

A QUEUEING MODEL FOR LOCAL TRAFFICS TO JOIN A MAIN STREAM UNDER A LEADING SPACE CONDITION

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Abstract We consider congestion of traffics that are randomly produced in a bounded area. Those traffics start at local lines which are connected to a main line, and they have a common destination located at the end of the main line. We assume that a stream on the main line runs with a constant speed, while local traffics can join the main stream only if a certain free space condition is satisfied. So, there may arise queues at junctions where the local lines meet the main line. Our primary interest is to see congestion at the junctions. To this end, we formulate this model as a discrete time queueing process, and compute the stationary distributions of the queue lengths at the junctions, provided stability conditions are satisfied. In particular, we give inductive formulas to compute the mean queue lengths. This model may be applied to road traffics and synchronous data transfers in telecommunication networks. From the theoretical point of view, the model may be considered as an extension of priority queues. Some numerical examples are presented as well.

Keywords: Queue, transportation, priority service, network flow, stationary distribution

1. Introduction

When we drive a car, we often experience congestion. A typical one among them occurs where a local stream joins a main stream. This may also arise in a telecommunication system, in which local lines are connected to a synchronous bus line such as DQDB (Distributed Queue Dual Bus) and SONET (Synchronous Optical Network) (e.g. see [7]). We consider this type of congestion when traffics are randomly produced in a bounded area. Namely, they are generated according to a Poisson process, and their original positions are independently located subject to an arbitrary distribution on the bounded area. These traffics start at local lines, and go to junctions where the local lines meet a main line. The traffics on the bounded area have a common destination located at the end of the main line, and run with a constant speed once they join the main stream. Our major interest is to evaluate congestion at the junctions.

We may need extra free space to join a main stream from a local line. For example, when we drive a car, we take enough space to join a main road for safety. In a data transmission link, the space may be considered as a set up time. So, it is natural to consider some extra space for the local traffics to join the main stream. For this, we impose the following space condition. Local traffics waiting at its junction can start to join the main stream only if a fixed length of free space has passed at the junction, then they continuously join the main stream as long as the free space continuously comes. We refer to this joining condition as a *leading space* condition. Assuming stability conditions, we consider the stationary distributions of queues at the junctions where the stability conditions may change according

to junctions. This model was originally proposed in [3] without the leading space condition. However, the arguments there are heuristic, and seem to be not correct. In this paper, we take a different approach incorporating the leading space condition, which seems to be important for applications.

This traffic model is originally of continuous time. However, we formulate it as a discrete time model by taking a sufficiently small time unit if necessary. We also measure a distance on the main line by this time unit in such a way that each traffic on the main stream runs with a unit speed and all the distances between the junctions are multiple times of the unit distance. So we can partition the main line by this unit distance. Since we are only concerned with the marginal distribution of the queue length at each junction, we can remove the unit intervals of the main line which have no junctions without loss of generality. From the assumption on the traffic generation, the number of arrival traffics at each junction during one time unit is independent of everything else and identically distributed, but its distribution may depend on the junction.

Thus, we will formulate the traffic model as a finite number of the discrete time queues that have independent compound Bernoulli arrival processes. If customers find available servers, i.e., the leading space condition is satisfied at their arriving junctions, then they are singly served by a unit of time there. Obviously, the local traffics have higher priority to get service if their junctions are closer to the origin, i.e., upper junction. So, this model can be viewed as a priority queue. However, the model is not a simple priority queue because of the leading space condition. Thus, we carefully consider regeneration times at each junction when it becomes empty, and use them as possible service completion times for one lower junction.

This paper is composed of six sections. In Section 2, we describe the above queueing model as a discrete time Markov chain. In Section 3, we consider embedded queue lengths processes just before departure epochs at the junctions. In Section 4, we compute the time stationary distributions and their means inductively. In Section 5, we exemplify numerical computations for the mean queue lengths as well as check them by simulations. We finally give some remarks in Section 6.

2. Queueing Models

Let us formally describe the discrete time model for queues at junctions which has been introduced in Section 1. Let N be the total number of the junctions, which are numbered as $1, 2, \dots, N$ in the order of the distance from the starting point of the main line, where a smaller number is given to a higher junction. We frequently refer to junction k as level k . The queue at junction k are also numbered as k . Let

$$L = \{1, 2, \dots, N\}.$$

We assume the following dynamics for this discrete time model.

- 1) Let c be a positive integer. At each level except for level 1, waiting traffics can join the main stream only when $c + 1$ units of free space consecutively arrive from upper levels, where the last space, i.e., the $c + 1$ -th space, is used for service. More specifically, the traffics start to join into free space after c units of free space have passed, and they continuously join the main stream as long as free space continuously comes from upper levels. As we mentioned, this condition is referred to as the leading space condition. At level 1, traffics can join the stream without free space.
- 2) It takes one unit time that the traffic on the main line runs from level $(k - 1)$ to level k .

- 3) Let $A_k(m, n)$ be the number of traffics arriving at level k during the time interval (m, n) , where $m < n$. It is assumed that $A_k(n, n + 1)$ for $n = 0, 1, \dots$ are independently and identically distributed with a finite mean and variance. Note that $\{A_k(n, n + 1); n \geq 0\}$ is the compound Bernoulli process, and $A_k(m, n) = \sum_{\ell=m}^{n-1} A(\ell, \ell + 1)$.

Figure 1 below illustrates the dynamics of this model. Note that $A_k(n, n + 1)$ has a Poisson distribution for each k in the discussions of Section 1. However, we do not make this Poisson assumption since our arguments are independent of the distribution of $A_k(n, n + 1)$. Thus, $A_k(n, n + 1)$ is assumed to have a generic distribution.

