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ESTIMATION METHODS BY STOCHASTIC MODEL IN BINARY AND TERNARY AHP

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Abstract New stochastic models for binary and ternary AHP are proposed, and further minimax and least square estimation methods (with parameter θ) for these models are proposed. The solutions of both methods are proved to be mathematically equivalent although the principles are different. Another method based on the well-known likelihood function is applied to our model with the parameter, which can expand the application limit of the conventional likelihood method. Various examples are solved by these proposed methods and we have successful results for all.

Keywords: AHP, binary AHP, ternary AHP, Bradley-Terry model

1. Introduction

The essence of AHP is to evaluate the weight of the object, which cannot be measured by numerical values, based on paired comparisons. Conventionally the paired comparison value a_{ij} of object i and j is to take the integers $1, 2, \cdots, 9$ (and their inverses)[10].

But the comparisons are often based on human intuitive feelings, so in such cases we can say at the utmost that

$$\text{“}i \text{ is better than } j\text{” (or equivalently “}j \text{ is worse than } i\text{”)} \tag{1.1}$$

or

$$\text{“}i \text{ is equivalent to } j\text{”.} \tag{1.2}$$

The case where the result of paired comparison is restricted to only (1.1) is called binary AHP, and the case which includes only (1.1) and (1.2) is called ternary AHP.

Considering the above-mentioned essence of AHP, we can say that the intrinsic feature of AHP is rather in binary and ternary AHP than general AHP where a_{ij} can take values of real numbers.

We can see typical binary and ternary AHP in sport games (including intellectual games such as “chess” or “go,” etc.) where “player i defeats j” corresponds to (1.1) and “player i ties j” corresponds to (1.2). Later on we often use such clear and concrete terms in sport games instead of (1.1) and (1.2).

Here we propose a stochastic model for binary and ternary AHP.

Firstly we consider binary AHP. Let $u_i(\geqq 0)$ be the true strength of object i ($i = 1, \cdots, n$), and p_{ij} be the probability for i to defeat j, then we assume

$$p_{ij} = u_i/(u_i + u_j). \tag{1.3}$$

This model is called Bradley-Terry model [1]. Of course, we have $p_{ij} + p_{ji} = 1$.

For ternary AHP, p_{ij} represents the probability for i to defeat j plus $1/2 \times$ probability for i to tie j.

Past researches in this field are classified mainly into two groups. Researches [4, 8, 14] can be said to constitute one group, where they take the compared value a_{ij} as a if player i defeats j (and a_{ji} as $1/a$) ($a > 1$ is a parameter) and take the principal eigenvector of $\boldsymbol{A} = [a_{ij}]$ as the weight vector like the usual AHP analysis. Their models do not include stochastic interpretations, and correspond to the special case ($r_{ij} = 1$ for all pairs (i, j), that is discussed in §2) of our model.

Researches [2, 3, 6, 7, 9] belong to the other group. These are papers in statistical field and are based on Bradley-Terry model. Their analyzing methods are based on maximum likelihood (ML) method. We can see basic explanation for these analyses in [16]. Here we note that ML method on Bradley-Terry model, used in the latter group, has severe limitations and is not very useful except for large sample case. Actually some examples in this paper cannot be solved by their method. (But [9] transfer the comparison value a_{ij} in AHP to Bradley-Terry model by special device, the above discussions do not fit for [9]).

Further, as for researches by combinational approaches, we have [5, 15] which do not belong to any of the above groups.

Our proposed method is based on Bradley-Terry model with our special device (see (2.6)). Our analyzing principle is minimax (MM) method (§3), but interestingly the solution of this minimax principle completely coincides with that of least square principle (§3) in our problem. Another proposed method is based on semi-maximum likelihood principle which modifies ordinary ML method by special device (2.6) and becomes free from the above-mentioned limitations (§4). Further, we discuss ternary AHP which we can treat almost the same as binary cases (§5). We introduce likelihood functions for our model to select the values of parameter θ in (2.6)(§6). Various examples are solved by our methods in §7.

2. The Approximation Formula with Parameter

There are n players $1, 2, \cdots, n$ and player i is matched with j by r_{ij} times (n objects $1, 2, \cdots, n$ and object i is compared with j by r_{ij} times).

As mentioned in §1, here we express a paired comparison in AHP by a match in a sport game. The former is compared by an evaluator, but in the latter case there are no such evaluators. Now r_{ij} comparisons between object i and j in AHP correspond to r_{ij} matches between player i and j, where each match of r_{ij} matches can be considered to be independently carried out. But the structure of r_{ij} comparisons in AHP is generally not so simple. There may be a case where these are compared by an evaluator and another case where these are compared by several separate evaluators. But here we simply assume that each comparison is always independently carried out under the same condition like in sport games. This assumption might not be valid, but if the simple model grasps the essential feature, it is more useful for theoretical development. Theory needs simplification.

First, consider the binary AHP. Let x_{ij} be the number of times for i to defeat j (for i to be better than j), and $\dot{x}_{ij} = x_{ij}/r_{ij}$, then we have

$$x_{ij} + x_{ji} = r_{ij}, \quad \dot{x}_{ij} + \dot{x}_{ji} = 1. \tag{2.1}$$

Here of course $r_{ij} = r_{ji}$ and for pair (i, j) with $r_{ij} = 0$ we have no data and $\dot{x}_{ij} = 0$, so (i, j) element of basic matrix $\boldsymbol{X}$ (3.6) is 0.

The given data are $x_{ij} \geqq 0$, $r_{ij} \geqq 0$ $(i, j = 1, \cdots, n)$ by which we want to estimate u_i (strength of player i or goodness of object i) through the model (1.3).

For rather large values of r_{ij} the value of $\dot{x}_{ij}$ must be near to p_{ij}, so we have such reliable approximation formula as $\dot{x}_{ij} \fallingdotseq u_i/(u_i+u_j)$. However for the smaller values of r_{ij}, say for $r_{ij}=1$, if i defeats j the above approximation formula gives $1 \fallingdotseq u_i/(u_i+u_j)$ or $0 \fallingdotseq u_j/(u_i+u_j)$, which is too extreme in judgment. To say that the probability for i to defeat j is 1 based on only one game is too excessive in judgment. Furthermore for the larger value of r_{ij}, if $x_{ij}=r_{ij}$ then $\dot{x}_{ij}=1$, so it brings the same formula $1 \fallingdotseq u_i/(u_i+u_j)$ as the case of $r_{ij}=1$. Instead of such extreme ones we had better have milder formula such as

$$1-\theta \fallingdotseq u_i/(u_i+u_j), \quad \theta \fallingdotseq u_j/(u_i+u_j), \tag{2.2}$$

where θ is a relaxation parameter with $0<\theta<1/2$.

