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OPTIMAL TIMING FOR INVESTMENT DECISIONS

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Abstract The net present value (NVP) is an important concept in investment decisions. As Ingersoll and Ross [7] have pointed out, the future fluctuation of interest rates is expected to have significant effects on the present value (PV) of the project concerned. If interest rates are expected to fall off in the next year, deferring an investment for yet another year is likely to be more gainful even if its current NPV is positive. The effects of deferment can be valued from its corresponding American option value. Berk [1] proposed a simple criterion for investment decisions which incorporate this American option value of investment. The simplicity of this model is obtained from the appropriate usage of a callable bond. It is admirable that this model does not postulate any assumptions on the behavior of interest rates. But this construction of the model has the pros and cons. It is easy to implement this model in business because the only adjustment required in this model is to replace the interest rate in NPV with the callable rate. On the other hand, the properties of this criterion have not been clarified.

In this paper we analyze Berk's model under the assumption that interest rates follow the geometric Brownian motion (GBM). By assuming the movement of interest rates, we can derive an analytical solution for the optimal timing for the investments in terms of the parameters of the GBM. This enables us to perform comparative statics and simulation. These results extract some properties of Berk's model and help the decision makers in implementing Berk's model.

Keywords: Finance, decision making, NPV, geometric Browniam motion, American option, real option

1. Introduction

Investment decisions are one of the most important issues in business. The traditional textbooks say "carry out the project whose NPV is positive" or "invest in the project with the highest NPV among alternative projects". Most of the elementary literature treats the investment decision methods with a certain idealized situation in mind. For example, NPV is obtained by subtracting the cost of investment from the PV of future profit discounted by the risk free rate on the assumption that the interest rate is known and unchangeable throughout the whole period of the project.

But we may have flexibilities in investment – e.g. to defer the project. The deferred project is quite important as an alternative under uncertainty. Once the project is undertaken, the deferred project will cease to exist. This may lead to a decrease in profitability because the profitability of the deferred project may be greater than the one undertaken now. Thus the traditional NPV criterion cannot find the optimal timing for an investment. One should undertake a project now only if the project has a larger profitability than that of deferred projects.

Some papers have analyzed the flexibility of investment decisions based on real option analysis. They deal with several sources of uncertainty which should be considered – profitability of the project, cost of the project, construction period, possibility of suspension, and so on. One of the fundamental uncertainties is profitability or cost. Some papers assume that the uncertain factors which affect the value of a project follow the GBM.

McDonald and Siegel [10] assume that both the project's return and the investment cost are uncertain. The homogeneity of return and cost enables them to extract the project's value in terms of return to cost ratio. Carlton, Fisher, and Giammarino [5] discuss the amount of investment which determines the size of a firm. Firm size may alter the cost structure of a firm, for example investment increases the fixed cost. Thus uncertainty has an effect on the cost even if the only source of uncertainty is the project's value. As fixed cost increases, operating leverage gets higher. Higher operating leverage leads to higher risks for the firm. This in turn reduces the incentive to expand the firm, ceteris paribus. They consider the effects of operating leverage on a firm's value, its optimal size, and asset returns. Majd and Pindyck [11] derive the decision rule under the flexibility of the investment rate at which construction proceeds. In their model, the manager can choose the amount to invest within the maximum investment rate. They show the 'bang-bang' type of investment rule and give the project's value. The rule says to invest at a maximum rate if the value of the project is greater than the threshold; otherwise do not invest.

Uncertainty affects the project's value in different ways. Pindyck [13] considers the technical uncertainty and cost uncertainty when the project takes time to complete and is irreversible. The technical uncertainty is only resolved as the investment proceeds. One can control the amount of investment and one has an option to discontinue the project. Many R&D projects involve this type of uncertainty. He models the total amount of investment as a stochastic process that is controlled by the rate of investment.

Exchange rate has a large effect on profitability too, especially for multinational enterprises. Botteron, Chesney, and Gibson-Asner [4] and Milne and Whalley [12] evaluate the options to sell a part of its product abroad and to delocalize its production.

