{3.11}$$

For the case of the Ehrenfest process, the associated orthogonal functions defined for integers x, $x = 0, 1, ..., 2V$ are obtained via induction from (2.3) as

$$y_n(x) = \frac{1}{\binom{2V}{n}} \sum_{j=0}^{n} \binom{2V-x}{n-j}\binom{x}{j}(-1)^j, \quad n \in \mathcal{N}, \tag{3.12}$$

where $\binom{m}{n} = 0$ if $n > m$. It is rather subtle to see that the infinitesimal generator matrix $\underline{\underline{Q}}_{2V}$ given in (3.13) below has the eigenvalues $-x = 0, -1, ..., -2V$. The reader is referred to Karlin and McGregor [7–10] for further details.

$$\underline{\underline{Q}}_{2V} = \begin{pmatrix} -V & V & 0 & \dots & 0 & 0 & 0 \\ 1/2 & -V & V-1/2 & \dots & 0 & 0 & 0 \\ 0 & 1 & -V & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -V & 1 & 0 \\ 0 & 0 & 0 & \dots & V-1/2 & -V & 1/2 \\ 0 & 0 & 0 & \dots & 0 & V & -V \end{pmatrix}. \tag{3.13}$$

Corresponding to (3.12), the measure for orthogonality is specified by

$$\mathrm{d}\psi(x) = \binom{2V}{x} 2^{-2V}, \quad x = 0, 1, ..., 2V, \tag{3.14}$$

so that

$$p_{2V:mn}(t) = \frac{\binom{2V}{n}}{2^{2V}} \sum_{j=0}^{2V} \binom{2V}{j} y_m(j)\, y_n(j)\, \mathrm{e}^{-jt}. \tag{3.15}$$

In this case, the polynomials $\{y_n(x)\}_{n\in\mathcal{N}}$ are called the Krawtchouk polynomials. It is clear from the independence of $\{J_j(t) : t \geq 0\}$ that the ergodic distribution $\underline{e}^{\top}$ of $\{N_{2V}(t) : t \geq 0\}$ is given by

$$\underline{e} = [e_n]^{\top}_{n\in\mathcal{N}}\,; \quad e_n = \binom{2V}{n} 2^{-2V}, \quad n \in \mathcal{N}. \tag{3.16}$$

We note from (3.15) and (3.16) that

$$\lim_{t\to\infty} p_{2V:mn}(t) = \frac{\binom{2V}{n}}{2^{2V}} \binom{2V}{0} y_m(0)\, y_n(0) = e_n, \quad m, n \in \mathcal{N} \tag{3.17}$$

as expected.

For the first passage time structure of the Ehrenfest process $\{N_{2V}(t) : t \geq 0\}$, one sees from (2.14) that

$$g_{n+1}(s) = \frac{2}{2V - n}\left[(s+V)\, g_n(s) - \frac{n}{2}\, g_{n-1}(s)\right], \tag{3.18}$$

with $g_{-1}(s) = 0$ and $g_0(s) = 1$. From (3.12) and (2.16), the orthogonal polynomials $g_n(s)$ defined for the eigenvalues $s = 0, -1, \cdots, -2V$ of $\underline{\underline{Q}}_{2V}$ are given explicitly by

$$g_n(s) = \frac{1}{\binom{2V}{n}} \sum_{j=0}^{n} \binom{2V+s}{n-j}\binom{-s}{j}(-1)^j, \quad 0 \leq n \leq 2V. \tag{3.19}$$

In order to evaluate the first passage times $s_{mn}(\tau)$ $(m < n)$ with corresponding Laplace transforms $\sigma_{mn}(s) = \sigma^+_m(s)\cdots\sigma^+_{n-1}(s) = g_m(s)/g_n(s)$ from (2.17), the zeros of $g_n(s)$ are needed. These zeros in turn enables one to quantify the historical maximum through (2.21). In principle, the zero search of $g_n(s)$ can be accomplished via a straightforward bisection approach since the zeros of $g_n(s)$ and $g_{n+1}(s)$ interleave because of the underlying orthogonality. In case of the Ehrenfest process, the amount of effort required for the zero search can be considerably reduced by the following properties.

Theorem 3.1 *Let $h_n(s) = g_n(s - V)$, $n \geq 0$. Then $h_n(s) = (-1)^n\, h_n(-s)$, $n \geq 0$, i.e.,*

$$h_n(s) \text{ is } \begin{cases} \text{odd} & \text{when } n \text{ is odd,} \\ \text{even} & \text{when } n \text{ is even.} \end{cases}$$

Proof. Equation (3.18) can be rewritten in terms of $h_n(s)$ as

$$h_{n+1}(s) = \frac{2}{2V - n}\left[s\, h_n(s) - \frac{n}{2}\, h_{n-1}(s)\right], \quad n \geq 0, \tag{3.20}$$

with $h_{-1}(s) = 0$ and $h_0(s) = 1$. The result then follows by induction on n. □

The next corollary is immediate from Theorem 3.1.

Corollary 3.1

(a) If $g_n(-x) = 0$, then $g_n(x - 2V) = 0$.

(b) If n is odd, then $g_n(-V) = 0$.

Theorem 3.1 implies that the zeros of $h_n(s)$ are symmetric about 0 and, correspondingly from Corollary 3.1, the zeros of $g_n(s)$ are symmetric about $-V$. Hence we need to find only $\lceil (n-1)/2 \rceil$ zeros, where $\lceil x \rceil$ is the minimum integer which is greater than or equal to x. Furthermore, since $h_n(s)$ is either odd or even, there are only $1 + \lceil (n-1)/2 \rceil$ terms in each $h_n(s)$, while $g_n(s)$ has $(n+1)$ terms as can be seen from (3.19). Consequently, the computational time of the zero search can be reduced approximately by a factor of 4. This property of the Ehrenfest process is due to the fact that the local growth rate is constant as specified in (1.12). Indeed, the results similar to Theorem 3.1 and Corollary 3.1 are available for general birth-death processes whenever $\nu_n = \lambda_n + \mu_n = \nu$ for all n.

Because of the peculiarity of the Ehrenfest process, its first passage time structure possesses certain interesting properties. Although these new results are of interest in their own right, they are not directly related to the main theme of this paper and are summarized in Appendix.