Generally we propose the following approximation formula; if $\dot{x}_{ij}>1/2$ then

$$\dot{x}_{ij}(1-\theta^{r_{ij}}) \fallingdotseq u_i/(u_i+u_j), \tag{2.3}$$

$$1-\dot{x}_{ij}(1-\theta^{r_{ij}}) \fallingdotseq u_j/(u_i+u_j) \tag{2.4}$$

and if $\dot{x}_{ij}=1/2$ then

$$1/2 \fallingdotseq u_i/(u_i+u_j)=u_j/(u_i+u_j). \tag{2.5}$$

Clearly for larger values of r_{ij} (2.3) and (2.4) are near to the above-mentioned usual ones.

Next, denote the left-hand sides of (2.3) and (2.4) by $\dot{x}_{ij}(\theta)$ and $\dot{x}_{ji}(\theta)$, respectively, that is

$$\begin{aligned}
\dot{x}_{ij}(\theta) &= \dot{x}_{ij}(1-\theta^{r_{ij}}) \quad \text{for } \dot{x}_{ij}>1/2,\\
\dot{x}_{ji}(\theta) &= 1-\dot{x}_{ij}(\theta) \quad \text{for } \dot{x}_{ji}<1/2,\\
\dot{x}_{ij}(\theta) &= \dot{x}_{ji}(\theta)=1/2 \quad \text{for } \dot{x}_{ij}=1/2,\\
\dot{x}_{ij}(\theta) &= \dot{x}_{ji}(\theta)=0 \quad \text{for } \dot{x}_{ij}=0.
\end{aligned} \tag{2.6}$$

Here $\dot{x}_{ii}$ has no actual meaning, but we can see $p_{ii}=1/2$, so we can assume $\dot{x}_{ii}=1/2$. Adapting the symbol to (2.6) we denote it by

$$\dot{x}_{ii}(\theta)=1/2. \tag{2.7}$$

Here we note that if $\dot{x}_{ij}>1/2$ then also $\dot{x}_{ij}(\theta)>1/2$ (the proof is in appendix) which shows that $\dot{x}_{ij}(\theta)$ is an appropriate approximation formula.

We propose $\dot{x}_{ij}(\theta)$ instead of ordinary $\dot{x}_{ij}$ as approximation formula for $u_i/(u_i+u_j)$, that is

$$\dot{x}_{ij}(\theta) \fallingdotseq u_i/(u_i+u_j) \quad (0<\theta<1/2) \tag{2.8}$$

(later we state how to select the value of θ on actual implementation of our method), and given the data $\{\dot{x}_{ij}(\theta)\}$ we estimate $u_1,\cdots,u_n$ based on (2.8).

As the estimation methods we propose minimax (MM) method and maximum likelihood (ML) method (another important one, least square (LS) method, completely coincides with MM) (see §3 for details).

For both methods the property of the matrix $[\ \dot{x}_{ij}(\theta)\ ]$ is fundamentally important, that is, $[\ \dot{x}_{ij}(\theta)\ ]$ must be irreducible in order for these methods to have solutions. (The basic matrix $\boldsymbol{X}$ in §3 is different from $[\ \dot{x}_{ij}(\theta)\ ]$, but the irreducibility of both matrix is equivalent.)

The above logic is valid for the analysis on $[\ \dot{x}_{ij}\ ]$. For example, if at least one player wins a complete victory or is totally defeated then $[\ \dot{x}_{ij}\ ]$ is reducible, so we cannot have solutions [16]. Generally for problems with rather smaller r_{ij} the matrix $[\ \dot{x}_{ij}\ ]$ becomes often reducible (see Example 1). But for even such cases the matrix [$\dot{x}_{ij}(\theta)$] is always irreducible except the case where the graph (see, for example, Figure 1 or Figure 2) for [$\dot{x}_{ij}(\theta)$] is disconnected, when the problem itself is originally meaningless. Because if $\dot{x}_{ij}(\theta) > 0$, then always $\dot{x}_{ji}(\theta) > 0$ by definition of $\dot{x}_{ij}(\theta)$ (see (2.6)).

Through the above discussions we can say that by using $\dot{x}_{ij}(\theta)$ instead of $\dot{x}_{ij}$ we can expand the application scope of MM or ML method.

Further, for the special case of all $r_{ij} = 1$, [4, 8, 14] use parameter $a(> 1)$ such that

$$a \fallingdotseq u_i/u_j \qquad \text{if and only if } i \text{ defeats } j. \tag{2.9}$$

This is equivalent to our model, and the relation of both parameters is

$$\theta = \frac{1}{1+a} \text{ or } a = \frac{1}{\theta} - 1. \tag{2.10}$$

Remark In the above discussion we said that if [$\dot{x}_{ij}$] is reducible the problem cannot be solved by MM method. But exactly speaking such problems can be solved by generalized method [11, 12], whose basic idea is shown in Theorem 2 of [12]. Of course, this generalized method is rather more complex than our direct method.

3. Minimax and Least Square Method for Binary AHP

Looking at (2.8) we have for each i

$$\dot{x}_{ij}(\theta)(u_i + u_j) \fallingdotseq u_i, \text{ for } j = 1, \cdots, n. \tag{3.1}$$

Averaging (3.1) with weight

$$\dot{r}_{ij} = r_{ij}/r_i \quad (r_i = \sum_{j=1}^{n} r_{ij}). \tag{3.2}$$

(here we assume formally $r_{ii} = 1$) we have

$$\sum_{j=1}^{n} \dot{r}_{ij}\dot{x}_{ij}(\theta)(u_i + u_j) \fallingdotseq u_i, \quad i = 1, \cdots, n. \tag{3.3}$$

We denote the left-hand side by $\hat{u}_i$, that is

$$\hat{u}_i = \sum_{j=1}^{n} \dot{r}_{ij}\dot{x}_{ij}(\theta)(u_i + u_j), \quad i = 1, \cdots, n. \tag{3.4}$$

