Journal of the Operations Research
Society of Japan
2006, Vol. 49, No. 4, 332-341

INVERSE SENSITIVE ANALYSIS OF PAIRWISE COMPARISON MATRICES

Kouichi Taji
Nagoya university

Keiji Matsumoto
Fujitec co., ltd.

(Received October 20, 2005; Revised July 25, 2006)

Abstract In the analytic hierarchy process (AHP), the consistency of pairwise comparison is measured by the consistency index (CI) value of pairwise comparison matrices. The CI value is defined by the size and the principal eigenvalue of comparison matrix, and the larger the CI value is, the less consistent pairwise comparison is. In this paper, we propose the estimation method of consistency intervals, in which the pairwise comparison value can move without exceeding given CI threshold. The proposed method is based on the eigenvalue method but does not calculate the principal eigenvalue and eigenvector at all, and hence, the method enables quick estimation. The method also enables us to detect which pairwise comparison leads to inconsistency of pairwise comparison and what extent the consistency can be improved, if the CI value of an original comparison matrix is large. Several numerical examples show that the proposed method is practically efficient.

Keywords: AHP, pairwise comparison, consistency interval, eigenvalue method

1. Introduction

The AHP (Analytic Hierarchy Process), proposed by Saaty [5], is one of decision making models which consists of three parts, namely, making hierarchy structure of the problem, evaluating local weights by pairwise comparison and calculating the global weights by additional sum. An AHP has been widely used because it can deal with unquantifiable objects and its implementation is very easy. In this paper, we concentrate pairwise comparison, the second stage of the AHP. It has been considered that an AHP is the method evaluating alternatives in a ratio scale [2], hence, the pairwise comparison value a_{ij} means the ratio of the weights w_i and w_j of alternatives i and j. When the exact weights of all alternatives are already known, each comparison value a_{ij} equals to w_i/w_j exactly. In this case, a pairwise comparison matrix A can be written as

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & 1 & & w_2/w_n \\ \vdots & & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & 1 \end{pmatrix}.$$

We call this situation *completely consistent*, which corresponds to the case that the rank of A is one. But in practical settings, such a case seldom occurs. Moreover, if the most popular linear scale, such as $1/9, 1/7, 1/5, 1/3, 1, 3, 5, 7, 9$, is used for a comparison value, it is hardly expected that the resulting comparison matrix has rank one.

Therefore, Saaty [5] has introduced so-called *consistency index value* (CI value) to measure the consistency of pairwise comparison. The CI value is defined as

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n-1}, \tag{1.1}$$

where n is the size of a comparison matrix A and $\lambda_{\max}$ is the principal eigenvalue of A. It is well known that $\lambda_{\max} \geq n$ holds for a pairwise comparison matrix and that $\lambda_{\max} = n$ if and only if the corresponding comparison matrix is completely consistent. Hence, in general the more CI value is, the less consistent a pairwise comparison matrix is, and Saaty [5] indicates that a comparison matrix can be thought to be consistent if its CI value is less than 0.1 (or in some case, 0.15).

When an AHP is applied to practical problems, pairwise comparisons are often reviewed and performed iteratively. Also, because of some mistake and inaccurate data, there may be an unusual and false observations [4] which deteriorate consistency of pairwise comparison. In such cases, it is practically useful to estimate consistency intervals, in which a pairwise comparison value can move without exceeding a given CI threshold, or to indicate a pairwise comparison which causes inconsistency the most.

In this paper, based on the eigenvalue method, we propose the estimation method of consistency intervals. The proposed method does not calculate an eigenvalue and an eigenvector at all, but calculates an upper bound of a given CI threshold. Moreover, by using an optimization method, the method detects a pairwise comparison error and suggests how much the consistency can be improved, when the pairwise comparison is unacceptably inconsistent. For these issue, based on the geometric mean method, Aguaròn et al. [1] proposed a calculation method of geometric consistency intervals and Lipovetsky and Conklin [4] proposed the method detecting a pairwise comparison error. But so far as the authors know, there has not been the method based on the eigenvalue method which is the most popular way of evaluating local weights from pairwise comparison matrix in AHP.

This paper is organized as follows. In section 2, we present the Frobenius' theorem which is a key of our method. We give the main results in section 3. We apply the proposed method to some numerical examples and show its practical effectiveness in section 4. Finally in section 5 ,we conclude the paper.