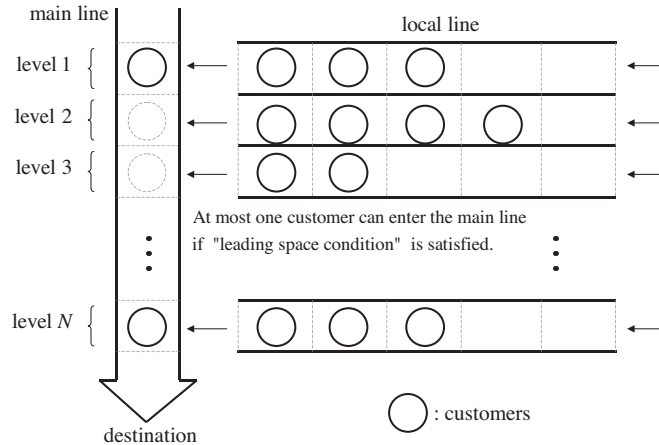


Figure 1: Traffic model for queues at junctions

We introduce a stochastic process for describing this traffic model. For $k \in L$, let $X_k^-(n)$ be the number of waiting traffics at level k just before time $n \geq 0$. For convenience, we assume that $X_k^-(n) = 0$ for $n \leq 0$. Define the following events for each $k \in L \cup \{0\}$ and $n \geq 0$:

$$G_k(n) = \{X_k^-(n - \ell) = 0, \quad 0 \leq \ell \leq c\},$$

$$F_k(n) = G_k(n) \cap F_{k-1}(n - 1),$$

where $F_0(n)$ is the whole sample space Ω . Note that event $F_k(n)$ means that $c + 1$ units of free space have been consecutively observed at level k by time n , so $X_k^-(n)$ is regenerated at this time instant. Furthermore, by the leading space condition, a waiting customer at level k can join the main stream at time n , i.e., n is a possible service instant of level k , only when event $F_{k-1}(n - 1)$ occurs. Hence, we have, for $n \geq 1$,

$$X_k^-(n + 1) = \max\{X_k^-(n) - \mathbf{1}_{F_{k-1}(n-1)}, 0\} + A_k(n, n + 1), \quad k \in L,$$

where $\mathbf{1}_A$ is the indicator function of event A . These equations show that $\{X_k^-(n); n \geq 0\}$ is inductively determined from the input processes $\{A_\ell(n, n + 1); n \geq 0\}$ for $1 \leq \ell \leq k$.

Note that the joint process $(X_1^-(n - (N - 1)), X_2^-(n - (N - 2)), \dots, X_N^-(n))$ can be considered as a discrete time priority queue with single slots of service for all customers. However, this is not a simple priority queue because of the leading space condition (see, e.g., Takagi (1993) for the standard discrete time priority queues). Furthermore, the joint process is not a Markov chain for $c \geq 2$.

3. Embedded Stationary Distributions

In this paper, our primary interest is to get the stationary distributions of $\{X_k^-(n); n \geq 0\}$ for $k \in L$. Since these processes are not Markov chains except for $k = 1$, we first consider its embedded processes just before possible departure epochs of the corresponding queues. Then, these embedded processes can be considered as Markov chains. Their transition probabilities require the distributions of the numbers of arrivals between the embedded epochs, which are obtained from the Markov chains for one upper levels. Hence, we can inductively compute the embedded stationary distributions of level k from level 1. So far, we first consider the embedded processes in this section, and then compute the time stationary distributions of $\{X_k^-(n); n \geq 0\}$ in Section 4.

First consider the level 1. In this case, we have

$$X_1^-(n+1) = \max\{X_1^-(n) - 1, 0\} + A_1(n, n+1). \quad (1)$$

This equation represents the same dynamics as the number of customers just before departures in the $M/G/1$ queue. Define the following generating functions for $0 < z \leq 1$,

$$\tilde{a}_k(z) = E[z^{A_k}], \quad k \in L,$$

where A_k is a random variable subject to the same distribution as $A_k(n, n+1)$. For $k \in L$, let X_k^- be a random variable representing the number of waiting customers at level k under steady state and $\tilde{x}_k(z)$ be its generating function, i.e., $\tilde{x}_k(z) = E[z^{X_k^-}]$. Then, the well-known Pollaczek-Khinchine formula (e.g., see (5.8) in chapter 2 of [1]) yields the following result.

Lemma 3.1 If $E[A_1] < 1$, then $\{X_1^-(n); n \geq 0\}$ has the stationary distribution, and its generating function $\tilde{x}_1(z)$ is given by

$$\tilde{x}_1(z) = (1 - E[A_1]) \frac{(z-1)\tilde{a}_1(z)}{z - \tilde{a}_1(z)}. \quad (2)$$

For each $k \in L \setminus \{1\}$, $\{X_k^-(n); n \geq 0\}$ does not have Markov property because it is affected by the upper junctions as we mentioned. We introduce the following random variables so as to have a Markov structure. For $k \in L$ and $n \geq 1$, let

$$\begin{aligned} \sigma_k(0) &= 0, \quad \sigma_k(n) = \inf_{\ell \geq 0} \{\ell > \sigma_k(n-1); F_k(\ell) \text{ holds}\}, \\ T_k(n) &= \sigma_k(n) - \sigma_k(n-1), \\ Y_k^-(n) &= X_k^-(\sigma_{k-1}(n) + 1). \end{aligned}$$

We also let $T_0(n) = 1$ for $n = 1, 2, \dots$. Note that $F_k(0)$ always holds by our convention. So, for $k = 0, 1, \dots, N$, $\{T_k(n); n \geq 1\}$ are i.i.d renewal random variables, and $\{\sigma_k(n); n \geq 0\}$ is the renewal process. Note that $\sigma_k(n)$'s are the regeneration epochs of the process $\{X_k^-(n)\}$. Furthermore, $\sigma_{k-1}(n) + 1$ for $n = 0, 1, \dots$ are only time when the leading space condition is observed at level k . Hence, $\{Y_k^-(n)\}$ is the embedded queue length process at level k just before the possible service instants. We also define

$$U_k(n) = A_k(\sigma_{k-1}(n-1) + 1, \sigma_{k-1}(n) + 1),$$

which is the total number of arriving customers during n -th consecutive possible service instants at level k .