4. Convergence of the Ehrenfest Process to the O-U Process

As we pointed out in Section 1, the Ehrenfest process $\{N_{2V}(t) : t \geq 0\}$ with shifting and scaling as specified in (1.15) satisfies the sufficient condition of Stone [20, 21] so that the convergence of $\{X_V(t) : t \geq 0\}$ in the weak Markov sense to the O-U process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ as $V \to \infty$ is assured. The proof of Stone's theorem [20, 21] is sophisticated because of its generality. Since the special case discussed in this paper is simple, we provide an independent proof that $X_V(t)$ given in (1.15) converges in law to $X_{\mathrm{OU}}(t)$ as $V \to \infty$.

As we saw in (1.2), the state probability density of the O-U process $\{\, X_{\mathrm{OU}}(t) \,:\, t \geq 0 \,\}$ with initial condition $X_{\mathrm{OU}}(0) = x_0$ is normally distributed with mean $x_0\, \mathrm{e}^{-t}$ and variance $1 - \mathrm{e}^{-2t}$ for any $t > 0$. The corresponding Laplace transform with respect to x is then given by

$$\gamma(x_0, s, t) = \exp\left\{ -x_0\, \mathrm{e}^{-t} s + \frac{1}{2}(1 - \mathrm{e}^{-2t})\, s^2 \right\}. \tag{4.1}$$

Let $\{\, X_V(t) \,:\, t \geq 0 \,\}$ be a stochastic process defined by (1.15). We note that $\{\, X_V(t) \,:\, t \geq 0 \,\}$ has a discrete support on $\{\, r(0), ..., r(2V) \,\}$ where

$$r(n) = \sqrt{\frac{2}{V}}\, n - \sqrt{2V}, \quad n = 0, 1, ... \tag{4.2}$$

Clearly $r(n+1) - r(n) = \sqrt{\frac{2}{V}} \to 0$ as $V \to \infty$. For notational convenience, we define

$$\eta_V(x) = \left\lceil \sqrt{\frac{V}{2}} x \right\rceil. \tag{4.3}$$

Theorem 4.1 *Let $\{\, X_{\mathrm{OU}}(t) \,:\, t \geq 0 \,\}$ be the O-U process with $X_{\mathrm{OU}}(0) = x_0,\ -\infty < x_0 < \infty$. Let $\{X_V(t) \,:\, t \geq 0\,\}$ be as in (1.15) with $X_V(0) = \sqrt{\frac{2}{V}}\, \eta_V(x_0)$ where V is chosen large enough so that $-\sqrt{2V} \leq X_V(0) \leq \sqrt{2V}$. Then $X_V(t)$ converges in law to $X_{\mathrm{OU}}(t)$ for all $t,\ t \geq 0$, as $V \to \infty$.*

Proof. Let $\varphi_V(x_0, w, t) = \mathrm{E}\left[\mathrm{e}^{-w X_V(t)} \,\middle|\, X_V(0) = \sqrt{\frac{2}{V}}\, \eta_V(x_0) \right]$. One sees from (3.10) and (1.15) that

$$\varphi_V(x_0, w, t) = \mathrm{e}^{w\sqrt{2V}} \beta_{N_{2V}(0)}\left(t, \mathrm{e}^{-w\sqrt{\frac{2}{V}}} \right) \tag{4.4}$$

where $N_{2V}(0) = V + \eta_V(x_0)$. We wish to show that $\varphi_V(x_0, w, t) \to \gamma(x_0, w, t)$ as V $\to \infty$. Equation (4.4) can be rewritten by (3.11) as

$$\begin{aligned} \varphi_V(x_0, w, t) \;\; = \;\; & \mathrm{e}^{w\sqrt{2V}} \left[\left\{ f(t) + g(t)\, \mathrm{e}^{-w\sqrt{\frac{2}{V}}} \right\} \left\{ g(t) + f(t)\, \mathrm{e}^{-w\sqrt{\frac{2}{V}}} \right\} \right]^V \\ & \times \left[\frac{g(t) + f(t) \mathrm{e}^{-w\sqrt{\frac{2}{V}}}}{f(t) + g(t) \mathrm{e}^{-w\sqrt{\frac{2}{V}}}} \right]^{\eta_V(x_0)}. \end{aligned} \tag{4.5}$$

Since $f(t) + g(t) = 1$, the first two factors on the right hand side of (4.5) can be written as

$$\mathrm{e}^{w\sqrt{2V}} \beta_V(t, \mathrm{e}^{-w\sqrt{\frac{2}{V}}}) = \left[1 + 2f(t)\, g(t) \left\{ \cosh\left(w\sqrt{\frac{2}{V}} \right) - 1 \right\} \right]^V. \tag{4.6}$$

For sufficiently small $|Re(w)|$, one has

$$f(t)\, g(t) \left| \cosh\left(w\, \sqrt{\frac{2}{V}} \right) - 1 \right| < \frac{1}{2} \tag{4.7}$$

so that from (4.6),

$$\begin{aligned}\log\left[\mathrm{e}^{w\sqrt{2V}}\beta_V(t,\mathrm{e}^{-w\sqrt{\frac{2}{V}}})\right] &= V\log\left[1+2\,f(t)\,g(t)\left\{\cosh\left(w\sqrt{\tfrac{2}{V}}\right)-1\right\}\right]\\ &= V\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k-1}\left\{2\,f(t)\,g(t)\right\}^k\left\{\cosh\left(w\sqrt{\frac{2}{V}}\right)-1\right\}^k.\end{aligned}$$

It then follows that

$$\log\left[\mathrm{e}^{w\sqrt{2V}}\beta_V(t,\mathrm{e}^{-w\sqrt{\frac{2}{V}}})\right]=\frac{1}{2}(1-\mathrm{e}^{-2t})\,w^2+O(V^{-1}). \tag{4.8}$$

The second factor on the right hand side of (4.5) can be rewritten as

$$\begin{aligned}\left[1-\frac{\{f(t)-g(t)\}\left(1-\mathrm{e}^{-w\sqrt{\frac{2}{V}}}\right)}{f(t)+g(t)\,\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)} &= \left[1-\frac{\mathrm{e}^{-t}w\sqrt{\frac{2}{V}}+O\left(V^{-1}\right)}{1+O\left(V^{-\frac{1}{2}}\right)}\right]^{\eta_V(x_0)}\\ &= \left[1-\frac{x_0\,\mathrm{e}^{-t}w\;+\;x_0\,O\left(V^{-\frac{1}{2}}\right)}{x_0\sqrt{\frac{V}{2}}\left\{1+O(V^{-\frac{1}{2}})\right\}}\right]^{x_0\sqrt{\frac{V}{2}}\left(\frac{\eta_V(x_0)}{x_0\sqrt{\frac{V}{2}}}\right)}.\end{aligned}$$