2. Frobenius' Theorem

Throughout the paper, we denote R_+ as a set of positive real number, $R_+ = \{x \in R | x > 0\}$, and R_+^n as a set of n dimensional positive real vectors, $R_+^n = \{x = (x_1, \ldots, x_n)^T \in R^n | x_i > 0,\ i = 1, \ldots, n\}$. We assume the size of a comparison matrix A is greater than or equal to 3.

For any $n \times n$ matrix $A = (a_{ij})$, we call *positive* if all elements a_{ij} is positive, and we denote A_i as i-th row vector of A. Then the well-known Perron-Frobenius' theorem [6, Theorem 1] says that there exists the unique positive eigenvalue $\lambda_{\max}$, called *principal eigenvalue* and its corresponding eigenvector w^*, called *principal eigenvector*, whose elements are all positive. Because a pairwise comparison matrix in AHP is of course positive, this fact guarantees that the uniqueness of the local weights estimated by the eigenvalue method.

We also have the following Frobenius' min-max theorem [6, Theorem 3].

Theorem 2.1 *Let A be an $n \times n$ positive matrix and let its principal eigenvalue be denoted by $\lambda_{\max}$. Then, for any positive vector $w \in R_+^n$, we have*

$$\min\left\{\frac{A_1 w}{w_1}, \cdots, \frac{A_n w}{w_n}\right\} \leq \lambda_{\max} \leq \max\left\{\frac{A_1 w}{w_1}, \cdots, \frac{A_n w}{w_n}\right\}. \tag{2.1}$$

Moreover, the both inequality holds with equality if and only if the vector w is the principal eigenvector w^*, that is,

$$\min\left\{\frac{A_1 w^*}{w_1^*}, \cdots, \frac{A_n w^*}{w_n^*}\right\} = \lambda_{\max} = \max\left\{\frac{A_1 w^*}{w_1^*}, \cdots, \frac{A_n w^*}{w_n^*}\right\}. \tag{2.2}$$

From this theorem, it can be seen that, for a given positive vector $w \in R_+^n$, the right hand side of the equation (2.1) gives an upper bound of the principal eigenvalue. Hence, from the equation (1.1), the CI value is guaranteed to be less than given $\overline{\text{CI}}$, if there is a positive vector $w \in R_+^n$ such that

$$\max\left\{\frac{A_1 w}{w_1}, \cdots, \frac{A_n w}{w_n}\right\} \leq n + \overline{\text{CI}} \times (n-1). \tag{2.3}$$

3. Main Results

In this section, we propose the method estimating consistency intervals and the method improving consistency.

Suppose that an (i,j)-element a_{ij} of a pairwise comparison matrix A is multiplied by $\delta > 0$ with satisfying reciprocal property, that is, a (j,i)-element a_{ji} is multiplied by $1/\delta$. Suppose also that all element except a_{ij} and a_{ji} is fixed. Then, we can estimate the consistency interval of a_{ij} if we obtain the lower and upper bound of δ satisfying the inequality (2.3). From the equation (2.2) in Theorem 2.1, to estimate more accurately, it is desirable that the positive vector w in (2.3) is chosen as close as the eigenvector of a perturbed comparison matrix.

One candidate is the eigenvector w^* of the original comparison matrix. In this case, to estimate the consistency interval for a_{ij}, it is sufficient from the equation (2.3) to solve $\hat{\delta}$ and $\tilde{\delta}$ of the equations

$$\frac{a_{i1}w_1^* + \cdots + \hat{\delta} a_{ij} w_j^* + \cdots + a_{in} w_n^*}{w_i^*} = n + \overline{\text{CI}} \times (n-1)$$

and

$$\frac{a_{j1}w_1^* + \cdots + \tilde{\delta} a_{ji} w_i^* + \cdots + a_{jn} w_n^*}{w_j^*} = n + \overline{\text{CI}} \times (n-1),$$

respectively, then either $[a_{ji}/\tilde{\delta}, \hat{\delta} a_{ij}]$ or $[\hat{\delta} a_{ij}, a_{ij}/\tilde{\delta}]$ yields consistency interval of a_{ij} for given consistency threshold $\overline{\text{CI}}$. However, this is not a good way as shown in the following example.