We illustrate an example of a sample path for levels $k - 1$ and k with leading space condition $c = 3$ in Figure 2. Observe that there are " $c + 1 = 4$ " free spaces in level $(k - 1)$ at times 0, 5, 6 and 12, so they are the renewal epochs of $\{X_{k-1}^-(n); n \geq 0\}$. Thus, we have $\sigma_{k-1}(0) = 0, \sigma_{k-1}(1) = 5, \sigma_{k-1}(2) = 6$ and $\sigma_{k-1}(3) = 12$, which implies $T_{k-1}(1) = 5, T_{k-1}(2) = 1$ and $T_{k-1}(3) = 6$. Similarly, the renewal times for level k are $\sigma_k(0) = 0, \sigma_k(1) = 1$ and $\sigma_k(2) = 6$, and the renewal intervals are computed as $T_k(1) = 1$ and $T_k(2) = 5$. Note that a customer in level k can be served at time n if and only if time $n - 1$ is the renewal time of level $(k - 1)$. Since $U_k(1) = 0, U_k(2) = 3$ and $U_k(3) = 1$ are the numbers of arriving customers during the renewal intervals at level k , the number of customers in level k just before these service available times are computed as $Y_k^-(0) = Y_k^-(1) = 0$ and $Y_k^-(2) = Y_k^-(3) = 3$.

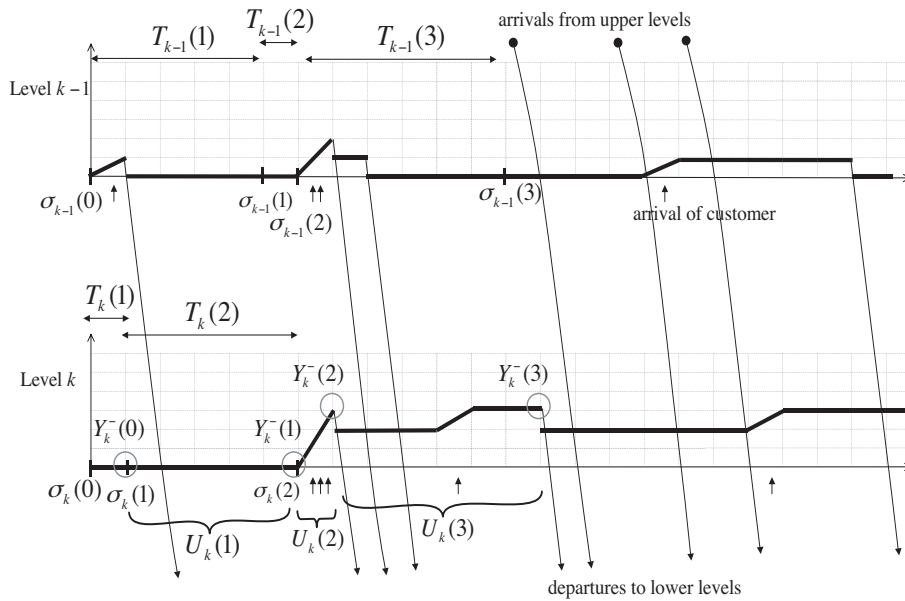


Figure 2: Sample path of level $k - 1$ and $k, c = 3$

Note that $\{U_k(n); n \geq 1\}$ is a sequence of i.i.d. random variable. Let $\tilde{u}_k(z)$ and $\tilde{g}_k(z)$ be the generating functions of $U_k(n)$ and $T_k(n)$, respectively, which are independent of n . Since $U_k(1) = \sum_{m=1}^{T_{k-1}(1)} A_k(m, m + 1)$, we have

$$\begin{aligned} \tilde{u}_k(z) &= E \left(z^{\sum_{m=1}^{T_{k-1}(1)} A_k(m, m+1)} \right) \\ &= \sum_{\ell=1}^{\infty} \tilde{a}_k^\ell(z) P(T_{k-1}(1) = \ell) = \tilde{g}_{k-1}(\tilde{a}_k(z)). \end{aligned} \tag{3}$$

From (3), we have

$$E[U_k] = E[A_k]E[T_{k-1}], \tag{4}$$

where U_k and T_{k-1} are random variables having the same distributions as $U_k(n)$ and $T_{k-1}(n)$, respectively.

Since one customer at level k can be served just after the time $\sigma_{k-1}(n) + 1$, arriving customers are queued up until just before the next possible service time, i.e., $\sigma_{k-1}(n + 1) + 1$. Hence, we have

$$Y_k^-(n + 1) = \max\{Y_k^-(n) - 1, 0\} + U_k(n + 1).$$

Thus, $\{Y_k^-(n); n \geq 1\}$ can be considered as the queueing length process of the discrete time version of the $M/G/1$ queue. As is well known, $\{Y_k^-(n); n \geq 1\}$ has the stationary distribution if $E[U_k] < 1$, which is referred to as a stability condition. Let Y_k^- be a random variable $Y_k^-(n)$ subject to this stationary distribution, and denote its generating function by $\tilde{y}_k(z)$. Similar to Lemma 3.1, we have the following result.

Lemma 3.2 If $E[U_k] < 1$, then the generating function of Y_k^- is given by

$$\tilde{y}_k(z) = (1 - E[U_k]) \frac{(z - 1)\tilde{u}_k(z)}{z - \tilde{u}_k(z)}, \quad k \in L. \tag{5}$$

Thus, all what we need is to get $\tilde{u}_k(z)$, which is obtained by $\tilde{g}_{k-1}(z)$ by (3). In the rest of this section, we compute $\tilde{g}_k(z)$ for $k \in L$. Define $\eta_k(n)$ and τ_k for $k \in L$ as

$$\begin{aligned} \eta_k(n) &= \sigma_{k-1}(n) + 1, \quad n \geq 0, \\ \tau_k &= \min \{ \eta_k(\ell) - 1; X_k^-(\eta_k(\ell)) = \max \{ X_k^-(\eta_k(0)) - 1, 0 \}, \ell \geq 1 \}. \end{aligned}$$

In words, $\eta_k(n)$ is the n -th possible service instant at level k , and, for each fixed $m = 0, 1, \dots$, τ_k represents the first time just before the queue length equals m measured from the possible service instant just after the queue length becomes m . Since the distribution of this time interval is independent of m , we denote it by τ_k . Remark that, for $m \geq 1$, τ_k represents the time that is required for the queue length at level k to decrease by one.