Another source of uncertainty is interest rate. The models mentioned above assume interest rate is known and constant over the whole period of the project. Some of them assume that the uncertainty of the project's return or the cost directly affect the project's value. On the contrary, the interest rate's uncertainty affects the project's value in different ways – i.e. through discounting. As Ingersoll and Ross [7] point out "even naïve investors who ignore the embedded option in the projects ...... may well be sensitive to the options inherent in possible changes in financing cost": even the smallest change of the interest rate could strongly affect the profitability of the project. Interest rates fluctuate randomly in the real world. Thus they may go down in the future, leading to an increase of the project's NPV. This also prompts one to postpone an investment because deferment of the investment may increase the NPV.

Berk [1] proposed investment decision model under the uncertainty of interest rates. As there are no assumptions on the future development of interest rates, this model provides a simplified rule which is easy to implement in business. On the other hand, the properties of the model concerning the future movement of interest rates are unclear. Practitioners would be keen to know the analytical properties of this model.

In this paper, we assume that interest rate follows the GBM. This simplifying assumption seems to be quite restrictive. But it may still be acceptable due to its tractability for practitioners. Especially in situations where they evaluate the option on interest rate using Black's model [3, **?**]. Moreover if we set $\mu = 0$ it coincides to the model in Dothan [6].

Our contributions on firms' decisions are summarized below. First, we can compare Berk's criterion with the traditional NPV in terms of the parameters of the GBM. Second, this analytical solution gives us some useful insight into the criterion of Berk's model especially on a firm's financing activity. Several economic implications on financing stem from this problem, because it is closely related to the ways of funding and cost of capital. Third, we can easily show numerical examples. Finally the solution enables us to show comparative statics.

This paper consists of the following. In section 2 we overview the model of Berk. Section 3 is devoted to developing the explicit American option value based on Berk's model. In section 4 comparative statics and numerical examples are shown, and we conclude our paper in section 5.

2. Optimal Investment Timing after Berk's Model

Let us consider a situation where the decision maker faces a project. For simplicity let us assume that the investment of an amount 1 in this project will yield a certain cash flow V forever. *

If the current consol rate is $r(t)$, the PV of the profit gained by the project is $\frac{V}{r(t)}$. According to the traditional NPV criterion, the requirement for undertaking the project is

$$\frac{V}{r(t)} \geq 1. \tag{2.1}$$

As stated earlier, we have an American option to defer the project. The tendency of the future interest rate's movement may prompt one to defer the project. The project whose NPV is positive should be undertaken right now, only if it has a larger profitability than that of a deferred project. Berk [1] proposed a condition for the optimal investment timing in terms of the *callable* consol bond. As the future movement of interest rates affect the value of the project, the option value of deferment can be evaluated by the value of the callable consol bond. Optimal timing of investment is given as follows:

Proposition 2.1 (Berk) *When the requirement*

$$\frac{V}{r_m(t)} \geq 1, \tag{2.2}$$

is met for the first time t^*, the optimal investment timing will be t^*, where $r_m(t)$ is the callable consol rate.

Since the callable consol rate is always higher than that of a consol bond, undertaking the project requires a higher profitability than in those cases taken by the traditional NPV criterion.

The intuition of this criterion is as follows. Let us assume the amount invested 1 is financed by a callable consol bond. If the consol rate decreases to a certain level, the investor can call the bond by issuing a non-callable consol bond. Hence we can hedge the future uncertainty of the interest rate movement by financing with the callable consol bond.

Berk gave an intuitive proof of equation (2.2). Let us consider the profitability of the project in two cases. One is an investment before t^*, the other is after t^*.

Case 1: Investment in the project prior to t^*, where $V < r_m(t)$ holds.

Let the timing of this investment be t such that $t < t^*$. Investment in a callable consol bond always yields a larger profit than in the project prior to t^* because $V < r_m(t)$. Hence it is suboptimal to invest prior to t^*.

Case 2: Investment in the project after t^*.