Example 3.1 Let consider the comparison matrix of size $n = 3$,

$$\begin{pmatrix} 1 & 2 & 2 \\ 1/2 & 1 & 2 \\ 1/2 & 1/2 & 1 \end{pmatrix}.$$

The CI value of this matrix is CI $= 0.02681$. When the above mentioned method is applied to this matrix with $\overline{\text{CI}} = 0.1$, consistency intervals are estimated as

$$\begin{array}{ll} a_{12} : [1.68858, 2.23236] & \text{CI} : [0.01529, 0.03604] \\ a_{13} : [1.79182, 2.36885] & \text{CI} : [0.03604, 0.01529] \\ a_{23} : [1.68858, 2.23236] & \text{CI} : [0.01529, 0.03604], \end{array}$$

where CI denotes the CI values of the comparison matrix with which the comparison values are replaced by the corresponding interval boundaries. This is poor result because the CI values at the lower and upper bounds of the interval are far from the threshold and because the only matrix satisfying the given consistency threshold is the original one if we limit the comparison value to an integer or its reciprocal.

Therefore, we adopt the row geometric mean of a comparison matrix as a positive vector w in (2.3), because the row geometric mean is a good approximation of a principal eigenvector and is often used as local weights. Moreover, it is a principal eigenvector in the case a matrix size less than 4.

In the following, we denote v as the row geometric mean of the original comparison matrix $A = (a_{ij})$, that is, the i-th element of v is

$$v_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}}.$$

We also denote $v(\delta_{ij})$ as the row geometric mean of a perturbed comparison matrix, in which a_{ij} is replaced by δa_{ij}. It follows from elementary calculation and the reciprocal property that

$$v_i(\delta_{ij}) = \delta^{\frac{1}{n}} v_i, \;\; v_j(\delta_{ij}) = \delta^{-\frac{1}{n}} v_j, \;\; v_k(\delta_{ij}) = v_k \;(k \neq i, j).$$

Furthermore, let $f_k(\delta_{ij})$ be denoted as the k-th element of the left hand side of (2.3) in which a_{ij} is replaced by δa_{ij}, then direct calculations show that

$$\begin{aligned} f_i(\delta_{ij}) &= \frac{a_{ij} v_j}{v_i} \delta^{1-\frac{2}{n}} + \frac{\sum_{l \neq i,j} a_{il} v_l}{v_i} \delta^{-\frac{1}{n}} + 1 \\ f_j(\delta_{ij}) &= \frac{a_{ji} v_i}{v_j} \delta^{-1+\frac{2}{n}} + \frac{\sum_{l \neq i,j} a_{jl} v_l}{v_j} \delta^{\frac{1}{n}} + 1 \\ f_k(\delta_{ij}) &= \frac{a_{ki} v_i}{v_k} \delta^{\frac{1}{n}} + \frac{a_{kj} v_j}{v_k} \delta^{-\frac{1}{n}} + \frac{\sum_{l \neq i,j} a_{kl} v_l}{v_k} \quad (k \neq i, j). \end{aligned}$$

Hence, if the interval of δ_{ij} satisfying

$$f(\delta_{ij}) = \max_{1 \leq k \leq n} f_k(\delta_{ij}) \leq n + \overline{\mathrm{CI}} \times (n-1) \tag{3.1}$$

is found, then we can estimate the consistent interval of a_{ij} for given consistent threshold $\overline{\mathrm{CI}}$.

In the case of size $n = 3$, we have from direct calculations that

$$f_1(\delta_{12}) = f_2(\delta_{12}) = f_3(\delta_{12}) = 1 + \delta^{1/3} \left(\frac{a_{12} a_{23}}{a_{13}} \right)^{1/3} + \delta^{-1/3} \left(\frac{a_{13}}{a_{12} a_{23}} \right)^{1/3},$$

and hence,

$$f(\delta_{12}) = \max_{1 \leq k \leq 3} f_k(\delta_{12}) = 1 + \delta^{1/3} \left(\frac{a_{12} a_{23}}{a_{13}} \right)^{1/3} + \delta^{-1/3} \left(\frac{a_{13}}{a_{12} a_{23}} \right)^{1/3}.$$