We want to decide the value of $u_i > 0$ $(i = 1, \cdots, n)$ making $\hat{u}_i$ nearest $u_i(i = 1, \cdots, n)$. One of principles for this is to decide $u_i(i = 1, \cdots, n)$ such that

$$\min_{u_1, \cdots, u_n} \max_i \{\hat{u}_i/u_i\}. \tag{3.5}$$

The solution of this minimax principle is the principal eigenvector of the following matrix $\boldsymbol{X}$, if this matrix is irreducible [11, 13].

$$\boldsymbol{X} = \begin{bmatrix} \dot{\delta}_1 & \dot{r}_{12}\dot{x}_{12}(\theta) & \cdots & \dot{r}_{1n}\dot{x}_{1n}(\theta) \\ \dot{r}_{21}\dot{x}_{21}(\theta) & \dot{\delta}_2 & \cdots & \dot{r}_{2n}\dot{x}_{2n}(\theta) \\ \vdots & \vdots & \vdots & \vdots \\ \dot{r}_{n1}\dot{x}_{n1}(\theta) & \dot{r}_{n2}\dot{x}_{n2}(\theta) & \cdots & \dot{\delta}_n \end{bmatrix} \tag{3.6}$$

$$\dot{\delta}_i = \dot{r}_{ii} + \sum_{j \neq i} \dot{r}_{ij} \dot{x}_{ij}(\theta) \quad (\dot{r}_{ii} = 1/r_i).$$

Let us call this method as minimax (MM) method.

Here we note the intrinsic feature of matrix $\boldsymbol{X}$,

Theorem 1. *Let $\boldsymbol{R}$ be a diagonal matrix whose i-th diagonal element is $r_i (i = 1, \cdots, n)$ ((3.2)), then $\boldsymbol{RXR}^{-1}$ is a column stochastic matrix, that is, the sum of elements of each column is 1. So $\boldsymbol{RXR}^{-1}$ has the maximum eigenvalue equal to 1. Consequently $\boldsymbol{X}$ also has the maximum eigenvalue equal to 1.*

Proof: We have

$$\boldsymbol{RXR}^{-1} = \begin{bmatrix} \delta_1/r_1 & r_{12}\dot{x}_{12}(\theta)/r_2 & \cdots & r_{1n}\dot{x}_{1n}(\theta)/r_n \\ r_{21}\dot{x}_{21}(\theta)/r_1 & \delta_2/r_2 & \cdots & r_{2n}\dot{x}_{2n}(\theta)/r_n \\ \vdots & \vdots & \vdots & \vdots \\ r_{n1}\dot{x}_{n1}(\theta)/r_1 & r_{n2}\dot{x}_{n2}(\theta)/r_2 & \cdots & \delta_n/r_n \end{bmatrix}$$

$$(\delta_i = r_i \dot{\delta}_i, \quad i = 1, \cdots, n).$$

Let s_i be the sum of elements of i-th column of $\boldsymbol{RXR}^{-1}$, then

$$\begin{aligned} s_i &= (\dot{\delta}_i + \sum_{j \neq i} r_{ji} \dot{x}_{ji}(\theta))/r_i \\ &= (r_{ii} + \sum_{j \neq i} r_{ij} \dot{x}_{ij}(\theta) + \sum_{j \neq i} r_{ji} \dot{x}_{ji}(\theta))/r_i. \end{aligned}$$

Here $r_{ij} = r_{ji}$ and $\dot{x}_{ij}(\theta) + \dot{x}_{ji}(\theta) = 1$ for all $(i, j) \in \{(i, j) | r_{ij} > 0\}$, so

$$s_i = (r_{ii} + \sum_{j \neq i} r_{ij})/r_i = \sum_j \frac{r_{ij}}{r_i} = 1.$$

$\boldsymbol{X}$ has the same eigenvalues as $\boldsymbol{RXR}^{-1}$, so $\boldsymbol{X}$ has the maximum eigenvalue equal to 1.□

$\boldsymbol{X}$ itself is not necessarily column stochastic, but we have

Corollary *If the total number of matches of every player is same ($r_1 = \cdots = r_n$), then matrix $\boldsymbol{X}$ in (3.6) is column stochastic matrix.*

Proof: The sum of the i-th column of $\boldsymbol{X}$ is

$$s_i = \dot{\delta}_i + \sum_{j \neq i} \dot{r}_{ji} \dot{x}_{ji}(\theta) = \dot{r}_{ii} + \sum_{j \neq i} \dot{r}_{ij} \dot{x}_{ij}(\theta) + \sum_{j \neq i} \dot{r}_{ji} \dot{x}_{ji}(\theta).$$

Here $\dot{r}_{ij} = r_{ij}/r_i = r_{ji}/r_j = \dot{r}_{ji}$ and $\dot{x}_{ij}(\theta) + \dot{x}_{ji}(\theta) = 1$ for all $(i, j) \in \{(i, j) | r_{ij} > 0\}$, so

$$s_i = \dot{r}_{ii} + \sum_{j \neq i} \dot{r}_{ij} = \sum_{j=1}^{n} \dot{r}_{ij} = \sum_j \frac{r_{ij}}{r_i} = 1.□$$

For actual implementation of our method we propose three stages of relaxation; weak relaxation ($\theta = 1/8$), middle ($\theta = 2/8$) and strong ($\theta = 3/8$), and for each stage calculate the weights of objects (eigenvector of $\boldsymbol{X}$). The problem which stage is taken should be left to the decision maker. If you want to decide the unique solution at any cost, you had better select one of three solutions with the maximum likelihood. This problem will be discussed in §6.

Example 1 $(n = 3,\ r_{ij} = 1,\ r_i = 3\ (i, j = 1, 2, 3))$
Let the given data be $x_{12} = 1$, $x_{13} = 1$, $x_{23} = 1$ which are graphically shown in Figure 1, where $x_{ij} = 1$ (player i defeats j) is shown by the arrow (i, j).