*The assumption can easily be relaxed. For an uncertain cash flow case, see Berk [2].

Let us compare two cash flows: one from the optimally invested project, the other is one from wait longer say to $t > t^*$. Suppose the optimally invested firm finance an amount 1 by issuing a callable consol bond. It is important to mention that if the firm optimally invested calls the consol bond at t by refinancing with a non-callable consol bond, then the two cash flows will be identical to each other after t. Besides, the cash flow from the optimally invested project from t^* to t is always nonnegative because the net cash flow is $V - r_m(t^*) \geq 0$. It is suboptimal to wait beyond t^* because one misses the nonnegative cash flow.

The conclusions of case 1 and 2 show that t^* is the optimal timing.

The formula (2.2) is identical to the traditional NPV criterion with the exception of the discount rate used for determining the PV of the future cash flow. According to the NPV criterion, a future cash flow is discounted with $r(t)$. On the other hand, Berk's model insists on discounting it with $r_m(t)$.

As Berk has pointed out, we can find a 30-year callable mortgage bond price which is traded in the United States. It can be used as a surrogate for a callable consol bond, thereby Berk's model can be easily implemented.

3. Determining Optimal Investment Timing after Modified Berk's Model

In this section, we derive the analytical solution for the optimal investment timing on the assumption that the interest rate on the consol bond with a face value "1" follows the GBM.

$$\frac{dr(t)}{r(t)} = \mu dt + \sigma dz, \tag{3.1}$$

where dz is a standard Brownian motion, and μ and σ are drift and volatility, respectively. As $r(t)$ satisfies the following equation

$$\int_t^\infty e^{-r(t)(s-t)} ds = \frac{1}{r(t)}, \tag{3.2}$$

the NPV of the project will be $\frac{V}{r(t)}$.

Let us consider the value of the callable consol bond issued at t. The issuer of a callable consol bond can call it when the exercise is profitable. If the issuer calls the bond at $s \geq t$, its face value "1" is paid in return for the future payment obligation $r_m(t)$. This is considered to be a perpetual American option. The PV of this obligation at s is $\frac{r_m(t)}{r(s)}$. Hence the option value of this callable consol bond, appraised with the risk neutral probability is given by

$$\max_{s \geq t} E\left[e^{-r(t)(s-t)} \max\left(\frac{r_m(t)}{r(s)} - 1, 0\right)\right]. \tag{3.3}$$

Let us value the risk neutral expectation.

Proposition 3.1 *The optimal timing to exercise the perpetual American call option issued at t is the first time s^* such that the following formula equalizes.*

$$\frac{1}{r(s)} \geq \frac{1}{r_m(t)} \frac{\rho}{\rho - 1}, \tag{3.4}$$

where $\rho = \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r(t)}{\sigma^2}} - \frac{\mu}{\sigma^2}$.

proof
We can evaluate (3.3) as follows.

$$\begin{aligned}\max_{s\geq t} E\left[e^{-r(t)(s-t)}\max\left(\frac{r_m(t)}{r(s)}-1,0\right)\right] &= \max_{\frac{r_m(t)}{r(s)}-1\geq 0}\left(\frac{r_m(t)}{r(s)}-1\right)E\left[e^{-r(t)(s-t)}\right] \\ &= \max_{\frac{r_m(t)}{r(s)}-1\geq 0}\left(\frac{r_m(t)}{r(s)}-1\right)\left(\frac{r(t)}{r(s)}\right)^{\rho}. \qquad (3.5)\end{aligned}$$

The second equation is derived from Karlin and Taylor [9]. The first-order condition is

$$\frac{1}{r(s)}\left(\frac{r(s)}{r(t)}\right)^{\rho}\left(\frac{r_m(t)}{r(s)}-\rho\left(\frac{r_m(t)}{r(s)}-1\right)\right)=0.$$

If $\rho > 1$, (3.3) has its maximum value at the first time s^* such that

$$\frac{1}{r(s^*)}=\frac{1}{r_m(t)}\frac{\rho}{\rho-1}. \qquad (3.6)$$

Otherwise it is on the monotonic increase. ∎

In the rest of this paper, we assume $\rho > 1$. As Berk has explained, the value of the project can be broken down into the following two components.