Thus, from (3.1), the consistent interval of a_{12} for $\overline{\mathrm{CI}}$ is given by

$$a_{12} : \left[\delta_{12}^{-} a_{12}, \delta_{12}^{+} a_{12} \right],$$

where $\delta_{12}^{\pm}$ are two solutions of the equation

$$1+\delta^{1/3}\left(\frac{a_{12}a_{23}}{a_{13}}\right)^{1/3}+\delta^{-1/3}\left(\frac{a_{13}}{a_{12}a_{23}}\right)^{1/3}=3+2\overline{\mathrm{CI}},$$

that is,

$$\delta_{12}^{\pm}=\frac{a_{13}}{a_{12}a_{23}}\left(1+\overline{\mathrm{CI}}\pm\sqrt{\left(1+\overline{\mathrm{CI}}\right)^2-1}\right)^3. \tag{3.2}$$

Similarly, $f(\delta_{13})$ and $f(\delta_{23})$ are given by

$$f(\delta_{13})=1+\delta^{1/3}\left(\frac{a_{13}}{a_{12}a_{23}}\right)^{1/3}+\delta^{-1/3}\left(\frac{a_{12}a_{23}}{a_{13}}\right)^{1/3}$$

and

$$f(\delta_{23})=f(\delta_{12})=1+\delta^{1/3}\left(\frac{a_{12}a_{23}}{a_{13}}\right)^{1/3}+\delta^{-1/3}\left(\frac{a_{13}}{a_{12}a_{23}}\right)^{1/3},$$

respectively. Hence, consistent intervals of a_{13} and a_{23} for $\overline{\mathrm{CI}}$ are given by

$$a_{13}:\ \left[\delta_{13}^{-}a_{13},\delta_{13}^{+}a_{13}\right]\ \text{and}\ a_{23}:\ \left[\delta_{12}^{-}a_{23},\delta_{12}^{+}a_{23}\right],$$

respectively, where $\delta_{13}^{\pm}$ are given by

$$\delta_{13}^{\pm}=\frac{a_{12}a_{23}}{a_{13}}\left(1+\overline{\mathrm{CI}}\pm\sqrt{\left(1+\overline{\mathrm{CI}}\right)^2-1}\right)^3 \tag{3.3}$$

and $\delta_{12}^{\pm}$ are given by (3.2).

By using the equations (3.2) and (3.3), we can calculate $\delta_{12}^{\pm}$ and $\delta_{13}^{\pm}$ for the comparison matrix of Example 3.1 with $\overline{\mathrm{CI}}=0.1$, such that

$$\delta_{12}^{+}=\delta_{23}^{+}=1.89186,\ \delta_{12}^{-}=\delta_{23}^{-}=0.13215\ \delta_{13}^{+}=7.56742,\ \delta_{13}^{-}=0.52858,$$

and consistency intervals for $\overline{\mathrm{CI}}=0.1$ are calculated as

$$\begin{aligned}a_{12}&:[0.26430,3.78371]\\ a_{13}&:[1.05716,15.13484]\\ a_{23}&:[0.26430,3.78371].\end{aligned}$$

Remark 3.1 Let $\lambda_{\max}(\delta_{ij})$ be denoted as a principal eigenvalue of comparison matrix with a_{ij} replaced by δa_{ij}. Then, in the case of $n=3$, the above argument and Theorem 2.1 lead to $\lambda_{\max}(\delta_{12})=\lambda_{\max}(\delta_{23})=f(\delta_{12})$ and $\lambda_{\max}(\delta_{23})=f(\delta_{23})$. Hence, at the lower and upper bound of the obtained consistent interval, a CI value of resulting comparison matrix is equal to $\overline{\mathrm{CI}}$. It is easy to show that $f(\delta_{12})$ and $f(\delta_{13})$ have unique minimum $\delta_{12}^{*}=\frac{a_{13}}{a_{12}a_{23}}$ and $\delta_{13}^{*}=\frac{a_{12}a_{23}}{a_{13}}$, respectively, which correspond to completely consistent matrices. This is well-known fact that a comparison matrix with size $n=3$ can be made completely consistent by replacing one of comparison value a_{12}, a_{13} or a_{23} by a_{13}/a_{23}, $a_{12}a_{23}$ or a_{13}/a_{12}, respectively.

When the size of a comparison matrix is greater than 3, there is no longer a closed form as the case $n=3$, and hence, we should resort to numerical solution methods. The function $f(\delta_{ij})$ defined by (3.1) has nice property, unimodal function, though $f(\delta_{ij})$ is not convex.