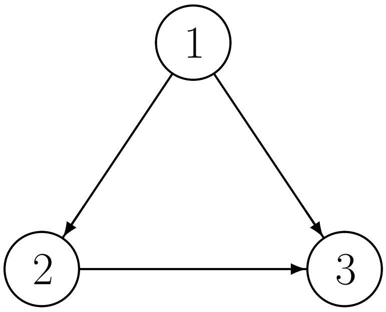

Figure 1: Simple example

Formula (3.1) for these data are;

$$\begin{array}{rrr} \frac{1}{2}(u_1+u_1)=u_1, & \theta(u_1+u_2)=u_2, & \theta(u_1+u_3)=u_3, \\ (1-\theta)(u_1+u_2)=u_1, & \frac{1}{2}(u_2+u_2)=u_2, & \theta(u_2+u_3)=u_3, \\ (1-\theta)(u_1+u_3)=u_1, & (1-\theta)(u_2+u_3)=u_2, & \frac{1}{2}(u_3+u_3)=u_3 \end{array}$$

and matrix $\boldsymbol{X}$ is;

$$\boldsymbol{X} = \frac{1}{3}\begin{bmatrix} 3-2\theta & 1-\theta & 1-\theta \\ \theta & 2 & 1-\theta \\ \theta & \theta & 1+2\theta \end{bmatrix}.$$

The principal eigenvector of the matrix $\boldsymbol{X}$, from our method, is shown in Table 1.

Table 1: The result of Example 1

	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.7777777	0.5999999	0.4545453
u_2	0.1555556	0.2571430	0.3146855
u_3	0.0666667	0.1428571	0.2307692
L	0.5372807	0.3634615	0.2261131
$L(\theta)$	0.2907131	0.1773511	0.1359961

(The meaning of L, $L(\theta)$, see §4)

Note that the ranking of u_1, u_2, u_3 is independent of choice of $\theta \in \{1/8, 2/8, 3/8\}$. The values of u_i are standardized with $\sum_i u_i = 1$.

Here we note that parameter a used in [4, 8, 14] (mentioned in §1 as the first group) corresponds to $(1-\theta)/\theta$ ((2.10)). So they can solve Example 1 by ordinary eigenvector method. For example, their solution for the case $\theta = 2/8 (a = 3)$ is $(u_1 = 0.585,\ u_2 = 0.280,\ u_3 = 0.135)$, which corresponds to the second column in Table 1.

Considering such a simple problem as Example 1, we must say that our method is almost equivalent to the conventional method by the first group. But they consider only the case where $r_{ij} = 1$ for all pairs (i, j). On the contrary, we can treat the cases where r_{ij} takes any integer values ($\geqq 0$).□

The request to make $\hat{u}_i$ (see (3.4)) close to $u_i (i = 1, \cdots, n)$ is accomplished by another principle, least square (LS), where the sum S of squares of errors $e_i = \hat{u}_i - u_i (i = 1, \cdots, n)$ is minimized under the condition $u_1, \cdots, u_n \geqq 0$. To avoid trivial solution $u_i = 0 (i = 1, \cdots, n)$ we set a restriction

$$u_1 + \cdots + u_n = 1. \tag{3.7}$$

That is, our LS solution is $\boldsymbol{u}$ ($\boldsymbol{u}^T = (u_1, \cdots, u_n)$) minimizing

$$S = \sum_i (\hat{u}_i - u_i)^2 \tag{3.8}$$

under the condition (3.7).

Now from (3.4) we have

$$\begin{aligned} \hat{u}_i &= \dot{r}_{ii}\dot{x}_{ii}(\theta)(u_i + u_i) + \sum_{j \neq i} \dot{r}_{ij}\dot{x}_{ij}(\theta)(u_i + u_j) \\ &= \dot{r}_{ii}u_i + \sum_{j \neq i} \dot{r}_{ij}\dot{x}_{ij}(\theta)u_i + \sum_{j \neq i} \dot{r}_{ij}\dot{x}_{ij}(\theta)u_j \text{ (see (2.7))} \\ &= \delta_i u_i + \sum_{j \neq i} \dot{r}_{ij}\dot{x}_{ij}(\theta)u_j \text{ (see (3.6))} \\ &= \sum_j X_{ij}u_j, \end{aligned} \tag{3.9}$$

where X_{ij} is (i, j) element of $\boldsymbol{X}$.

Therefore from Theorem 1 the principal eigenvector $\boldsymbol{u}$ (standardized by (3.7)) of $\boldsymbol{X}$ makes the value of S zero, which is the minimum value of $S (\geqq 0)$. So the standardized principal eigenvector of $\boldsymbol{X}$ is a solution of LS problem (3.7) and (3.8).

Conversely if $S = 0$, each term of the right-hand side of (3.8) is zero, so from (3.9) we have $\sum_j X_{ij}u_j = u_i$ for $i = 1, \cdots, n$, which means LS solution is standardized principal eigenvector of $\boldsymbol{X}$ which is unique [11, 12].

Summarizing the above we have

Theorem 2. *Under the irreducibility of* $\boldsymbol{X}$ *(3.6), the solution of LS ((3.7) and (3.8)) coincides with the minimax (MM) solution, that is the standardized principal eigenvector of* $\boldsymbol{X}$ *which is unique.*□

Above we proved Theorem 2, but this is a special case of Theorem 7.5 in [11]. However Theorem 2 is proved by Theorem 1 which is our original result.

In any case Theorem 2 shows the superior property of MM method. MM itself minimizes the maximum discrepancy of $\hat{\boldsymbol{u}}$ and $\boldsymbol{u}$, and at the same time it minimizes the total discrepancies, that is, MM is based on these two important principles.

4. Maximum Likelihood (ML) Method for Binary AHP

ML method applied to Bradley-Terry model is well known in the statistical field [16], which is of course to find positive $u_1, \cdots, u_n$ maximizing

$$L = \prod_{i<j} \binom{r_{ij}}{x_{ij}} (\frac{u_i}{u_i + u_j})^{x_{ij}} (\frac{u_j}{u_i + u_j})^{x_{ji}} \tag{4.1}$$

(under the standardized condition $u_1 + \cdots + u_n = 1$). The fundamental condition for this is

$$\sum_{j \neq i} r_{ij} \frac{u_i}{u_i + u_j} = \sum_{j \neq i} x_{ij}, \quad i = 1, \cdots, n \tag{4.2}$$

which can be solved by successive approximation under several restrictions which is summarized for $[x_{ij}]$ to be irreducible [16]. But these restrictions are very severe. For example, if $x_{ij} = 0$ (or $x_{ij} = r_{ij}$) for all j ($\neq i$) for some i then we cannot solve (4.2)[16]. Except for very large sample cases we have often such cases.