Component A: Cash flow from the project.

It is identical to a long position in a non-callable consol bond.

Component B: Callable consol bond.

A callable consol bond consists of a long position in a non-callable consol bond and a short position in an American call option.

The project consists of a long position on component A and finance, which is a short position on component B. By subtracting component B from component A, we obtain a long position in an American call option. Thus investment should be undertaken at the time when this American call option value is maximized.

Let us consider this American call option value.

Corollary 3.1 *The value of this American option is*

$$\max_{s\geq t} E\left[e^{-r(t)(s-t)}\max\left(\frac{r_m(t)}{r(s)}-1,0\right)\right]=\frac{1}{\rho-1}. \qquad (3.7)$$

proof
At optimal timing we have (3.6). Substituting it into American option value (3.5), we have (3.7). ∎

Corollary 3.2 *The optimal time for investment is the first time t^* that the following condition is met.*

$$NPV=\frac{V}{r(t)}-1\geq\frac{1}{\rho-1}, \qquad (3.8)$$

or equivalently

$$\frac{V}{r(t)}\geq\frac{\rho}{\rho-1}. \qquad (3.9)$$

proof
The investment should be undertaken when the project's NPV is greater than the value of this American call option which is a continuation value. Note that immediate exercise of the callable bond is also optimal at that time. We obtain the following relationship.

$$\begin{aligned} \frac{V}{r(t)} - 1 &\geq \frac{1}{\rho - 1} \\ &= \frac{\rho}{\rho - 1} - 1. \end{aligned} \tag{3.10}$$

By comparing the LHS and the last equality we have (3.9). ∎

This inequality clarifies Berk's criterion. Because the LHS of (3.8) is NPV of the project it insists that NPV must be greater than $\frac{1}{\rho-1}$. From the assumption of $\rho > 1$, we have $\frac{1}{\rho-1} > 0$. Equivalently the PV of cash flow from the project should be greater than $\frac{\rho}{\rho-1} > 1$. This means that to undertake the project, NPV should be larger than the threshold value $\frac{1}{\rho-1} > 0$. This also explains why the traditional NPV criterion cannot find the optimal timing. In other words, traditional NPV is a looser condition. If one uses the traditional NPV criterion, a less profitable project is likely to be undertaken. Once the project is undertaken, the deferred project will cease to exist. This will lead to a decrease in profitability. Finally this value $\frac{1}{\rho-1}$ can be applicable as a hurdle rate.

Proposition 3.2 *The optimal time to undertake a project is the first time t^* that the following condition is satisfied.*

$$\frac{V}{r_m(t)} \geq 1. \tag{3.11}$$

proof
Let us evaluate (3.10) at optimal exercise time s^* of a callable consol bond.

$$\begin{aligned} \frac{V}{r(t)} - 1 &\geq \frac{\rho}{\rho - 1} - 1 \\ &= \frac{r_m(t)}{r(s^*)} - 1. \end{aligned}$$

The second line is given by (3.6). Note that $r(s^*) = r(t)$ holds at the optimal time for investment. Connecting the two equations, we obtain the optimal condition (3.11). ∎

Proposition 3.2 gives the same criterion as Berk's model assuming that $r(t)$ follows the GBM. Moreover condition (3.9) relates the optimal timing to the parameters of the interest rate process. The number of the parameters contained in (3.9) is small and the parameters are easy to estimate from the movement of the interest rate. In the situation where the decision maker assumes the process of the interest rate as the GBM, the condition (3.9) or (3.11) would help practitioners because they are still simple enough to be implemented in business. Additionally equation (3.9) enables us to perform comparative statics for the interest rate process. We can extract some useful findings from comparative statics.

4. Comparative Statics and Numerical Examples

In this section we will perform comparative statics and provide some numerical examples for equation (3.9).