Theorem 3.1 *The function $f(\delta_{ij}) : R_+ \to R$ defined by (3.1) is unimodal and has the unique minima in R_+.*

Proof. Let $g_k : R_+ \to R,\ (k = 1, \ldots, n)$ be a function defined as

$$g_k(x) = \begin{cases} \frac{a_{ij}v_j}{v_i}x^{n-2} + \frac{\sum_{l\neq i,j} a_{il}v_l}{v_i}x^{-1} + 1 & \text{if } k = i \\ \frac{a_{ji}v_i}{v_j}x^{-n+2} + \frac{\sum_{l\neq i,j} a_{jl}v_l}{v_j}x + 1 & \text{if } k = j \\ \frac{a_{ki}v_i}{v_k}x + \frac{a_{kj}v_j}{v_k}x^{-1} + \frac{\sum_{l\neq i,j} a_{kl}v_l}{v_k} & \text{otherwise,} \end{cases}$$

and let $g : R_+ \to R$ be a function defined as

$$g(x) = \max_{1\leq k\leq n} g_k(x). \tag{3.4}$$

Then $f(\delta_{ij})$ is expressed as $f(\delta_{ij}) = g(\delta^{1/n})$, and it is suffices to show that the function g is unimodal and has the unique minima.

We first note that each function g_k becomes infinity as $x \to +\infty$ or $x \downarrow 0$, and hence, the function g also has the same property. Moreover, each g_k is a continuous strict convex function on R_+. It is easy to show that the pointwise maximum of finitely many number of continuous and strict convex functions is also continuous strictly convex. Therefore, the function g defined by (3.4) is a continuous strictly convex function on R_+, and must have unique minima at R_+. □

From this theorem, the consistent interval satisfying the inequality (3.1) for given threshold $\overline{\text{CI}}$ is evaluated by the following consistent interval estimation procedure.

Procedure CIE

Step 1 Choose an $\alpha > 0$ sufficiently large such that $f(\alpha) > n + \overline{\text{CI}} \times (n-1)$ and $f(1/\alpha) > n + \overline{\text{CI}} \times (n-1)$.

Step 2 Apply the minimizing method for one-dimensional unimodal function (e.g. a golden section method [3]) to find a δ^*_{ij} minimizing $f(\delta_{ij})$ over $[1/\alpha, \alpha]$.

Step 3 If $f(\delta^*_{ij}) > n + \overline{\text{CI}} \times (n-1)$ then Stop. The interval satisfying (3.1) does not exist. Otherwise, find solutions δ^-_{ij} in $[1/\alpha, \delta^*_{ij}]$ and δ^+_{ij} in $[\delta^*_{ij}, \alpha]$ of the equation

$$f(\delta_{ij}) = n + \overline{\text{CI}} \times (n-1).$$

The interval $[\delta^-_{ij}a_{ij}, \delta^+_{ij}a_{ij}]$ is required one.

Note that Step 3 of this procedure is easily performed by a bisection method for example.

As stated in Remark 3.1, $\lambda_{\max}(\delta_{ij})$ is denoted as a principal eigenvalue of comparison matrix with a_{ij} replaced by δa_{ij}. Then it is clear from (1.1) that minimizing $\lambda_{\max}(\delta_{ij})$ leads to improving consistency. From the construction of the function f, the function $f(\delta_{ij})$ is only an upper bound of $\lambda_{\max}(\delta_{ij})$, that is, $\lambda_{\max}(\delta_{ij}) \leq f(\delta_{ij})$ in the case $n \geq 4$. But it is expected to improve consistency when the upper bound $f(\delta_{ij})$ is minimized. In particular, if there is a pair (i, j) such that the minimum $f(\delta^*_{ij})$ is less than the original principal eigenvalue $\lambda_{\max} = \lambda_{\max}(1)$, then the consistency index is also guaranteed to be improved.

Based on the above observation, we propose the method detecting the comparison error when the CI value of the original comparison matrix is relatively large.

Procedure ED

Step 1 Search a $\delta^*_{ij} \in R^*_+$ minimizing the function $f(\delta_{ij})$ for $(i, j = i, \ldots, n)$.

Step 2 Choose the pair (i^*, j^*) as a candidate of a comparison error such that $f(\delta^*_{i^*j^*})$ is the smallest among $f(\delta^*_{ij})$. The comparison value $a_{i^*j^*}$ is replaced by $\delta^*_{i^*j^*}a_{i^*j^*}$ to improve consistency.