Let $\{\tau_k(\ell); \ell \geq 1\}$ be a sequence of independent random variables that have the same distribution as τ_k . Observe one possible service period at level k , which corresponds with one recurrence period of level $k - 1$, and is distributed as T_{k-1} . If j customers arrive in this period, then $\tau_k(1) + \dots + \tau_k(j)$ units of time are required so that the queue length is either decreased by one or return to zero. Hence, we have

$$\tau_k \stackrel{d}{\sim} T_{k-1} + \sum_{\ell=1}^{A_k(1, T_{k-1}+1)} \tau_k(\ell), \tag{6}$$

where $\stackrel{d}{\sim}$ stands for equality in distribution. Figure 3 illustrates a sample path for (6).

In this figure, $\sigma_{k-1}(0) = 0$, $\sigma_{k-1}(1) = 15$, $\sigma_{k-1}(2) = 16$, $\sigma_{k-1}(3) = 21$ and $\sigma_{k-1}(4) = 22$ are renewal times at level $(k - 1)$. Hence, the possible service times at level k are $\eta_k(0) = 1$, $\eta_k(1) = 16$, $\eta_k(2) = 17$, $\eta_k(3) = 22$ and $\eta_k(4) = 23$. Since $X_k^-(\eta_k(0)) = 3$, $X_k^-(\eta_k(1)) = 2$, $X_k^-(\eta_k(2)) = 3$ and $X_k^-(\eta_k(3)) = 2$, we have $\tau_k = \eta_k(4) - \eta_k(0) = 22$.

Let $\tilde{\tau}_k(z)$ be the generating function of τ_k , then we have the following result.

Lemma 3.3 For $0 < z < 1$, $\tilde{\tau}_k(z)$ is uniquely determined by

$$\tilde{\tau}_k(z) = \tilde{g}_{k-1}(z\tilde{a}_k(\tilde{\tau}_k(z))), \quad k \in L, \tag{7}$$

where $\tilde{g}_0(z) = z$.

PROOF. Taking the generating function of (6), we have

$$\begin{aligned} \tilde{\tau}_k(z) &= E[z^{T_{k-1} + \sum_{\ell=1}^{A_k(1, T_{k-1}+1)} \tau_k(\ell)}] \\ &= \sum_{u=0}^{\infty} \sum_{j=1}^{\infty} E[z^{j + \sum_{\ell=1}^u \tau_k(\ell)}] P(T_{k-1} = j) P(A_k(1, j + 1) = u) \\ &= \sum_{j=1}^{\infty} \{z\tilde{a}_k(\tilde{\tau}_k(z))\}^j P(T_{k-1} = j). \end{aligned}$$

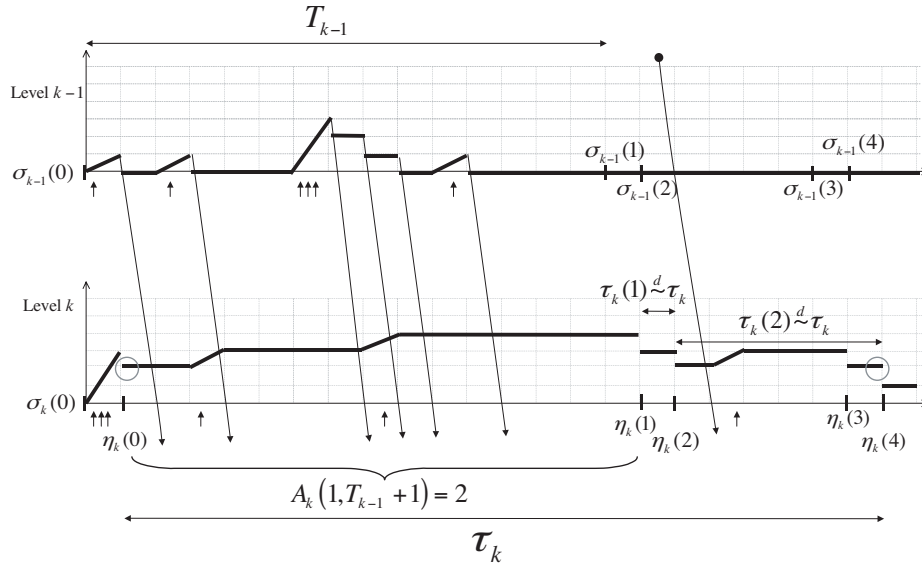


Figure 3: Sample path for τ_k with $c = 3$

Thus, we get (7). We next prove the uniqueness of $\tilde{\tau}_k(z)$. For $z \in (0, 1)$, let $\xi = \tilde{\tau}_k(z)$, then (7) is written as

$$\xi = \tilde{g}_{k-1}(z\tilde{a}_k(\xi)). \tag{8}$$

We show that this equation uniquely determines ξ in $(0, 1)$. Note that $\tilde{g}_{k-1}(z\tilde{a}_k(\xi))$ is a convex and increasing function of ξ for each z . Since $\tilde{a}_k(0) = P(A_k = 0) > 0$ and $\tilde{a}_k(\xi)$ is a convex and increasing in ξ , we have

$$0 < \tilde{g}_{k-1}(z\tilde{a}_k(0)) < \tilde{g}_{k-1}(z\tilde{a}_k(1)) = \tilde{g}_{k-1}(z) < 1, \quad 0 < z < 1.$$

This implies that (8) has only one solution in $(0, 1)$. □

From Lemma 3.3, we can numerically compute $\tilde{\tau}_k(z)$ for each z by following algorithms.

(Algorithm 1 for $\tilde{\tau}_k(z)$)

Step 1 Set $\xi_0 = 0$.

Step 2 Iterate $\xi_n = \tilde{g}_{k-1}(z\tilde{a}_k(\xi_{n-1}))$, $n \geq 1$ until $|\xi_n - \xi_{n-1}| < \epsilon$, where ϵ is a sufficiently small positive number.