From (4.3), $\frac{\eta_V(x_0)}{x_0\sqrt{\frac{V}{2}}}\to 1$ as $V\to\infty$ while $(1+\frac{\beta}{\alpha})^{\alpha}\to\mathrm{e}^{\beta}$ as $\alpha\to\infty$. It then follows that

$$\left[\frac{g(t)+f(t)\,\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}{f(t)+g(t)\,\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)}\to\exp\{-x_0\,\mathrm{e}^{-t}\,w\}\;\text{ as }\;V\to\infty. \tag{4.9}$$

From (4.5), (4.8) and (4.9), one concludes that

$$\varphi_V(x_0,w,t)\to\exp\left\{-x_0\,\mathrm{e}^{-t}\,w+\frac{1}{2}(1-\mathrm{e}^{-2t})\,w^2\right\}$$

as $V\to\infty$, completing the proof. □

The next corollary is immediate from Theorem 4.1.
Corollary 4.1 *For any x_0, $x\in(-\infty,\infty)$, let $m=V+\eta_V(x_0)$ and $n=V+\eta_V(x)$. Then*

$$\sqrt{\frac{V}{2}}\,p_{2V:mn}(t)\;\to\;g(x_0,x,t)\;\text{ as }V\to\infty$$

for all t, $t\geq 0$.

Corollary 4.1 may be seen alternatively in the following manner. Let $\left(H_{e_j}(x)\right)_{j=0}^{\infty}$ be the set of Hermite polynomials defined by the Rodrigues formula

$$H_{e_j}(x)=\mathrm{e}^{\frac{x^2}{2}}\,(-1)^j\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j\left(\mathrm{e}^{-\frac{x^2}{2}}\right),\quad j\geq 0, \tag{4.10}$$

where

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{2}}\,H_{e_i}(x)\,H_{e_j}(x)\,\mathrm{d}x=\delta_{ij}\,\sqrt{2\pi}\,j!. \tag{4.11}$$

The classical decomposition theorem, see e.g. Magnus, Oberhettinger and Soni [16], states that

$$\frac{1}{\sqrt{1-z^2}}\mathrm{e}^{-\frac{(x-yz)^2}{2(1-z^2)}} = \mathrm{e}^{-\frac{x^2}{2}}\sum_{j=0}^{\infty} H_{e_j}(x)\, H_{e_j}(y)\, \frac{z^j}{j!}. \tag{4.12}$$

Applying (4.12) to (1.2), one finds that

$$g(x_0,x,\tau) = \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\sum_{n=0}^{\infty} H_{e_n}(x_0)\, H_{e_n}(x)\, \frac{\mathrm{e}^{-n\tau}}{n!}. \tag{4.13}$$

From (3.15), (2.16) and Lemma A.1, one obtains that

$$\sqrt{\frac{V}{2}}\, p_{2V:mn}(t) = \sqrt{\frac{V}{2}}\,\frac{\binom{2V}{n}}{2^{2V}}\sum_{j=0}^{2V}\binom{2V}{j} y_j(m)\, y_j(n)\, \mathrm{e}^{-jt}. \tag{4.14}$$

It is known, see e.g. Szegö [26], that

$$\lim_{V\to\infty}\sqrt{\binom{2V}{j}}\, y_j(m) = \frac{1}{\sqrt{j!}}\, H_{e_j}(x_0); \quad \lim_{V\to\infty}\sqrt{\binom{2V}{j}}\, y_j(n) = \frac{1}{\sqrt{j!}}\, H_{e_j}(x).$$

The first factor $\sqrt{\frac{V}{2}}\frac{\binom{2V}{n}}{2^{2V}}$ in (4.14) converges to $\mathrm{e}^{-\frac{x^2}{2}}/\sqrt{2\pi}$ as $V\to\infty$ from Starling formula and Corollary 4.1 follows.

It is natural to expect that a first passage time of $\{X_V(t) : t\geq 0\}$ also converges in law to the corresponding first passage time of $\{X_{\mathrm{OU}}(t) : t\geq 0\}$ as $V\to\infty$, which we prove next.

Theorem 4.2 *Let m and n be as in Corollary 4.1. Let $T_{r(m)r(n)} = \inf\{\tau : X_V(\tau) = r(n) \,|X_V(0) = r(m)\}$ and $T_{x_0\, x} = \inf\{\tau : X_{\mathrm{OU}}(\tau) = x \,|\, X_{\mathrm{OU}}(0) = x_0\}$. Then $T_{r(m)r(n)}$ converges in law to $T_{x_0\, x}$ as $V\to\infty$.*

Proof. Let $l(x_0,x,\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}[T_{x_0\, x}\leq\tau]$ with $\lambda(x_0,x,s) = \int_0^\infty \mathrm{e}^{-s\tau} l(x_0,x,\tau)\,\mathrm{d}\tau = \mathrm{E}\left[\mathrm{e}^{-sT_{x_0\, x}}\right]$. From the consistency relations, one sees that

$$g(x_0,x,\tau) = \int_0^\tau l(x_0,x,\tau-y)\, g(x,x,y)\,\mathrm{d}y. \tag{4.15}$$

Taking the Laplace transform on both sides of (4.15) with respect to τ and solving for $\lambda(x_0,x,s)$, it follows that

$$\lambda(x_0,x,s) = \frac{\gamma(x_0,x,s)}{\gamma(x,x,s)}. \tag{4.16}$$

For the counter part of (4.15) for the Ehrenfest process $\{N_{2V}(t) : t\geq 0\}$, one has

$$p_{2V:mn}(t) = \int_0^t s_{mn}(t-y)\, p_{2V:nn}(y)\,\mathrm{d}y \tag{4.17}$$

where $s_{mn}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}[T_{mn}\leq\tau]$ with $T_{mn} = \inf_{t\geq 0}\{N_{2V}(t) = n | N_{2V}(0) = m\}$. Let $\pi_{mn}(s) = \int_0^\infty \mathrm{e}^{-s\tau} p_{2V:mn}(\tau)\,\mathrm{d}\tau$ and $\sigma_{mn}(s) = \int_0^\infty \mathrm{e}^{-s\tau} s_{mn}(\tau)\,\mathrm{d}\tau = \mathrm{E}\left[\mathrm{e}^{-sT_{mn}}\right]$. Corresponding to (4.16), Equation (4.17) then yields that

$$\sigma_{mn}(s) = \frac{\pi_{mn}(s)}{\pi_{nn}(s)} = \frac{\sqrt{\frac{V}{2}}\pi_{mn}(s)}{\sqrt{\frac{V}{2}}\pi_{nn}(s)}. \tag{4.18}$$