However if we introduce

$$\sum_{j \neq i} r_{ij} \frac{u_i}{u_i + u_j} = \sum_{j \neq i} r_{ij} \dot{x}_{ij}(\theta), \quad i = 1, \cdots, n \tag{4.3}$$

instead of (4.2), then we can solve (4.3) freely from the above restrictions, through the following successive application;

Select arbitrary initial values $u_1, \cdots, u_n$ (with $u_1 + \cdots + u_n = 1$) and calculate

$$u_i' = \sum_{j \neq i} r_{ij} \dot{x}_{ij}(\theta) / \sum_{j \neq i} \frac{r_{ij}}{u_i + u_j}, \quad i = 1, \cdots, n. \tag{4.4}$$

Standardize $u_1', \cdots, u_n'$ to have $u_i'' = u_i' / \sum_k u_k' (i = 1, \cdots, n)$, and take u_i'' as new $u_i (i = 1, \cdots, n)$ and repeat the process till the convergence is attained.

Equation (4.3) is obtained as the condition for $u_1, \cdots, u_n$ to maximize

$$L(\theta) = \prod_{i<j} \binom{r_{ij}}{x_{ij}} \left(\frac{u_i}{u_i + u_j}\right)^{r_{ij}\dot{x}_{ij}(\theta)} \left(\frac{u_j}{u_i + u_j}\right)^{r_{ij}(1 - \dot{x}_{ij}(\theta))}. \tag{4.5}$$

Strictly speaking, (4.5) is not likelihood function. But if we take (4.5) as semi-likelihood function for our model, then solution of (4.3) has meaning of semi-maximum likelihood (semi-ML) methods. But later we often call semi-ML method itself as ML.

The above procedure is the same as that in [16] except the right-hand side of (4.3) which are real numbers instead of integers in [16]. Of course if $\theta = 0$ then the solution of (4.3) is the same as that of (4.2). So by the content on p.33 of [16] the condition of the convergence is only the irreducibility of the fundamental matrix $\boldsymbol{X}$ (3.6). The irreducibility of $\boldsymbol{X}$ is very natural and most meaningful problems have this property (see §2). Actually, all examples in this paper can be solved by the above procedure.

Solving Example 1 by our ML method, we have the following result in Table 2.

Table 2: The result of Example 1 by ML method

	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.7690338	0.5918096	0.4518311
u_2	0.1859869	0.2777946	0.3206353
u_3	0.0449793	0.1303957	0.2275335
L	0.6126050	0.3795271	0.2275447
$L(\theta)$	0.3012641	0.1781490	0.1360250

Comparing this with Table 1, we see that both methods give very near results.

We have had solutions by MM and ML. Each method is based on the respective reasonable principle, and we cannot decide which is superior. But each has its own peculiar property.

The right-hand side $\sum_{j \neq i} r_{ij} \dot{x}_{ij}(\theta) (= t_i)$ of (4.3) is a kind of total score of player i, and the solution of ML is decided by only t_i $(i = 1, \cdots, n)$. So ML gives the solution weighted on scores.

On the other hand, the solution by MM satisfies the relation

$$u_i = \frac{1}{1-\dot{\delta}_i}\sum_{j\neq i}\dot{r}_{ij}\dot{x}_{ij}(\theta)u_j \quad (i=1,\cdots,n)$$

because of Theorem 1, which shows that u_i is not determined by only scores $\sum r_{ij}\dot{x}_{ij}(\theta)$, but is determined by scores weighted by u_j, that is, even if, for example, player 1 and 2 have the same total score, if player 1 has higher scores and player 2 has lower scores for strong players u_j, that is $r_{1j}\dot{x}_{1j}(\theta) > r_{2j}\dot{x}_{2j}(\theta)$, then player 1 has higher evaluation than player 2 ($u_1 > u_2$). After all, MM attaches importance to the total relation to defeat or to be defeated.

We can see the above-mentioned situations in later examples.

5. Ternary AHP

Ternary AHP includes cases (1.1) and (1.2) ("equivalent" or "tie"). So beside x_{ij} (mentioned in §2) we consider "tie." Let $t_{ij} = t_{ji}$ be the number of times for i to tie j. Then we have

$$x_{ij}+x_{ji}+t_{ij}=r_{ij}, \quad \dot{x}_{ij}+\dot{x}_{ji}+\dot{t}_{ij}=1 \quad (\dot{t}_{ij}=\frac{t_{ij}}{r_{ij}}). \tag{5.1}$$

Here of course $t_{ii}=0$ and for pair (i,j) with $r_{ij}=0$, $\dot{x}_{ij}=0$ and $\dot{t}_{ij}=0$.

The value $\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij}$ approximates $p_{ij}=u_i/(u_i+u_j)$, so we propose the following approximation formula with relaxation parameter $\theta(0<\theta<1/2)$;
if $\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij} > 1/2$ then

$$(\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij})(1-\theta^{r_{ij}}) \fallingdotseq u_i/(u_i+u_j), \tag{5.2}$$

$$1-(\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij})(1-\theta^{r_{ij}}) \fallingdotseq u_j/(u_i+u_j) \tag{5.3}$$

and if $\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij}=1/2$ then

$$1/2 \fallingdotseq u_i/(u_i+u_j)=u_j/(u_i+u_j). \tag{5.4}$$

Let us denote the left-hand sides of (5.2) and (5.3) by $\dot{x}_{ij}(\theta)$ and $\dot{x}_{ji}(\theta)$, respectively, that is

$$\begin{aligned}
\dot{x}_{ij}(\theta) &= (\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij})(1-\theta^{r_{ij}}) \quad \text{for } (\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij})>1/2,\\
\dot{x}_{ji}(\theta) &= 1-\dot{x}_{ij}(\theta) \quad \text{for } (\dot{x}_{ji}+\frac{1}{2}\dot{t}_{ij})<1/2,\\
\dot{x}_{ij}(\theta) &= \dot{x}_{ji}(\theta)=1/2 \quad \text{for } (\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij})=1/2,\\
\dot{x}_{ij}(\theta) &= \dot{x}_{ji}(\theta)=0 \quad \text{for } (\dot{x}_{ij}+\frac{1}{2}\dot{t}_{ij})=0.
\end{aligned} \tag{5.5}$$

If we use symbols defined in (5.5), the analysis of ternary AHP is completely the same as those of binary AHP mentioned in §3 (the minimax analysis case) and §4 (the maximum likelihood analysis case).