For the simplicity purpose, the Procedure ED chooses only the pair (i^*, j^*) at which the value $f(\delta^*_{i^*j^*})$ is the smallest. But another pair (i', j'), at which the value $f(\delta^*_{i'j'})$ is second smallest, may also be a comparison error and a candidate for improving consistency. In fact, our numerical results in the next section indicate that the second smallest pair can improve consistency, but the degree is far from that by the smallest one. Finally, we note that δ^*_{ij} and $f(\delta^*_{ij})$ appeared in Step 1 of the procedure ED are already obtained in the Step 2 of Procedure CIE and are not necessary evaluated again.

4. Numerical Experiences

In this section, we present some computational results of the proposed methods in the previous section.

We first show the results of the matrix with size $n = 3$.

Example 4.1 The following comparison matrix with size $n = 3$ is an example of Aguarón et al. [1]:

$$\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{pmatrix}.$$

The CI value of this matrix is CI = 0.00185. We evaluated the consistency intervals with threshold $\overline{\text{CI}} = 0.1$. From the equations (3.2) and (3.3), we obtained

$$\delta^+_{12} = \delta^+_{23} = 3.15309,\ \delta^-_{12} = \delta^-_{23} = 0.22024\ \delta^+_{13} = 4.54045,\ \delta^-_{13} = 0.317145,$$

and hence, consistency intervals were

$$\begin{aligned} a_{12} &: [0.44049, 6.30618] \\ a_{13} &: [1.58575, 22.70225] \\ a_{23} &: [0.66073, 9.45927]. \end{aligned}$$

In this example, the upper bound of the interval for a_{13} is larger than 9, the upper bound of pairwise comparison value usually used in AHP. However such a value can be computationally accepted according to the definition of CI (1.1). We note that, at the boundary of each obtained interval, the CI value of the corresponding comparison matrix is equal to 0.1, because the row geometric mean coincides with the principal eigenvector in the case $n = 3$.

Example 4.2 The next example is the comparison matrix with $n = 4$ such that

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 1/2 & 1 & 2 & 2 \\ 1/2 & 1/2 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}$$

whose CI value is CI = 0.04044. We applied the Procedure CIE with $\overline{\text{CI}} = 0.1$, and we obtained $\delta^\pm_{ij}$ as

$$\begin{aligned} &\delta^+_{12} = \delta^+_{23} = \delta^+_{34} = 1.57914, && \delta^-_{12} = \delta^-_{23} = \delta^-_{34} = 0.15831 \\ &\delta^+_{13} = \delta^+_{24} = 2.17732, && \delta^-_{13} = \delta^-_{24} = 0.45928, \\ &\delta^+_{14} = 6.31657, && \delta^-_{14} = 0.63326. \end{aligned}$$

Hence, the consistent intervals and the corresponding CI values of their boundaries were

$$\begin{array}{ll} a_{12}, a_{23}, a_{34} : [0.31663, 3.15828] & \text{CI} : [0.07681, 0.07681] \\ a_{13}, a_{24} : [0.91856, 4.35464] & \text{CI} : [0.06748, 0.06748] \\ a_{14} : [1.26651, 12.63314] & \text{CI} : [0.07681, 0.07681]. \end{array} \quad (4.1)$$

Since the size of the matrix is greater then 3, the function $f(\delta_{ij})$ is only an upper bound of $\lambda_{\max}(\delta_{ij})$ and each interval obtained is expected to be inside of the corresponding exact interval. This is also observed from the results because the CI values at the boundaries of the obtained intervals are less than $\overline{\text{CI}} = 0.1$. For the comparison purpose, we calculated the consistent interval less than CI $= 0.1$ by substitution method where we calculated the principal eigenvalue and eigenvector of the matrix whose comparison value a_{ij} was replaced by some positive integer or its reciprocal. Then we have the consistent intervals as follows

$$a_{12}, a_{23}, a_{34} : [1/3, 3], \quad a_{13}, a_{24} : [1, 6], \quad a_{14} : [2, 15].$$

By comparing this with the results (4.1) of the proposed method, it can be seen that the proposed estimation method yields good estimation.