Hence from (4.16), (4.18) and Corollary 4.1, one has $\sigma_{mn}(s) \to \lambda(x_0, x, s)$ as $V \to \infty$, i.e. T_{mn} converges in law to $T_{x_0,x}$ as $V \to \infty$. It is clear that $T_{mn} = T_{r(m)r(n)}$ almost surely, completing the proof. □

Similarly, we can prove that the historical maximum defined in (2.19) also converges in law to that of the O-U process.

Theorem 4.3 *Let m be as in Corollary 4.1. Let $M(r(m), \theta) = \max_{0 \le t \le \theta} \{ X_V(t) | X_V(0) = r(m) \}$ and $M(x_0, \theta) = \max_{0 \le t \le \theta} \{ X_{\mathrm{OU}}(t) | X_{\mathrm{OU}}(0) = x_0 \}$. Then $M(r(m), \theta)$ converges in law to $M(x_0, \theta)$ as $V \to \infty$.*

Proof. For the O-U process, for $x > x_0$, one sees that $F_{x_0,\theta}(x) = \mathrm{P}[M(x_0, \theta) \le x] = \mathrm{P}[T_{x_0,x} > \theta] = \overline{S}_{x_0,x}(\theta)$. Hence

$$F_{x_0,\theta}(x) = \begin{cases} 0 & \text{if} \quad x < x_0 \\ \overline{S}_{x_0,x}(\theta) & \text{if} \quad x \ge x_0, \end{cases} \tag{4.19}$$

where $\overline{S}_{x_0,x_0}(\theta) \stackrel{\text{def.}}{=} \lim_{\Delta \to 0} \overline{S}_{x_0,x_0+\Delta}(\theta) = \mathrm{P}\left[X_{\mathrm{OU}}(\tau) \le x_0,\ 0 \le \tau \le \theta \,|\, X_{\mathrm{OU}}(0) = x_0 \right]$. The theorem then follows from Theorem 4.2 and (2.21). □

5. Development of Algorithms and Numerical Results

In this section, we develop numerical algorithms for computing transition probabilities, first passage times, and the historical maximum of $\{X_V(t) : t \ge 0\}$ based on the theoretical results discussed in the previous sections. Numerical results are also exhibited, demonstrating the accuracy and efficiency of these algorithms.

Before going into the discussion of numerical algorithms, it is appropriate to summarize state conversions among $\{N_{2V}(t) : t \ge 0\}$, $\{X_V(t) : t \ge 0\}$ and $\{X_{\mathrm{OU}}(t) : t \ge 0\}$, see Table 1 below. We note that when the state of $\{N_{2V}(t) : t \ge 0\}$ moves from 0 to $2V$, the state of $\{X_V(t) : t \ge 0\}$ moves from $-\sqrt{2V}$ to $\sqrt{2V}$.

Table 1: State conversions

Process	State conversion		State space
	$x \in \mathbb{R} \to m \in \mathcal{N}$	$m \in \mathcal{N} \to x \in \mathbb{R}$	
$N_{2V}(t)$	$m = \eta_V(x) + V$	m	$\mathcal{N} = \{0, 1, ..., 2V\}$
$X_V(t) = \sqrt{\frac{2}{V}} N_{2V}(t) - \sqrt{2V}$	$\sqrt{\frac{2}{V}} \eta_V(x)$	$r(m) = \sqrt{\frac{2}{V}} m - \sqrt{2V}$	$\{-\sqrt{2V}, ..., \sqrt{2V}\}$
$X_{\mathrm{OU}}(t)$	x	$x = r(m)$	$\mathbb{R} = (-\infty, \infty)$

Remark : $\eta_V(x) = \left\lceil \sqrt{\frac{V}{2}} x \right\rceil$.

5.1. Transition probabilities and tail probabilities

Given x_0, x, t and V, the transition probability $p_{2V:mn}(t)$ can be computed by employing the state conversion in Table 1 and the discrete convolution algorithm based on (3.11). Formally one has from (3.11),

$$p_{2V:mn}(t) = \sum_{r=0 \vee (n-m)}^{n \wedge (2V-m)} a_{m,r}(t)\, b_{m,n-r}(t) \tag{5.1}$$

where $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, and

$$a_{m,r}(t) = \binom{2V-m}{r} f(t)^{2V-m-r} g(t)^r; \quad b_{m,r}(t) = \binom{m}{r} f(t)^r g(t)^{m-r}. \tag{5.2}$$

From (1.15) and (5.1), the transition probability density function of $\{X_V(t) : t \geq 0\}$ is then given by

$$g_V(m, n, t) = \sqrt{\frac{V}{2}} p_{2V:mn}(t) \tag{5.3}$$

where

$$m = \eta_V(x_0) + V; \quad n = \eta_V(x) + V \quad \text{with} \quad \eta_V(x) = \left\lceil \sqrt{\frac{V}{2}} x \right\rceil . \tag{5.4}$$

Accordingly, $g_V(m, n, t)$ approximates $g(x_0, x, t)$ of (1.2) through the state conversion determined by (5.4).