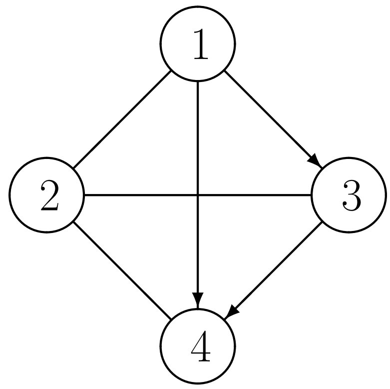

Figure 2: Example 2

Example 2 $(n = 4,\ r_{ij} = 1,\ r_i = 4\ (i, j = 1, 2, \cdots, 4))$
Consider a simple example shown in Figure 2, where "tie" is shown by edges without arrows.

The same example is shown in [14], where the solutions by method in [14] are $u_1 = 0.3431$, $u_2 = 0.2426$, $u_3 = 0.2426$, $u_4 = 0.1716$, which show that u_2 and u_3 are equally weight.

On the other hand, from our MM method, we have

$$\boldsymbol{X} = \frac{1}{4}\begin{bmatrix} 3/2 + 2(1-\theta) & 1/2 & (1-\theta) & (1-\theta) \\ 1/2 & 5/2 & 1/2 & 1/2 \\ \theta & 1/2 & 5/2 & (1-\theta) \\ \theta & 1/2 & \theta & 3/2 + 2\theta \end{bmatrix}, \tag{5.6}$$

and we have Table 3.

Table 3: The result of Example 2 by MM method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.5833333	0.4807691	0.3928570	0.3166666
u_2	0.2500000	0.2500000	0.2500000	0.2500000
u_3	0.1166667	0.1771256	0.2182541	0.2421570
u_4	0.0500000	0.0921053	0.1388889	0.1911765
L	0.3419060	0.3346335	0.2705675	0.1943114
$L(\theta)$	0.3419060	0.2213892	0.1608763	0.1330829

(The meaning of L, $L(\theta)$, see §6)

By Theorem 2 the solution by LS coincides with that of Table 3. By our proposed method $u_2 > u_3$ (for any value of θ) which is different from $u_2 = u_3$ in [14]. Although the score (3 ties) of player 2 are different from that (1 win, 1 loss, 1 tie) of player 3, their weights by [14] are the same.□

Next we apply ML method to this problem to have solutions in Table 4.

6. Likelihood Function to Cope with Ranking Changes

Our proposed method includes the relaxation parameter θ. So we are anxious about how to decide the value of θ. As mathematicians we would like to give the unique optimal solution.

Table 4: The result of Example 2 by ML method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.6160973	0.4922337	0.3956756	0.3169595
u_2	0.1688215	0.2093601	0.2333520	0.2460322
u_3	0.1688215	0.2093601	0.2333520	0.2460322
u_4	0.0462596	0.0890460	0.1376204	0.1909760
L	0.3869846	0.3490691	0.2740197	0.1946484
$L(\theta)$	0.3869846	0.2276523	0.1616045	0.1331166

(The meaning of L, $L(\theta)$, see §6)

But as operations research workers we should give several alternatives and leave the choice to the decision maker. So we propose three stages of relaxation as mentioned in §3.

If for all of three stages the rankings of $(u_1, \cdots, u_n)$ are the same as seen in almost all our examples, then the decision maker does not hesitate. However we might have the case where the rankings change depend on the value of θ as seen in the following example.

Example 3

C. Genest, F. Lapointe and S. W. Drury [4] discussed the following comparison matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1/a & a & a & a & a & a \\ a & 1 & a & a & 1/a & a & 1/a \\ 1/a & 1/a & 1 & a & a & a & a \\ 1/a & 1/a & 1/a & 1 & a & a & a \\ 1/a & a & 1/a & 1/a & 1 & 1/a & a \\ 1/a & 1/a & 1/a & 1/a & a & 1 & a \\ 1/a & a & 1/a & 1/a & 1/a & 1/a & 1 \end{bmatrix} \tag{6.1}$$

For this problem parameter a corresponds to $(1-\theta)/\theta$ as mentioned in (2.10). So the values of a corresponding to $\theta = 1/8$, $2/8$ and $3/8$ are $a = 7$, 3 and $5/3$, respectively.

The principal eigenvectors of $\boldsymbol{A}$ for these values of a are shown in Table 5.

Table 5: The principal eigenvector of Example 3

	$a = 7$	$a = 3$	$a = 5/3$
u_1	0.2157579	0.2047278	0.1803020
u_2	0.2184638	0.2022938	0.1721184
u_3	0.1577404	0.1593033	0.1569208
u_4	0.1153239	0.1239575	0.1365716
u_5	0.1203864	0.1199375	0.1257676
u_6	0.0843132	0.0964541	0.1188613
u_7	0.0880144	0.0933261	0.1094583

As a result the ranking on $a = 5/3$ of $\boldsymbol{A}$ is $u_1 > u_2 > u_3 > u_4 > u_5 > u_6 > u_7$, and $u_2 > u_1 > u_3 > u_5 > u_4 > u_7 > u_6$, on $a = 7$. The rankings change on value of a.

On the other hand, applying our methods MM and ML to (6.1), we have the result for the above values of θ, shown in Table 6 and Table 7. (The case of $\theta = 0$ is included for reference.)

Table 6: The result of Example 3 by MM method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.3571427	0.2919902	0.2360053	0.1865486
u_2	0.2857143	0.2420307	0.2019704	0.1681416
u_3	0.1190477	0.1518351	0.1633884	0.1589119
u_4	0.0595238	0.0930601	0.1198181	0.1369930
u_5	0.0857143	0.0919302	0.1046074	0.1224664
u_6	0.0357143	0.0628785	0.0916256	0.1193164
u_7	0.0571429	0.0662752	0.0825848	0.1076220
L	0.0000064	0.0000092	0.0000058	0.0000021
$L(\theta)$	0.0000064	0.0000028	0.0000011	0.0000006

Table 7: The result of Example 3 by ML method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.4319426	0.3186626	0.2433406	0.1875725
u_2	0.1857125	0.1885715	0.1782554	0.1621149
u_3	0.1857125	0.1885715	0.1782554	0.1621149
u_4	0.0901932	0.1167747	0.1322081	0.1403194
u_5	0.0438032	0.0723138	0.0980558	0.1214542
u_6	0.0438032	0.0723138	0.0980558	0.1214542
u_7	0.0188328	0.0427920	0.0718289	0.1049698
L	0.0000241	0.0000166	0.0000071	0.0000022
$L(\theta)$	0.0000241	0.0000038	0.0000012	0.0000006

By our method, the rankings also change on value of θ in Table 6, $u_1 > u_2 > u_3 > u_4 > u_5 > u_7 > u_6$ on $\theta = 1/8$ and $u_1 > u_2 > u_3 > u_4 > u_5 > u_6 > u_7$ on $\theta = 3/8$. In Table 7 we can see subtle changes. Generally ranking changes by ML are milder than that of MM, we think. The weights on various values of θ are illustrated in Figure 3 for reference.□

The decision maker who encounters the case like Example 3 might be bewildered to take which value of θ. For such situation we propose a method by likelihood function to select the value of θ.