Step 3 Return ξ_n

We now consider $\tilde{g}_k(z)$ for $k \in L$. To this end, we need some extra notation. For each $k \in L$, let

$$q_k = \prod_{i=1}^k P(A_i = 0) = \prod_{i=1}^k \tilde{a}_i(0),$$

and let D_k be a random variable subject to the geometric distribution with parameter q_k , i.e.,

$$P(D_k = j) = (1 - q_k)q_k^j, \quad j \geq 0.$$

Let \hat{T}_k be a random variable T_k given the conditions that $A_k(0, 0) \geq 1$ or $T_{k-1} \geq 2$, i.e.,

$$P(\hat{T}_k = \ell) = P(T_k = \ell | A_k(0, 1) \geq 1 \text{ or } T_{k-1} \geq 2) \quad \text{for } \ell \geq 1,$$

where the conditional event is meant that there are customers at level k just before time 1 who either arrive from the outside or come from level $k - 1$, provided time 0 is the renewal epoch for level k . Then, we have the following key observation.

Lemma 3.4 For $k \in L$, we have

$$T_k \stackrel{d}{\sim} T_{k-1} + \begin{cases} 0, & \text{if } A_k(0, T_{k-1}) = 0, \\ \sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m) + c + 1, & \text{if } A_k(0, T_{k-1}) \geq 1, D_k \geq c + 1, \\ \sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m) + D_k + \hat{T}_k, & \text{if } A_k(0, T_{k-1}) \geq 1, D_k \leq c. \end{cases}$$

PROOF. Suppose that time 0 is the renewal epoch for level k . Then, time -1 is the renewal epoch for level $k - 1$. Hence, $(-1, T_{k-1} - 1]$ is the renewal interval for level $k - 1$, and T_{k-1} is the first possible service time after time 0 for level k . We first consider the case that $A_k(0, T_{k-1}) = 0$. In this case, there is no local arrival at level k during the time interval $(0, T_{k-1})$, so T_k must be T_{k-1} . We next consider the case that $A_k(0, T_{k-1}) \geq 1$. In this case, there are local arrivals at level k during that time interval. Since the number of these arrivals is $A_k(0, T_{k-1})$ and one customer is served at the end of the interval, the queue at level k is again empty after $\sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m)$ time units have passed. After this, if we observe consecutive $c + 1$ units of free space, then we have the next renewal instant for level k . This is the second case. Otherwise, we need to repeat similar intervals until we observe consecutive $c + 1$ units of free space. Since the repeated intervals are subject to the conditional distribution of T_k given that $A_k(0, 1) \geq 1$ or $T_{k-1} \geq 2$, we have the third case. \square

We need one more lemma to compute $\tilde{g}_k(z)$.

Lemma 3.5 Let $\hat{g}_k(z)$ be the generating function of \hat{T}_k , then we have

$$\hat{g}_k(z) = \frac{\tilde{g}_k(z) - zq_k}{1 - q_k}. \tag{9}$$

PROOF. From the definition of \hat{T}_k , we have

$$\begin{aligned} \hat{g}_k(z) &= E[z^{\hat{T}_k} | A_k(0, 1) \geq 1 \text{ or } T_{k-1} \geq 2] \\ &= \frac{1}{1 - P(A_k(0, 1) = 0, T_{k-1} = 1)} (E[z^{\hat{T}_k}] - E[z^{\hat{T}_k} \mathbf{1}_{(A_k(0, 1) = 0, T_{k-1} = 1)}]) . \end{aligned}$$

Since $P(A_k(0, 1) = 0, T_{k-1} = 1) = q_k$ and $T_k = 1$ if $A_k(0, 1) = 0$ and $T_{k-1} = 1$, we get (9). \square

We now arrive at the following result which determines $\tilde{g}_k(x)$ inductively from $k = 1$ to N using Lemma 3.3.

Lemma 3.6 The generating function \tilde{g}_k of T_k is

$$\tilde{g}_k(z) = 1 + \frac{(\tilde{\tau}_k(z) - 1)\tilde{g}_{k-1}(z\tilde{a}_k(0))}{\tilde{\tau}_k(z) - \gamma_k(z)f_k(z)}, \tag{10}$$

where

$$f_k(z) = \tilde{\tau}_k(z) - \tilde{g}_{k-1}(z\tilde{a}_k(0)), \quad \gamma_k(z) = \frac{1 - (zq_k)^{c+1}}{1 - zq_k}.$$

PROOF. From Lemma 3.4, we can decompose the generating function of T_k as

$$\begin{aligned}
 E[z^{T_k}] &= E[z^{T_{k-1}} \mathbf{1}_{\{A_k(0, T_{k-1})=0\}}] \\
 &\quad + E\left[z^{T_{k-1} + \sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m) + c + 1} \mathbf{1}_{\{A_k(0, T_{k-1}) \geq 1, D_k \geq c + 1\}}\right] \\
 &\quad + E\left[z^{T_{k-1} + \sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m) + D_k + \hat{T}_k} \mathbf{1}_{\{A_k(0, T_{k-1}) \geq 1, D_k \leq c\}}\right].
 \end{aligned}$$

The first term of the right hand side is computed as

$$\begin{aligned}
 E[z^{T_{k-1}} \mathbf{1}_{\{A_k(0, T_{k-1})=0\}}] &= \sum_{n=1}^{\infty} E[z^n \mathbf{1}_{\{A_k(0, n)=0\}}] P(T_{k-1} = n) \\
 &= \sum_{n=1}^{\infty} z^n \tilde{a}(0)^n P(T_{k-1} = n) \\
 &= \tilde{g}_{k-1}(z \tilde{a}_k(0)).
 \end{aligned}$$