By differentiating the RHS of the equation (3.9) with σ, we have

$$\frac{2\mu\left(\frac{r(t)}{\mu} - \rho\right)}{\sigma^3(1-\rho)^2\sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r(t)}{\sigma^2}}}. \tag{4.1}$$

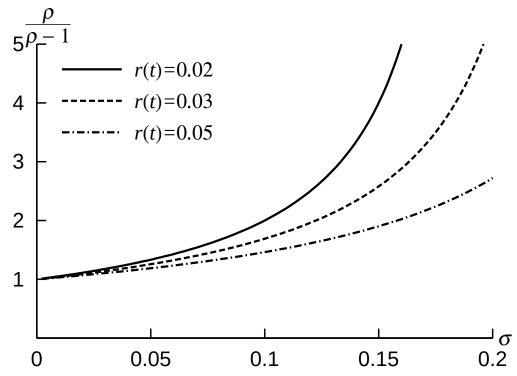

Figure 1: Threshold for PV ($\mu = 0$)

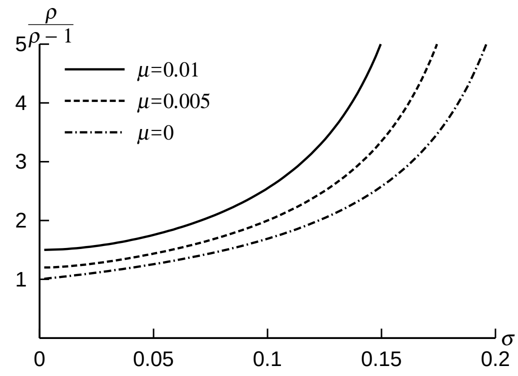

Figure 2: Threshold for PV ($r(t) = 0.03$)

The sign of (4.1) is determined by $\frac{r(t)}{\mu} - \rho$. If $\frac{r(t)}{\mu} - \rho > 0$, the increase of σ will bring a rise in the threshold of PV. Note that we have assumed $\rho > 1$. By combining two inequalities we obtain an alternative condition $r(t) > \mu$ for the numerator of (4.1) is positive. In other words, if $r(t) > \mu$, the increase of volatility leads to a rise in the threshold and vice versa.

By differentiating the RHS of the equation (3.9) with μ and $r(t)$, we have the following values, respectively.

$$\frac{\rho}{\sigma^2(1-\rho)^2\sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r(t)}{\sigma^2}}} > 0, \tag{4.2}$$

and

$$\frac{-1}{\sigma^2(1-\rho)^2\sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r(t)}{\sigma^2}}} < 0. \tag{4.3}$$

The former equation suggests that if the drift of interest rates increases, the threshold will rise. In other words, when the interest rate tends to increase, a project must have a higher profitability to be undertaken at the current time. Equation (4.3) means that if the current interest rate is considered to be high, the threshold is prone to be lowered. In that case, note that the NPV of the project will be lowered also. As the interest rate declines, the threshold rises but simultaneously the PV of the project rises. Later we will examine this effect in more detail.

Let us show the effect of the volatility of interest rates on the threshold of the PV of a project by using numerical examples. First, we set the drift of interest rates to zero. This means that the movement of interest rates has no tendency. It makes (4.1) nonnegative. Figure 1 shows the effect of volatility on the threshold of PV for some values of $r(t)$. The increase in the volatility of interest rates will cause the threshold of PV to rise. It is caused by an increase in the American option value. This is why the project should have a large enough NPV to be undertaken, as is shown in (3.8). The effect in question increases as the consol rate becomes lower.

If we are faced with an uncertainty of interest rates, deferment of investment is likely to be profitable. This situation is similar to the circumstances Japan faced in the late 1980's, and our results suggest the investment policy that should have been adopted. At that time Japanese firms should have postponed investments. Unfortunately, many rushed headlong into investments. If they had deferred investments until the uncertainty of interest rates had lessened, the depression of Japanese economy might have been milder. This is one of the reasons why Japan experienced a lengthy depression after the *Bubble* of the 1980's. Waiting would have been a better alternative.