First, we consider binary AHP with Bradley-Terry model. The likelihood function L of Bradley-Terry model for given data r_{ij}, $x_{ij}(i, j = 1, \cdots, n)$ is

$$L = \prod_{i<j} \binom{r_{ij}}{x_{ij}} (\frac{u_i}{u_i + u_j})^{x_{ij}} (\frac{u_j}{u_i + u_j})^{x_{ji}} \tag{6.2}$$

if each comparison (or game) between i and j is independently carried out (as already shown in (4.1)).

We propose the decision method to select one with maximum value of L (6.2) of three stages relaxation parameter θ. We attach the values of L and $L(\theta)$ in Tables showing weights of objects, but the values of $L(\theta)$ are only given for reference.

Example 4

Firstly consider a simple problem in Example 1.

The likelihood function of the data given in Example 1 is

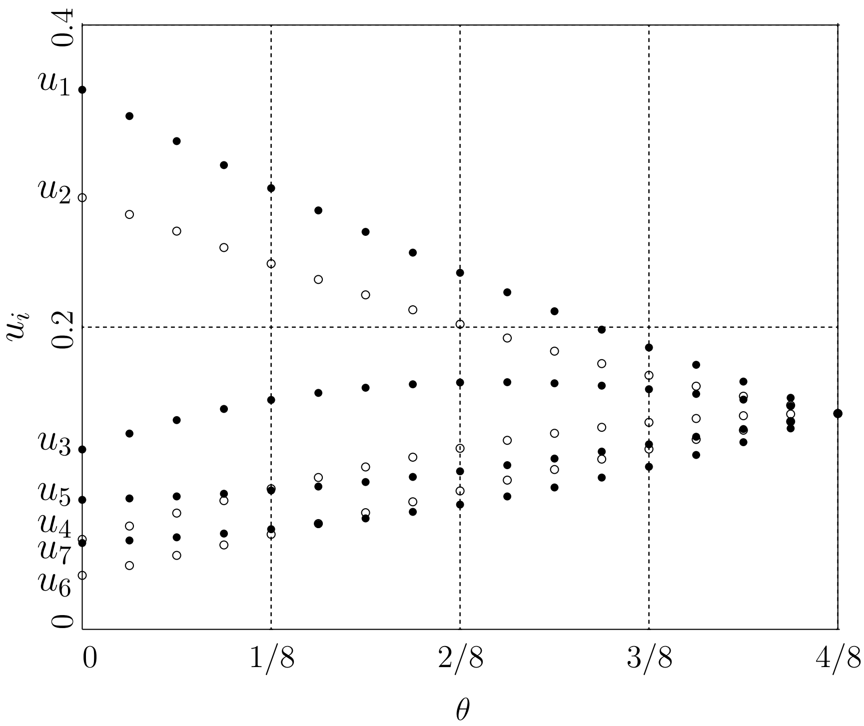

Figure 3: Weights of Example 3 on various values of θ

$$L = \frac{u_1}{u_1+u_2} \cdot \frac{u_1}{u_1+u_3} \cdot \frac{u_2}{u_2+u_3}. \tag{6.3}$$

The values of L for solutions of three stages are;

$$\begin{aligned} L &= 0.5372807 \quad \text{for } \theta = 1/8, \\ L &= 0.3634615 \quad \text{for } \theta = 2/8, \\ L &= 0.2261131 \quad \text{for } \theta = 3/8. \end{aligned}$$

The result shows that the weak stage ($\theta = 1/8$) is desirable.

Next consider Example 3. The values of L for three stages are shown in Table 6 and Table 7, which shows that the solution for $\theta = 1/8$ among three stages is desirable. We include the values of $L(\theta)$(4.5) for reference.□

Statisticians want to use maximum likelihood (ML) method finding $(u_1, \cdots, u_n)$ to maximize L. But as mentioned in §1 and §2, there are many obstructions and we cannot solve practical problems except for fairly large sample cases. Furthermore, for general statistical problems ML method itself cannot necessarily gives good estimates. This cannot give unbiased estimates for some cases. So we cannot adopt ML method as the absolute criterion. We adopt likelihood function as a standard only to select one of three stages.

We define the (semi) likelihood function L and $L(\theta)$ for ternary AHP as follows;

$$L = \prod_{i<j} \binom{r_{ij}}{x_{ij} + \frac{1}{2}t_{ij}} (\frac{u_i}{u_i+u_j})^{x_{ij}+\frac{1}{2}t_{ij}} (\frac{u_j}{u_i+u_j})^{x_{ji}+\frac{1}{2}t_{ji}}, \tag{6.4}$$

$$L(\theta) = \prod_{i<j} \binom{r_{ij}}{x_{ij} + \frac{1}{2}t_{ij}} (\frac{u_i}{u_i+u_j})^{r_{ij}\hat{x}_{ij}(\theta)} (\frac{u_j}{u_i+u_j})^{r_{ij}(1-\hat{x}_{ij}(\theta))}, \tag{6.5}$$

where for odd t_{ij} we use the general formula

$$\binom{r}{x} = \frac{r(r-1)\cdots(r-x+1)}{x(x-1)\cdots 2\times 1}.$$

Of course, the decision method to select a relaxation stage is the same as in the binary case.

Here cases of $\theta = 0$ correspond to the conventional ML method (but are shown for reference), and other cases of $\theta \neq 0$ are solved by the new ML method. Example 1 conflicts with the restrictions, so we cannot solve by ML method for $\theta = 0$.

7. Various Examples

We illustrate three examples in this section.