In Figure 1, values of $g(x_0, x, t) - g_V(m, n, t)$ are plotted for $x_0 = 0, \ -5 \leq x \leq 5, \ t = 1$, and $V = 10, 20, 30, 40, 45, 47, 48, 49, 50$, demonstrating the stochastic convergence of $X_V(t)$ to $X_{\mathrm{OU}}(t)$. We see that differences among $g_V(m, n, t)$ for $45 \leq V \leq 50$ are almost negligible. Figure 2 exhibits graphically $g(x_0, x, t)$ represented by solid curves and $g_V(m, n, t)$ marked by $+, \circ, *$ for $t = 1, 3, 5$ respectively with $x_0 = 0, \ -5 \leq x \leq 5$, and $V = 50$. For tail

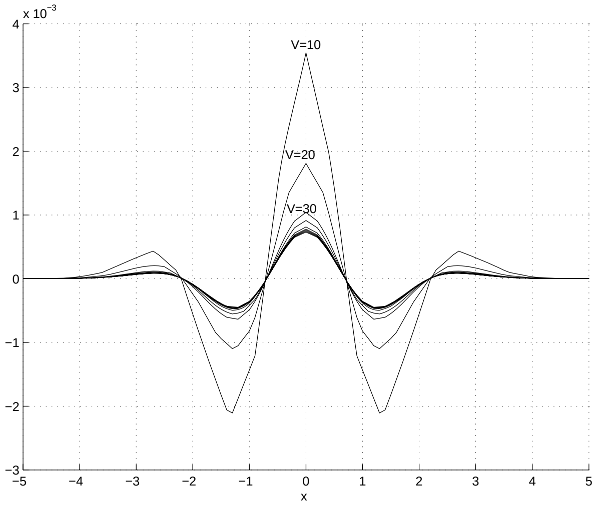

Figure 1: Difference $g(x_0, x, t) - g_V(m, n, t)$ of transition probabilities $(x_0 = 0, t = 1)$

probabilities of $g(x_0, x, t)$ with respect to x, we define

$$\overline{G}(x_0, x, \tau) = \int_x^\infty g(x_0, y, \tau) \, \mathrm{d}y. \tag{5.5}$$

Values of $\overline{G}(x_0, x, \tau)$ can be computed fairly accurately with speed using the Laguerre transform. The reader is referred to Sumita [22], where 12 digit accuracy was achieved for such computations. More readily accessible references are Sumita and Kijima [24, 25]. In order to approximate $\overline{G}(x_0, x, \tau)$, a Simpson's method is employed, i.e.

$$\overline{G}_V(m, n, \tau) = \frac{1}{2} \sum_{k=n}^{2V-1} \{p_{2V:mk}(\tau) + p_{2V:m,k+1}(\tau)\} + p_{2V:m,2V}(\tau) \tag{5.6}$$

where the last term represents the approximation for $\overline{G}(x_0, \sqrt{2V}, \tau)$. Numerical results for $\overline{G}(x_0, x, \tau)$ and $\overline{G}_V(m, n, \tau)$ are depicted in Figures 3 and 4, corresponding to Figures 1 and 2. Algorithmic relationships discussed above are summarized in Table 2.

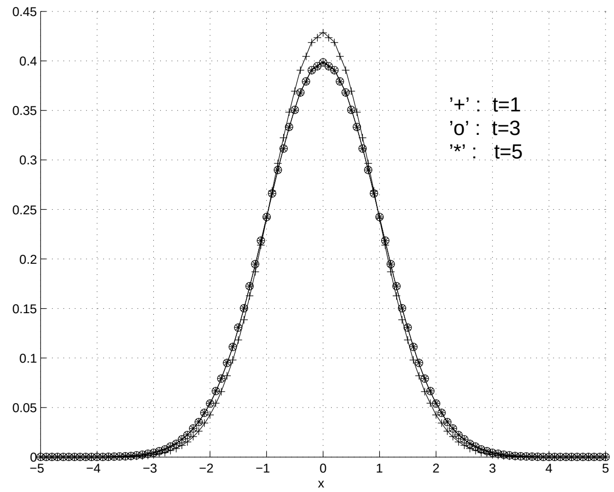

Figure 2: Transition probabilities : the O-U process vs the Ehrenfest process ($x_0 = 0, V = 50$)

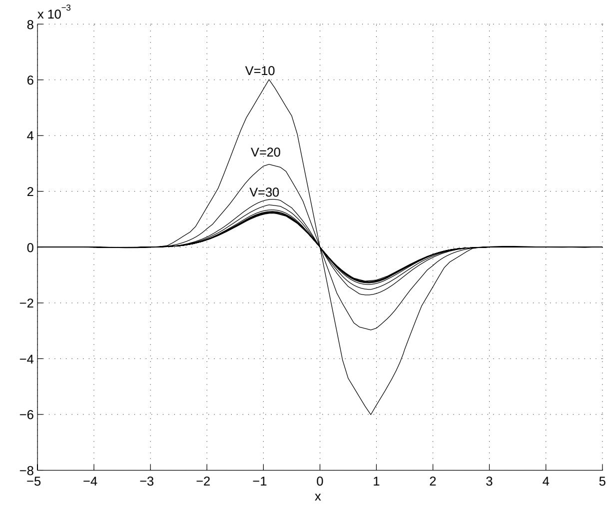

Figure 3: Difference $\overline{G}(x_0, x, \tau) - \overline{G}_V(m, n, \tau)$ of tail probabilities ($x_0 = 0, t = 1$)

5.2. Zeros of orthogonal polynomials for the Ehrenfest process

In order to evaluate the first passage time densities $s_{mn}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}[T_{mn} \le \tau],\ \ m < n$, with corresponding Laplace transforms $\sigma_{mn}(s) = \sigma_m^+(s) \cdots \sigma_{n-1}^+(s) = g_m(s)/g_n(s)$ from (2.17), the zeros of $g_n(s)$ are needed. These zeros in turn enable one to evaluate the corresponding survival functions and the distribution of the historical maximum. For the Ehrenfest process, the zeros of $g_n(s)$ are related to those of $h_n(s)$ as specified in Theorem 3.1 and the computational burden can be reduced by a factor of 4. More specifically, one can write

$$\begin{cases} h_{2m}(s) = \displaystyle\sum_{j=0}^{m} w_{2m,2j}\, s^{2j}, \quad m \ge 0, \\ h_{2m+1}(s) = \displaystyle\sum_{j=0}^{m} w_{2m+1,2j+1}\, s^{2j+1}, \quad m \ge 0, \end{cases} \tag{5.7}$$