Example 4.3 The next example is the comparison matrix of size $n = 4$ including circular priority such as

$$\begin{pmatrix} 1 & 1/2 & 1 & 2 \\ 2 & 1 & 1/2 & 1 \\ 1 & 2 & 1 & 1/2 \\ 1/2 & 1 & 2 & 1 \end{pmatrix}.$$

It can be seen from the matrix that an alternative 4 is more important than 3, 3 is more important than 2, 2 is more than 1 and 1 is more than 4, and hence, the preference is circular. Hence, the CI value of this matrix is large, CI $= 0.16667$, indicating that the pairwise comparison is inconsistent. We applied Procedure ED and obtained the minimum $f(\delta^*_{ij})$ and the corresponding δ^*_{ij} such that

$$\begin{array}{ll} f(\delta^*_{12}) = f(\delta^*_{23}) = f(\delta^*_{34}) = 4.41421, & \delta^*_{12} = \delta^*_{23} = \delta^*_{34} = 4 \\ f(\delta^*_{14}) = 4.41421, & \delta^*_{14} = 0.25 \\ f(\delta^*_{13}) = f(\delta^*_{24}) = 4.5, & \delta^*_{13} = \delta^*_{24} = 1. \end{array}$$

This results suggests that one of a_{12}, a_{23}, a_{34} and a_{14} may be considered as comparison error. For example, when a_{12} is replaced by $2 = a_{12} \times \delta^*_{12}$, the resulting matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1/2 & 1 & 1/2 & 1 \\ 1 & 2 & 1 & 1/2 \\ 1/2 & 1 & 2 & 1 \end{pmatrix}.$$

This matrix includes no circular priority and its CI value is CI $= 0.08333$ where consistency is improved significantly. We remark that, in this example, second smallest pairs of our results are $(1, 3)$ and $(2, 4)$ and these minimum are attained at $\delta^*_{13} = \delta^*_{24} = 1$. This does not necessarily indicates that the consistency cannot be improved when either a_{13} or a_{24} is perturbed. However, when a_{13}, for example, is replaced by 2 or 1/2, the CI value of the resulting matrix is 0.19104, greater than the original one.

For the comparison purpose, we applied another error detection and consistency improving method using the principal eigenvector such as the following.

Step 1 Calculate the principal eigenvector w^* of the comparison matrix A and then calculate $a_{ij}w_j^*/w_i^*$ for $(i, j = i, \ldots, n)$.

Step 2 Find the pair (i_0, j_0) such that $a_{i_0j_0}w_{j_0}^*/w_{i_0}^*$ is the largest (or smallest) among $a_{ij}w_j^*/w_i^*$. The comparison value $a_{i_0j_0}$ is regarded as a comparison error and $a_{i_0j_0}$ and $a_{j_0i_0}$ are replaced by $w_{i_0}^*/w_{j_0}^*$ and $w_{j_0}^*/w_{i_0}^*$, respectively.

In the following, we call this procedure ED2.

When the procedure ED2 is applied to this example, the principal eigenvector is $w^* = (0.25, 0.25, 0.25, 0.25)$ and a_{12}, a_{14}, a_{23} and a_{34} are also considered as comparison errors. But when an a_{12} is replaced by $w_1^*/w_2^* = 1$ for example, the CI value of the resulting matrix is 0.10337, which is larger than that of our method ED.

Example 4.4 This example is a comparison matrix of size $n = 8$ from Lipovetsky and Conklin [4] such that

$$\begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & 1/3 & 5 & 3 & 3 & 1/5 & 1/7 \\ 1/3 & 3 & 1 & 6 & 3 & 4 & 6 & 1/5 \\ 1/7 & 1/5 & 1/6 & 1 & 1/3 & 1/4 & 1/7 & 1/8 \\ 1/6 & 1/3 & 1/3 & 3 & 1 & 1/2 & 1/5 & 1/6 \\ 1/6 & 1/3 & 1/4 & 4 & 2 & 1 & 1/5 & 1/6 \\ 3 & 5 & 1/6 & 7 & 5 & 5 & 1 & 1/2 \\ 4 & 7 & 5 & 8 & 6 & 6 & 2 & 1 \end{pmatrix}.$$

The CI value of this matrix is 0.23841. Lipovetsky and Conklin [4] have detected that $a_{37} = 6$ is comparison error and have obtained by substitution method an appropriate value of a_{37} is 1/2 and the corresponding CI value is 0.11623. We applied Procedure ED and we found that the smallest value among $f(\delta_{ij}^*)$ was also $f(\delta_{37}^*) = 9.00245$ with $\delta_{37}^* = 0.06554$ The corresponding comparison value was $a_{37}^* = 0.39321$ and the CI value of that value matrix was 0.11623. These results are almost same as the results of [4].