The second term is

$$\begin{aligned}
 &E\left[z^{T_{k-1} + \sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m) + c + 1} \mathbf{1}_{\{A_k(0, T_{k-1}) \geq 1, D_k \geq c + 1\}}\right] \\
 &= z^{c+1} P(D_k \geq c + 1) \sum_{n=1}^{\infty} z^n \sum_{\ell=1}^{\infty} E\left[z^{\sum_{m=1}^{\ell-1} \tau_k(m)}\right] P(A_k(0, n) = \ell) P(T_{k-1} = n) \\
 &= (zq_k)^{c+1} \tilde{\tau}_k(z)^{-1} \{\tilde{g}_{k-1}(z \tilde{a}_k(\tilde{\tau}_k(z))) - \tilde{g}_{k-1}(z \tilde{a}_k(0))\} \\
 &= (zq_k)^{c+1} \tilde{\tau}_k(z)^{-1} f_k(z),
 \end{aligned}$$

where the last equality is obtained using (7). Similarly, the third term is

$$\begin{aligned}
 &E\left[z^{T_{k-1} + \sum_{m=1}^{A_k(0, T_{k-1})-1} \tau_k(m) + D_k + \hat{T}_k} \mathbf{1}_{\{A_k(0, T_{k-1}) \geq 1, D_k \leq c\}}\right] \\
 &= \sum_{n=1}^{\infty} \left\{ \sum_{\ell=1}^{\infty} \left\{ \sum_{d=0}^c z^n z^d E\left[z^{\sum_{m=1}^{\ell-1} \tau_k(m)}\right] \hat{g}_k(z) P(D_k = d) \right\} P(A_k(0, n) = \ell) \right\} P(T_{k-1} = n) \\
 &= \hat{g}_k(z) \tilde{\tau}_k(z)^{-1} \{\tilde{g}_{k-1}(z \tilde{a}_k(\tilde{\tau}_k(z))) - \tilde{g}_{k-1}(z \tilde{a}_k(0))\} \sum_{d=0}^c z^d P(D_k = d) \\
 &= (\tilde{g}_k(z) - zq_k) \tilde{\tau}_k(z)^{-1} f_k(z) \gamma_k(z)
 \end{aligned}$$

where the last equality is obtained using Lemma 3.5. Summing these three terms, we have

$$\begin{aligned}
 \tilde{g}_k(z) &= \tilde{g}_{k-1}(z \tilde{a}_k(0)) + (zq_k)^{c+1} \tilde{\tau}_k(z)^{-1} f_k(z) + (\tilde{g}_k(z) - zq_k) \tilde{\tau}_k(z)^{-1} f_k(z) \gamma_k(z) \\
 &= \tilde{\tau}_k(z)^{-1} (\tilde{g}_{k-1}(z \tilde{a}_k(0)) + (1 - (1 - zq_k) \gamma_k(z) + (\tilde{g}_k(z) - zq_k) \gamma_k(z)) f_k(z)) \\
 &= \tilde{\tau}_k(z)^{-1} (\tilde{g}_{k-1}(z \tilde{a}_k(0)) + (\tilde{g}_k(z) - 1) \gamma_k(z) f_k(z)).
 \end{aligned}$$

This yields (10). □

As a corollary of Lemmas 3.3 and 3.6, we give a recursive formula to compute $E[T_k]$.

Lemma 3.7 For $k \in L$, we have

$$E[T_k] = \frac{E[T_{k-1}]\tilde{g}_{k-1}(\tilde{a}_k(0))}{(1 - E[A_k]E[T_{k-1}])(1 - f_k(1)\gamma_k(1))}, \tag{11}$$

where $E[T_0] = 1$, $f_k(1) = 1 - \tilde{g}_{k-1}(\tilde{a}_k(0))$ and $\gamma_k(1) = \frac{1-q_k^{c+1}}{1-d_k}$.
 PROOF. Differentiating both sides of (7) at $z = 1$, we have

$$E[\tau_k] = (1 + E[\tau_k]E[A_k])E[T_{k-1}],$$

which yields

$$E[\tau_k] = \frac{E[T_{k-1}]}{1 - E[A_k]E[T_{k-1}]}. \tag{12}$$

Similarly, differentiating both sides of (10) at $z = 1$, we have

$$E[T_k] = \frac{E[\tau_k]\tilde{g}_{k-1}(\tilde{a}_k(0))}{1 - f_k(1)\gamma_k(1)}.$$

Hence, we get (11). □

4. The Stationary Distributions and Their Means

We are now ready to compute the stationary distribution of X_k^- under the stability condition, which is $E[U_k] < 1$ obviously. Let $\check{a}_k(\theta)$ and $\check{x}_k(\theta)$ be the moment generating functions of A_k and X_k^- , respectively, i.e., $\check{a}_k = \tilde{a}_k(e^\theta)$ and $\check{x}_k(\theta) = \tilde{x}_k(e^\theta)$.

Theorem 4.1 For $k \in L$, if $E[U_k] < 1$, then we have

$$\check{x}_k(\theta) = \frac{1 - E[A_k]E[T_{k-1}]}{E[T_{k-1}]} \frac{(e^\theta - 1)\check{a}_k(\theta)}{e^\theta - \tilde{g}_{k-1}(\check{a}_k(\theta))} \frac{1 - \tilde{g}_{k-1}(\check{a}_k(\theta))}{1 - \check{a}_k(\theta)}, \quad \theta \leq 0. \tag{13}$$

PROOF. Since $E[A_k]E[T_{k-1}] = E[U_k] < 1$ by (4) and the assumption, $E[T_{k-1}]$ is finite. So, we can apply the cycle formula with respect to $\{T_{k-1}(n)\}$ to get the stationary distribution of $X_k^-(n)$ (e.g., see Section 3.4.1 of [1]). Thus we get

$$\begin{aligned} E[z^{X_k^-}] &= \frac{1}{E[T_{k-1}]} E \left[\sum_{n=1}^{T_{k-1}} z^{X_k^-(n)} \right] \\ &= \frac{1}{E[T_{k-1}]} E \left[\sum_{n=1}^{T_{k-1}} z^{\max(Y_k^- - 1, 0) + A_k(0, n)} \right] \\ &= \frac{1}{E[T_{k-1}]} E \left[z^{\max(Y_k^- - 1, 0)} \right] E \left[\sum_{n=1}^{T_{k-1}} \tilde{a}_k^n(z) \right]. \end{aligned}$$