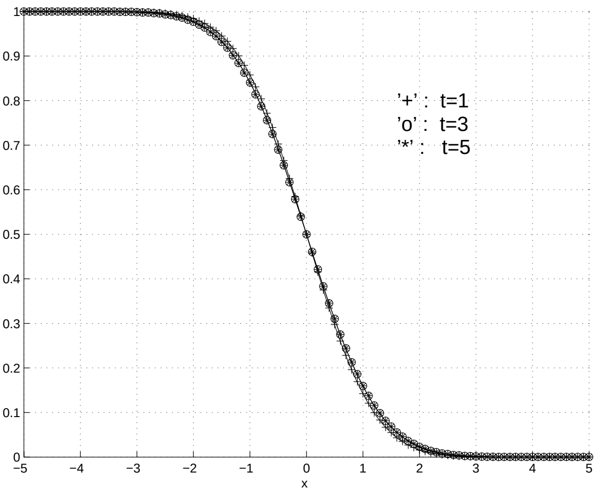

Figure 4: Tail probabilities : the Ehrenfest process vs the O-U process ($x_0 = 0, V = 50$)

Table 2: Probability conversions

Process	Transition probability	Tail probability
$N_{2V}(t)$	$p_{2V:mn}(t)$ via (5.1)	$\sum_{k=n}^{2V} p_{2V:mk}(t)$
$X_V(t) = \sqrt{\frac{2}{V}}N_{2V}(t) - \sqrt{2V}$	$g_V(m,n,t) = \sqrt{\frac{V}{2}}p_{2V:mn}(t)$	$\overline{G}_V(m,n,\tau)$ in (5.6)
$X_{\mathrm{OU}}(t)$	$g(x_0,x,t)$	$\overline{G}(x_0,x,t) = \int_x^\infty g(x_0,y,t)\mathrm{d}y$

Remark : $\cdot\ m = \eta_V(x_0) + V,\ n = \eta_V(x) + V,$

$\cdot\ p_{2V:mn}(t) = \mathrm{P}\left[\,N_{2V}(t) = n \,|\, N_{2V}(0) = m\,\right],$

$\cdot\ g(x_0,x,t) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[\,X_{\mathrm{OU}}(t) \le x \,|\, X_{\mathrm{OU}}(0) = x_0\,\right]$
$= \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2t})}} \exp\left\{-\frac{\left(x - x_0\,\mathrm{e}^{-t}\right)^2}{2(1-\mathrm{e}^{-2t})}\right\}.$

since $h_{2m}(s)$ is an even function and $h_{2m+1}(s)$ is an odd function from Theorem 3.1. It then follows from (3.20), for $m \ge 0$, that

$$\begin{cases} w_{2m,0} = -\dfrac{2}{2(V-m)+1}\left(m - \dfrac{1}{2}\right) w_{2m-2,0}, \\ w_{2m,2j} = \dfrac{2}{2(V-m)+1}\left\{w_{2m-1,2j-1} - \left(m - \dfrac{1}{2}\right) w_{2m-2,2j}\right\}, \quad j = 1, ..., m-1, \\ w_{2m,2m} = \dfrac{2}{2(V-m)+1} w_{2m-1,2m-1}, \end{cases} \tag{5.8}$$

and

$$\begin{cases} w_{2m+1,2j+1} = \dfrac{w_{2m,2j} - m\,w_{2m-1,2j+1}}{V-m}, \quad j = 0, ..., m-1, \\ w_{2m+1,2m+1} = \dfrac{w_{2m,2m}}{V-m}, \end{cases} \tag{5.9}$$

where $h_0(s) = w_{0,0} = 1$.

We note that $h_{2m+1}(0) = 0$ for $m \ge 0$. Furthermore, $h_n(s) = 0$ if and only if $h_n(-s) = 0$ for all $n \ge 0$. Hence for both $h_{2m}(s)$ and $h_{2m+1}(s)$, it suffices to search m zeros in $(0, \infty)$. For $h_n(s)$ with $1 \le n \le 4$, the zeros can be obtained explicitly by solving the underlying equations. For higher values of n, a straightforward bisection method can be employed by exploiting the fact that zeros of $h_{n+1}(s)$ interleave those of $h_n(s)$. Let ξ_{nj} $(0 \le j \le n-1)$

be zeros of $h_n(s)$. For notational convenience, let $-\alpha_{nj}$ $(0 \le j \le n-1)$ be zeros of $g_n(s)$. From Theorem 3.1, one then has

$$\alpha_{nj} = V - \xi_{nj}, \quad 0 \le j \le n-1. \tag{5.10}$$

5.3. First passage times and the historical maximum

Let $T_{V:mn}$ $(m < n)$ be the first passage time of the Ehrenfest process $\{N_{2V}(t) : t \ge 0\}$ with probability density function $s_{V:mn}(\tau)$ and its Laplace transform $\sigma_{V:mn}(s)$. Since $\sigma_{V:mn}(s) = \sigma^+_{V:m}(s) \cdots \sigma^+_{V:n-1}(s)$, one has from (2.17) that

$$\sigma_{V:mn}(s) = \frac{g_m(s)}{g_n(s)} = c_{mn} \frac{\prod_{j=0}^{m-1}(s+\alpha_{mj})}{\prod_{j=0}^{n-1}(s+\alpha_{nj})}; \quad c_{mn} = \frac{\prod_{j=0}^{n-1}\alpha_{nj}}{\prod_{j=0}^{m-1}\alpha_{mj}}. \tag{5.11}$$

As shown in Theorem 3.10 of Sumita and Masuda [23], $s_{V:mn}(\tau)$ is unimodal expressed as convolutions of completely monotone density functions. Since $\sigma_{V:mn}(s)$ is regular apart from singular points $-\alpha_{n,j}$, $0 \le j \le n-1$, Equation (5.11) can be rewritten as

$$\sigma_{V:mn}(s) = \sum_{j=0}^{n-1} A_{V:mn:j} \frac{\alpha_{nj}}{s+\alpha_{nj}}; \quad A_{V:mn:k} = \frac{\prod_{j=0}^{m-1}(1-\frac{\alpha_{nk}}{\alpha_{mj}})}{\prod_{j=0, j\ne k}^{n-1}(1-\frac{\alpha_{nk}}{\alpha_{nj}})}. \tag{5.12}$$

In real domain, Equation (5.12) leads to the probability function $s_{V:mn}(\tau)$ and its survival function $\overline{S}_{V:mn}(\tau) = \int_\tau^\infty s_{V:mn}(y)\,\mathrm{d}y$ given as

$$s_{V:mn}(\tau) = \sum_{j=0}^{n-1} A_{mn:j} \cdot \alpha_{nj} \mathrm{e}^{-\alpha_{nj}\tau}; \qquad \overline{S}_{V:mn}(\tau) = \sum_{j=0}^{n-1} A_{mn:j} \mathrm{e}^{-\alpha_{nj}\tau}. \tag{5.13}$$