The second smallest of this example by Procedure ED was the pair $(1, 3)$, and its minimum was $f(\delta_{13}^*) = 10.38167$ attained at $\delta_{13}^* = 0.16953$. The CI value of the matrix in which a_{13} was replaced by $0.50859 = 3 \times 0.16953$ was CI $= 0.20394$. Thus, the second smallest pair is hardly considered as comparison error in this example.

We also applied the procedure ED2 to this example, and the value a_{37} was also regarded as a comparison error. But the revised value indicated by ED2 is $w_3^*/w_7^* = 1.12753$ and the CI value of resulting matrix is 0.12691, which is inferior to our method.

Example 4.5 In the last computational experiences, we tested the proposed methods CIE and ED to all possible comparison matrices of size $n = 4$ with the linear scale, 1/9, 1/7, 1/5, 1/3, 1, 3, 5, 7, 9. The number of all such matrices is 531441, of which only 18681 matrices are consistent, that is, their CI values are less than 0.1, and the rest 512760 matrices are inconsistent.

We applied the procedure CIE to consistent cases. The CIE succeeded to calculate consistency intervals with $\overline{\text{CI}} = 0.1$ properly for 10461 matrices of 18681 consistent matrices. The CI values of original matrices of successful cases are 0.04390 in average and the CI values of the boundaries of obtained consistency intervals are 0.070477 in average.

On the other hand, for inconsistent 512760 matrices, whose CI values are greater than 0.1 and their average CI value is 0.97103, we applied the procedures ED and ED2. The both two methods could decrease successfully CI values for all 512760 matrices. However, the average CI value of matrices improved by ED is 0.13392, which is much less than that of ED2, 0.37059. Furthermore, in the method ED, there are 241524 cases of which the CI values of improved matrices are less than 0.1, while in the method ED2, there are only 66444 cases. Finally, we remark that the CI values of matrices improved by ED were less than those of ED2 in all 512760 cases.

We conclude this section by remarking that the computational time of our methods for these examples was negligibly small as expected.

5. Conclusion

In this paper, we have proposed the consistency interval estimation method. The proposed method is based on the eigenvalue method, but it calculates the upper bound of the principal eigenvalue instead of eigenvalues and eigenvectors, and hence quick estimation is possible. As a by-product of the method, we have also proposed a comparison error detection method when the CI value of an original comparison matrix is relatively large. Some numerical examples have shown that the proposed methods are practically efficient and useful.

In the proposed method, we have used a row geometric mean as an approximation to the principal eigenvector for computing upper bounds. This approximation is in general worse as the size of comparison matrix or the CI value is large, that is, the performance of the proposed methods, especially the procedure CIE, depend on the size and the CI value of the original comparison matrix. Therefore, to clarify the relationship between the size of a comparison matrix and $\overline{\text{CI}}$ in CIE yielding the target CI value is one of practically important subject. Finding an accurate vector whose computational burden is not so much is also a subject of future research.

References

[1] J. Aguarón, M.T. Escobar and J.M. Moreno-Jimenz: Consistency stability intervals for a judgment in AHP decision support systems. *European Journal of Operational Research*, **145** (2003), 382–393.

[2] P.T. Harker and L.G. Vargas: The Theory of Ratio Scale Estimation: Saaty's Analytic Hierarchy Process. *Management Science,* **33** (1987), 1383–1403.

[3] T. Ibaraki and M. Fukushima: *FORTRAN 77 Optimization Programming* (in Japanese, Iwanami-Shoten, Tokyo, 1991).

[4] S. Lipovetsky and W.M. Conklin: Robust estimation of priorities in the AHP. *European Journal of Operational Research*, **137** (2002), 110–122.

[5] T.L. Saaty: *The Analytic Hierarchy Process* (McGraw Hill, New York, 1980).

[6] K. Sekitani and N. Yamaki: A Logical Interpretation for Eigenvalue Method in AHP. *Journal of the Operations Research Society of Japan*, **42** (1999), 219–232.

Kouichi Taji
Department of Mechanical Science and Engineering
Graduate School of Engineering
Nagoya University
Furo-cho, Chikusa, Nagoya 464-8603, Japan
E-mail: taji@nuem.nagoya-u.ac.jp