Table 1: Threshold of PV $\frac{\rho}{\rho-1}$ $(\mu = 0)$

$\sigma \backslash r$	0.025	0.03	0.035	0.04	0.045	0.05
0.01	1.047	1.043	1.039	1.037	1.034	1.033
0.05	1.288	1.256	1.233	1.215	1.2	1.188
0.1	1.809	1.69	1.608	1.547	1.5	1.462
0.15	3.038	2.58	2.309	2.129	2	1.902
0.2	9.472	5.449	4.097	3.414	3	2.721

Table 2: Minimum value V^* $(\mu = 0)$

$\sigma \backslash r$	0.025	0.03	0.035	0.04	0.045	0.05
0.01	0.026	0.031	0.036	0.041	0.047	0.052
0.05	0.032	0.038	0.043	0.049	0.054	0.059
0.1	0.045	0.051	0.056	0.062	0.068	0.073
0.15	0.076	0.077	0.081	0.085	0.09	0.095
0.2	0.237	0.163	0.143	0.137	0.135	0.136

Table 3: Increment $V^* - V^0 (\mu = 0)$

$\sigma \backslash r$	0.025	0.03	0.04	0.05
0.01	0.001	0.001	0.001	0.002
0.05	0.007	0.008	0.009	0.009
0.1	0.02	0.021	0.022	0.023
0.15	0.051	0.047	0.045	0.045
0.2	0.212	0.133	0.097	0.086

Figure 2 shows the effect of volatility on the threshold of PV with the interest rate $r(t)$ fixed to 0.03. Drift μ in large value implies the tendency of interest rates to increase. The greater the μ is, the more profitable wait is. Additionally it is important to note that it is the movement of the interest rate's drift rather than that of interest rates that has larger effect on the threshold of PV.

Table 1 shows the threshold of PV for some volatility values and interest rates when μ is set to zero. The larger the volatility is or the smaller the interest rate is, the larger the threshold is. As mentioned earlier when interest rates decrease, then threshold rises. But decrease of the interest rate increases the PV of projects also. Let us consider which effect is dominating. If interest rate $r(t)$ changes from 0.05 to 0.025, then the PV of perpetual cash flow 0.1 doubles from 2 to 4. When volatility is small, say 0.01, the threshold changes from 1.033 to 1.047. The effect of discount dominates the change in option value. On the other hand, if we set volatility relatively high, say 0.2, the threshold is almost tripled. The effect from option value is larger than that of discounting because a larger volatility increases the option value. The dominating effect changes depending on the level of the interest rate's volatility.

Table 2 shows the minimum value of V^* with which the investment should be made. We can find the trade off between discounting and the option value in this table again. The Nikkei 225 recorded its historical high 38,915.87 yen on 29th Dec. 1989. The official bank rate in early Dec. 1989 was 3.75% and it was raised several times and finally reached 6% by the end of August 1990. Some projects with relatively small V would have lost their profitability quickly ex post.

Finally let assume that V^* is obtained from optimal timing and V^0 from NPV rule. The increment of the value $V^* - V^0$ is shown in table 3. As volatility or interest rates increase, the increment becomes greater. Greater uncertainty tends to require larger profitability if investment is to take place.

5. Conclusions

We have considered a traditional problem in investment decisions. As has been pointed out in many papers, NPV method does not lead to optimal timing for investment because NPV contains only current information about an investment. In a dynamic setting where the interest rate fluctuates, the future movement of interest rates must be taken into account. The project needs to have a higher value than that of a deferred project if the project is to undertaken. We have shown an analytical solution for this problem within Berk's framework on the assumption that interest rates follow the GMB. We also have insights into financing on Berk's model. Additionally we have been able to determine the optimal investment timing using the parameters of the interest rate process.

In addition to the analytical solution, comparative statics suggest some insights into the decisions under uncertainty of interest rates. Further, numerical examples have shown the effects of the movement of interest rates on the threshold.

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