Since $T_{V:mn}$ for $\{N_{2V}(\tau) : \tau \ge 0\}$ is, sample-path-wise, equal to $T_{r(m)r(n)}$ for $\{X_V(\tau) : \tau \ge 0\}$, $s_{V:mn}(\tau)$ and $\overline{S}_{V:mn}(\tau)$ provide approximations for $s_{x_0\,x}(\tau)$ and $\overline{S}_{x_0,x}(\tau)$ of $\{X_{\mathrm{OU}}(\tau) : \tau \ge 0\}$ from Theorem 4.2 with m and n as specified in Table 1. For $x_0 = 0$ and $x = 0.5$, 1.0, 1.5, 2.0, Figure 5 depicts $s_{V:mn}(\tau)$ with state conversion specified in Table 1, approximating $s_{x_0x}(\tau)$ with expected unimodality. Corresponding survival functions are plotted in Figure 6.

In order to test the accuracy of the Ehrenfest approximation, some numerical results are compared with those of Keilson and Ross [14] as shown in Table 3. Here we define $t^*(x_0, x, y\%) \stackrel{\text{def.}}{=} \overline{S}^{-1}_{V:mn}(y\%)$, where $m = \eta_V(x_0) + V$ and $n = \eta_V(x) + V$. The numerical results are for $V = 100$, $x_0 = 0$, $x = 0.5, 1, 1.5, 2$ and $y = 25, 50, 75$. The values for these cases are not found in the table of Keilson and Ross [14] and the linear interpolation is employed. It should also be noted that the approach of Keilson and Ross [14] cannot compute the survival functions of the first passage times beyond $t > 5$ accurately, while the Ehrenfest approximation proposed in this paper can deal with such tail probabilities without any difficulty.

Let $M(x_0, \theta)$ be the historical maximum of $\{X_{\mathrm{OU}}(\tau) : \tau \ge 0\}$ in the time interval $[0, \theta]$. As in (4.19), its distribution function $F_{x_0,\theta}(x)$ has a dual relationship with the survival function $\overline{S}_{x_0,x}(\theta)$. Hence $F_{x_0,\theta}(x)$ can be approximated by $F_{V:m\theta}(n) = \overline{S}_{V:m,n+1}(\theta)$ for $x_0 < x$, which implies $m < n$. When $x = x_0$, $\overline{S}_{x_0,x}(\theta) = \lim_{\Delta\to 0} \overline{S}_{x_0,x_0+\Delta}(\theta)$ can be approximated by $\overline{S}^+_{V:m}(\theta) = F_{V:m\theta}(m)$. With $m = \eta_V(x_0)+V$, and $n = \eta_V(x)+V$, $\overline{S}_{V:mn}(\theta)$ can be computed from (5.13). In Figure 7, $F_{V:m\theta}(n)$ are plotted for $x_0 = 0$, $\theta = 1, 3, 5$, and $V = 50$, where the stochastic ordering $T_{01} \prec T_{03} \prec T_{05}$ is observed as expected.

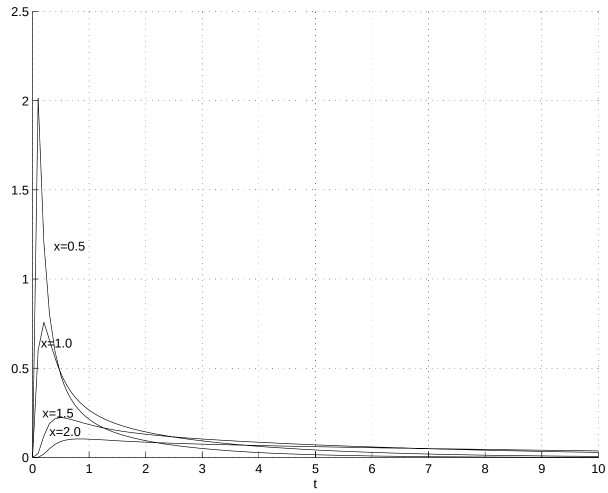

Figure 5: First passage time density functions $(x_0 = 0, V = 50)$

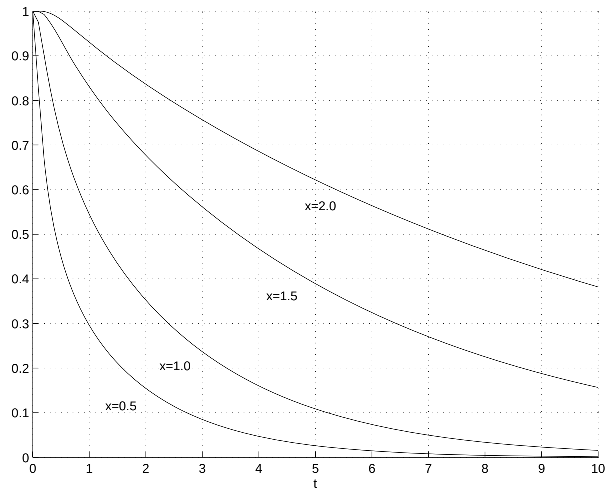

Figure 6: Survival functions of first passage times $(x_0 = 0, V = 50)$

Table 3: Comparison of the first passage times starting from $x_0 = 0$

x	$t^*(0, x, 25\%)$		$t^*(0, x, 50\%)$		$t^*(0, x, 75\%)$	
	Ehrenfest	Keilson&Ross	Ehrenfest	Keilson&Ross	Ehrenfest	Keilson&Ross
0.5	0.9309	0.9289	0.2776	0.2808	0.0925	0.0982
1	2.8714	2.8779	1.1853	1.1898	0.4415	0.4427
1.5	6.3951	-	3.0639	3.0752	1.2279	1.2388
2	14.3903	-	7.2389	-	3.0995	3.0946

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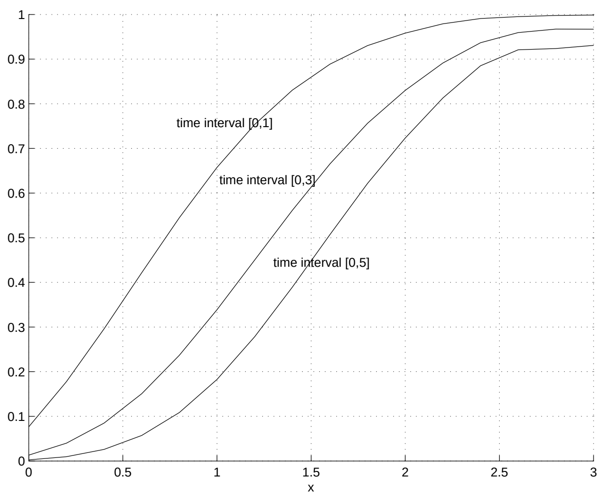

Figure 7: Distributions of the historical maximum ($x_0 = 0,\ \theta = 1, 3, 5, V = 50$)

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Appendix

In this appendix, some interesting new results are derived concerning the first passage time structure of the Ehrenfest process. We first show that $g_V(s)$ determined by (3.18) has negative odd integers as its root. A preliminary lemma is needed.