From (3) and (5) and using the fact that $\tilde{y}_k(0) = 1 - E[U_k]$,

$$\begin{aligned} E \left[z^{\max(Y_k^- - 1, 0)} \right] &= z^{-1} E \left[z^{Y_k^-} \mathbf{1}(Y_k^- \geq 1) \right] + P(Y_k^- = 0) \\ &= z^{-1}(\tilde{y}_k(z) + (z - 1)\tilde{y}_k(0)) \\ &= (1 - E[U_k]) \frac{z - 1}{z - \tilde{g}_{k-1}(\tilde{a}(z))}. \end{aligned}$$

On the other hand,

$$E \left[\sum_{n=1}^{T_{k-1}} \tilde{a}_k^n(z) \right] = \tilde{a}_k(z) E \left[\frac{1 - \tilde{a}_k^{T_{k-1}}(z)}{1 - \tilde{a}_k(z)} \right] = \frac{\tilde{a}_k(z)(1 - \tilde{g}_{k-1}(\tilde{a}_k(z)))}{1 - \tilde{a}_k(z)}.$$

Combining these computations and letting $z = e^\theta$, we have (13). □

Differentiating both sides of (13) at $\theta = 0$, we have the following result.

Corollary 4.1 For $k \in L$, we have,

$$E[X_k^-] = \frac{V[A_k]E[T_{k-1}] + E[A_k]^2V[T_{k-1}]}{2E[A_k]E[T_{k-1}](1 - E[A_k]E[T_{k-1}])} - \frac{V[A_k]}{2E[A_k]} + \frac{E[A_k]}{2}, \tag{14}$$

and, by Little’s law, the mean waiting time $E[W_k]$ at the level k is given by

$$E[W_k] = \frac{1}{E[A_k]} \times E[X_k^-].$$

These results show how the mean characteristics $E[X_k^-]$ and $E[W_k]$ are affected by upper levels through T_{k-1} . To evaluate them, we need to compute $E[T_{k-1}]$ and $V[T_{k-1}]$ from given distributions of A_1, A_2, \dots, A_{k-1} . By Lemma 3.7, we can compute $E[T_{k-1}]$ inductively, using $\tilde{g}_0(\tilde{a}_1(0)), \tilde{g}_1(\tilde{a}_2(0)), \dots, \tilde{g}_{k-2}(\tilde{a}_{k-1}(0))$. Similarly to Lemma 3.7, we can compute $V[T_{k-1}]$ using Lemma 3.6, which yields

$$V[T_{k-1}] = \frac{\tilde{g}_{k-2}(\tilde{a}_{k-1}(0))(V[\tau_{k-1}] + E[\tau_{k-1}]^2) + 2\tilde{a}_{k-1}(0)\tilde{g}'_{k-2}(\tilde{a}_{k-1}(0))E[\tau_{k-1}]}{1 - \gamma_{k-1}(1)f_{k-1}(1)} - \frac{2\tilde{g}_{k-2}(\tilde{a}_{k-1}(0))E[\tau_{k-1}]B_{k-1} + E[\tau_{k-1}]^2\tilde{g}_{k-2}(\tilde{a}_{k-1}(0))^2}{(1 - \gamma_{k-1}(1)f_{k-1}(1))^2}, \tag{15}$$

where $B_{k-1} = E[\tau_{k-1}] - \gamma'_{k-1}(1)f_{k-1}(1) - \gamma_{k-1}(1)f'_{k-1}(1)$. Here, $E[\tau_{k-1}]$ is computed by (12). Differentiating (7), $V[\tau_{k-1}]$ is similarly computed as

$$V[\tau_{k-1}] = \frac{V[T_{k-2}] + V[A_{k-1}]E[T_{k-2}]^3}{(1 - E[A_{k-1}]E[T_{k-2}])^3}. \tag{16}$$

Thus, $V[T_{k-1}]$ is obtained from $E[T_{k-2}]$, $V[T_{k-2}]$, $\tilde{g}_{k-2}(\tilde{a}_{k-1}(0))$ and $\tilde{g}'_{k-2}(\tilde{a}_{k-1}(0))$ since $f'_{k-1}(1) = E[\tau_{k-1}] - \tilde{a}_{k-1}(0)\tilde{g}'_{k-2}(\tilde{a}_{k-1}(0))$. So, we can inductively compute $V[T_{k-1}]$ if we know $\tilde{g}_\ell(\tilde{a}_{\ell+1}(0))$ and $\tilde{g}'_\ell(\tilde{a}_{\ell+1}(0))$ for $\ell = 0, 1, \dots, k - 2$. To get these values, we use a recursive computation algorithm, which is elaborated below.

Suppose $k \geq 3$. Otherwise, the computations are obvious. Let $\ell = k - 2$. We consider to numerically evaluate $\tilde{g}_\ell(z)$ and $\tilde{g}'_\ell(z)$ for $z = \tilde{a}_{\ell+1}(0)$. From Lemma 3.6, they require $\tilde{\tau}_\ell(z)$ and $\tilde{\tau}'_\ell(z)$ and $\tilde{g}_{\ell-1}(z)$ and $\tilde{g}'_{\ell-1}(z)$. Note that $\tilde{\tau}_\ell(z)$ is computed from $\tilde{g}_{\ell-1}(z)$ by Algorithm 1, which requires $\tilde{g}_{\ell-1}(u)$ for finitely many u other than z . On the other hand, differentiating (7), we have

$$\tilde{\tau}'_\ell(z) = \frac{\tilde{g}'_{\ell-1}(z\tilde{a}_\ell(\tilde{\tau}_\ell(z))) \cdot \tilde{a}_\ell(\tilde{\tau}_\ell(z))}{1 - z\tilde{g}'_{\ell-1}(z\tilde{a}_\ell(\tilde{\tau}_\ell(z))) \cdot \tilde{a}'_\ell(\tilde{\tau}_\ell(z))}.$$

So, $\tilde{\tau}'_\ell(z)$ is obtained from $\tilde{\tau}_\ell(z)$ and $\tilde{g}'_{\ell-1}(z\tilde{a}_\ell(\tilde{\tau}_\ell(z)))$. Hence, $\tilde{g}_\ell(z)$ and $\tilde{g}'_\ell(z)$ are computed from $\tilde{g}_{\ell-1}(u)$ and $\tilde{g}'_{\ell-1}(u)$ for finitely many u . In this way, we recursively call the procedure until ℓ is decreased to 1, where $\tilde{g}_0(z) = z$ and the recursive computations are completed.