Lemma A.1 *For $m, n \in \mathcal{N}$, one has $g_n(-m) = g_m(-n)$.*

Proof. Because of an elementary property of binomial coefficients, one sees that

$$\begin{aligned}\frac{\binom{2V-n}{m-j}\binom{n}{j}}{\binom{2V}{m}} &= \frac{(2V-n)!}{(m-j)!\,(2V-n-m+j)!}\cdot\frac{n!}{j!\,(n-j)!}\cdot\frac{m!\,(2V-m)!}{(2V)!}\\ &= \frac{\binom{2V-m}{n-j}\binom{m}{j}}{\binom{2V}{n}},\end{aligned}$$

and the result follows from (3.19). □

Theorem A.1

$$g_V(s) = \prod_{j=1}^{V} \frac{s+2j-1}{2j-1}$$

Proof. Corollary 3.1 b) states that $g_n(-V) = 0$ whenever $n \in \mathcal{N}$ is odd. Hence from Lemma A.1, one has $g_V(-n) = g_n(-V) = 0$ whenever $n \in \mathcal{N}$ is odd. Since $g_V(s)$ is a polynomial of degree V, the theorem follows. □

We are now in a position to evaluate the limiting behavior of T_{0V} as $V \to \infty$. For a random variable X with $F_X(x) = \mathrm{P}[\,X \leq x\,]$, $-\infty < x < \infty$, suppose the corresponding Laplace transform $\varphi_X(s) = \mathrm{E}[\,\mathrm{e}^{-sX}\,] = \int_{-\infty}^{\infty} \mathrm{e}^{-sx}\,\mathrm{d}F_X(x)$ has the convergence strip containing the imaginary axis on the complex plane. Then the conjugate transform Y of X is defined as

$$F_Y(y) = \mathrm{P}[\,Y \leq y\,] = \frac{\int_{-\infty}^{y} \mathrm{e}^{-sx}\,\mathrm{d}F_X(x)}{\varphi_X(s)}, \quad -\infty < y < \infty. \tag{A.1}$$

The reader is referred to Keilson [12] for more detailed discussions of the conjugate transform. The next theorem shows that T_{0V} with certain shifting and scaling converges in law to a conjugate transform of an extreme-value random variate.

Theorem A.2 *Let Y be a random variable having the probability density function*

$$f_Y(\tau) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\tau - \mathrm{e}^{-\tau}\right\}, \quad -\infty < \tau < \infty. \tag{A.2}$$

Then $2\,T_{0V} - \log V$ converges in law to Y as $V \to \infty$.

Proof. Let $Z = 2\,T_{0V} - \log V$. Then from Theorem A.1 and (2.13), one sees that

$$\varphi_Z(s) = \mathrm{E}[\,\mathrm{e}^{-sZ}\,] = V^s\,\sigma_{0V}(2s) = V^s \prod_{j=1}^{V} \frac{2j-1}{2s+2j-1}.$$

By simple algebra, this then leads to

$$\varphi_Z(s) = \frac{V^{\frac{1}{2}}}{2^{2V}} \binom{2V}{V} \left\{ V^{s-\frac{1}{2}} \prod_{j-1}^{V} \frac{j}{s-\frac{1}{2}+j} \right\}. \tag{A.3}$$

The factor inside the braces converges to $\Gamma(s+\frac{1}{2})$ as $V \to \infty$, while the rest converges to $\frac{1}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{\pi}}$, i.e., $\varphi_Z(s) \to \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})}$ as $V \to \infty$. It is known that $\Gamma(s+1)$ is the Laplace transform of the extreme value distribution with p.d.f. $\exp\{-\tau - \mathrm{e}^{-\tau}\}$, $-\infty < \tau < \infty$, and thus the theorem follows. □

We saw in (2.18) that the upward first passage time T_n^+ is a finite mixture of exponential variates for general birth-death processes. In case of the Ehrenfest process, the fact that the exit rate of each state $\nu_n = \lambda_n + \mu_n = V$ is constant enables one to show that T_n^+ can also be expressed as an infinite mixture of Gamma variates of odd order.

Theorem A.3 *For the Ehrenfest process, T_n^+ and T_n^- $(n \in \mathcal{N})$ are infinite mixtures of Gamma variates of odd order with Laplace transforms* $\Gamma(V, 2j+1)$, $j = 0, 1, 2, \ldots$

Proof. The recursive formula for $\sigma_n^+(s)$ in (2.11) can be rewritten as

$$\sigma_n^+(s) = \frac{r_n^+\,\varepsilon(s)}{1 - r_n^-\,\varepsilon(s)\,\sigma_{n-1}^+(s)}; \quad \varepsilon(s) = \sigma_0^+(s) = \frac{V}{s+V},\ n \geq 1, \tag{A.4}$$

where $r_n^+ = 1 - \frac{n}{2V}$ is the probability of going up given exit from n and $r_n^- = \frac{n}{2V}$ is that of going down given exit from n. For $Re(s) > 0$, Equation (A.4) has a series expression

$$\sigma_n^+(s) = r_n^+\,\varepsilon(s) \sum_{j=0}^{\infty} \left\{ r_n^-\,\varepsilon(s)\,\sigma_{n-1}^+(s) \right\}^j$$

and the result follows by induction for $\sigma_n^+(s)$. For $\sigma_n^-(s)$, it suffices to note that $\sigma_n^+(s) = \sigma_{2V-n}^-(s)$, completing the proof. □

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