5. Numerical Examples

Let us numerically compute the mean queue length $E[X_k^-]$ using Corollary 4.1 and the algorithm given after it. We first compare our numerical results with the corresponding simulations. Of course, such comparison is not meaningful in mathematics, but it is sometimes helpful to find mathematical errors in particular when analytical results are complicated. Table 1 gives such numbers for the case that $c = 3$ and customers arrive at level k subject to the Poisson process with rate λ_k , i.e., $A_k(n, n+1)$ has the Poisson distribution with mean λ_k , where each simulation is performed for 10^8 time units. Here, the 95% interval means the 95% confidence interval. Those confidence intervals are computed, assuming that the queue length process is uncorrelated. Hence, they would be much smaller than the exact confidence intervals particularly for high traffic intensities. Taking this into account, we can see that our numerical computations are fully compatible with the corresponding simulations.

Level k	1	2	3	4	5	6
λ_k	0.05	0.06	0.04	0.08	0.07	0.1
Numerical	0.051316	0.097791	0.110680	0.367563	0.792541	6.105358
Simulation	0.051332	0.097876	0.110809	0.367630	0.793398	6.133683
95% interval	$4.4 \cdot 10^{-5}$	$6.2 \cdot 10^{-5}$	$7.6 \cdot 10^{-5}$	$1.5 \cdot 10^{-4}$	$2.5 \cdot 10^{-4}$	$7.2 \cdot 10^{-4}$

Table 1: Comparison with simulation: $c = 3$, $N = 6$ and Poisson arrivals

We next present a numerical example to see how the mean queue length looks like in Figure 4. It is given for $c = 3, 4, 5, 6$, $N = 6$ and the Poisson arrivals. We observe that the

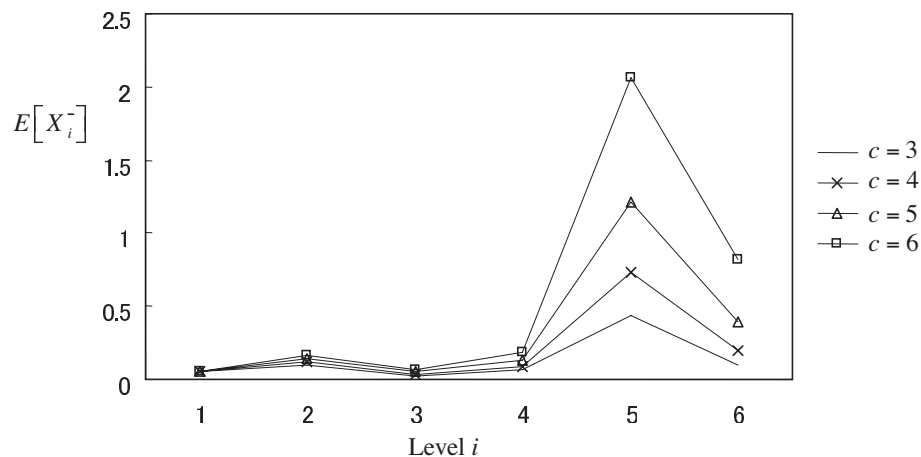


Figure 4: $\lambda_k = 0.05, 0.06, 0.01, 0.02, 0.1, 0.01$ for $k = 1, 2, \dots, 6$

mean queue length can be decreased if the mean arrival rates are sufficiently small. The reader may concern how long it takes for our numerical computations because the recursive algorithm is used. Up to $k = 5$, the mean number of customers at level k is immediately obtained. However, it takes about 40 seconds to compute one case for $c = 6$ and $k = 6$. For these computations, we have used language C on a personal computer with 2.8 GHz Pentium D CPU.

6. Concluding Remarks

In this paper, we made numerical computations only for the mean queue length. This is because our major interest is on analytical results. Of course, higher moments can be

computed by using Theorem 4.1 in a similar way to compute the mean. Note that computing the m -th moment of the queue length at level k requires to evaluate the n -th derivative $\tilde{\tau}_\ell^{(n)}(z)$ for $\ell = 1, 2, \dots, k-1$ and $n = 1, 2, \dots, m+1$. So, we need a lot of recursive computations.

Since the analytic results are so complicated, it may be interesting to consider approximations. For example, we may have a fluid flow model by rescaling time and queue lengths. For this, we need a priority queue with continuous valued priority level. Such a priority model is studied in [4] but for the standard priority rule and conventional customers. If we impose a complicated priority rule such as the leading space condition, then its stationary analysis may be a challenging problem. In the area of fluid approximations for load traffics, arguments are usually heuristic as is seen in [3]. There seem to be many theoretical issues in finding good fluid models (e.g., see [8]). We hope this study will provide useful information for such theoretical investigations.

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References

- [1] P. Bremaud: *Markov chains gibbs fields, monte carlo simulation, and queues*, (Springer, 1998).
- [2] A. Cobham: Priority assignment in waiting line problems. *Operations Research*, **2** (1954), 70-76.
- [3] H. Miura: A queueing model for transportation facilities. *The proceeding of the autumn meeting of Operations Research society of Japan*, (1997), 64-65 (in Japanese).
- [4] L. Takaćs: Priority Queues. *Operations Research* **12** (1964), 63-74.
- [5] H. Takagi: Priority queues with batch Poisson arrivals. *Operations Research Letters* **10** (1991) 225-232.
- [6] H. Takagi: *Queueing analysis, Vol 3: Discrete-time system*, (North-Holland, Amsterdam, 1993).
- [7] J. Walrand, P. Varaiya: *High-performance Communication Network, Second edition* (Morgan Kaufmann, San Francisco, California, 2000), 211-223.
- [8] W. Whitt: *Stochastic-Process Limits, An Introduction to Stochastic-Process Limits and Their Application to Queues* (Springer, 2001).

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