Example 5 group decision problem

R. E. Jensen [8] discussed rankings of five candidates by five evaluators. Comparison matrix $\boldsymbol{A_i}$ and its principal eigenvector $\boldsymbol{U_i}$ of evaluator i $(i = 1, \cdots, 5)$ are;

$$\boldsymbol{A_1} = \begin{bmatrix} 1 & 2 & 2 & 1/2 & 2 \\ 1/2 & 1 & 2 & 1/2 & 2 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 2 & 2 & 2 & 1 & 2 \\ 1/2 & 1/2 & 2 & 1/2 & 1 \end{bmatrix}, \quad \boldsymbol{U_1} = \begin{bmatrix} 0.2447 \\ 0.1854 \\ 0.1065 \\ 0.3229 \\ 0.1405 \end{bmatrix} \tag{7.1}$$

$$\boldsymbol{A_2} = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 1/2 & 1 & 2 & 1/2 & 2 \\ 1/2 & 1/2 & 1 & 1/2 & 2 \\ 1/2 & 2 & 2 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}, \quad \boldsymbol{U_2} = \begin{bmatrix} 0.3229 \\ 0.1854 \\ 0.1405 \\ 0.2447 \\ 0.1065 \end{bmatrix} \tag{7.2}$$

$$\boldsymbol{A_3} = \begin{bmatrix} 1 & 2 & 2 & 1/2 & 2 \\ 1/2 & 1 & 2 & 1/2 & 2 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 2 & 2 & 2 & 1 & 2 \\ 1/2 & 1/2 & 2 & 1/2 & 1 \end{bmatrix}, \quad \boldsymbol{U_3} = \begin{bmatrix} 0.2447 \\ 0.1854 \\ 0.1065 \\ 0.3229 \\ 0.1405 \end{bmatrix} \tag{7.3}$$

$$\boldsymbol{A_4} = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 1/2 & 1 & 1/2 & 1/2 & 2 \\ 1/2 & 2 & 1 & 1/2 & 2 \\ 1/2 & 2 & 2 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}, \quad \boldsymbol{U_4} = \begin{bmatrix} 0.3229 \\ 0.1405 \\ 0.1854 \\ 0.2447 \\ 0.1065 \end{bmatrix} \tag{7.4}$$

$$\boldsymbol{A_5} = \begin{bmatrix} 1 & 1/2 & 2 & 1/2 & 2 \\ 2 & 1 & 2 & 1/2 & 2 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 2 & 2 & 2 & 1 & 2 \\ 1/2 & 1/2 & 2 & 1/2 & 1 \end{bmatrix}, \quad \boldsymbol{U_5} = \begin{bmatrix} 0.1854 \\ 0.2447 \\ 0.1065 \\ 0.3229 \\ 0.1405 \end{bmatrix} \tag{7.5}$$

where the value of the parameter a (see §2 or (6.1)) is 2.

This is the so-called group decision problem. We have several summarizing methods of these data; arithmetic mean $\bar{\boldsymbol{U}}$ and geometric mean $\widehat{\boldsymbol{U}}$ of $\boldsymbol{U_i}$ $(i = 1, \cdots, 5)$ (component wise), and another is the principal eigenvector $\widehat{\widehat{\boldsymbol{U}}}$ of geometric mean $\boldsymbol{A}$ of $\boldsymbol{A_i}$ $(i = 1, \cdots, 5)$ (component wise);

$$\bar{\boldsymbol{U}} = \begin{bmatrix} 0.2642 \\ 0.1883 \\ 0.1291 \\ 0.2916 \\ 0.1269 \end{bmatrix}, \quad \widehat{\boldsymbol{U}} = \begin{bmatrix} 0.2586 \\ 0.1854 \\ 0.1258 \\ 0.2890 \\ 0.1258 \end{bmatrix}, \quad \widehat{\widehat{\boldsymbol{U}}} = \begin{bmatrix} 0.2607 \\ 0.1899 \\ 0.1271 \\ 0.2941 \\ 0.1282 \end{bmatrix},$$

$$\begin{aligned} &\bar{u}_4 > \bar{u}_1 > \bar{u}_2 > \bar{u}_3 > \bar{u}_5, \\ &\hat{u}_4 > \hat{u}_1 > \hat{u}_2 > \hat{u}_3 = \hat{u}_5, \\ &\hat{\hat{u}}_4 > \hat{\hat{u}}_1 > \hat{\hat{u}}_2 > \hat{\hat{u}}_5 > \hat{\hat{u}}_3. \end{aligned} \tag{7.6}$$

The rankings of object 4, 1 and 2 are stable, but the difference of object 3 and 5 is subtle. On the other hand, by our method, we have $r_{ij} = 5$ (for all i, j) and x_{ij}, as follows;

$$[x_{ij}] = \begin{bmatrix} 2.5 & 4 & 5 & 2 & 5 \\ 1 & 2.5 & 4 & 0 & 5 \\ 0 & 1 & 2.5 & 0 & 2 \\ 3 & 5 & 5 & 2.5 & 5 \\ 0 & 0 & 3 & 0 & 2.5 \end{bmatrix}. \tag{7.7}$$

The result of Example 6 with $\theta = 1/3$ corresponding to $a = 2$ by our MM method is

$$\boldsymbol{U^T} = [0.3476278 \;\; 0.0389970 \;\; 0.0037240 \;\; 0.6077877 \;\; 0.0018635]$$

and we have $u_4 > u_1 > u_2 > u_3 > u_5$.

The solutions for other values of θ are shown for reference in Table 8 and Table 9.

Table 8: The result of Example 5 by MM method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.3464934	0.3465022	0.3467718	0.3484659
u_2	0.0356963	0.0357209	0.0364838	0.0416103
u_3	0.0021443	0.0021560	0.0025189	0.0049934
u_4	0.6152876	0.6152314	0.6134946	0.6018763
u_5	0.0003784	0.0003894	0.0007309	0.0030542
L	0.0003965	0.0004184	0.0011558	0.0034832
$L(\theta)$	0.0003965	0.0004161	0.0009809	0.0012439

On any value of θ, our results are fairly different from conventional results ($\bar{\boldsymbol{U}}$, $\widehat{\boldsymbol{U}}$, $\widehat{\widehat{\boldsymbol{U}}}$). For example

$$\begin{aligned} u_1/u_5 &\fallingdotseq 2 \text{ (in the conventional result)}, \\ u_1/u_5 &\fallingdotseq 187 \text{ (in the result by MM, } \theta = 1/3). \end{aligned}$$

Although all five evaluators evaluate object 5 worse than three objects 1, 2 and 4, u_5 is not so bad by the conventional method. Our method improves such unfair evaluations.□

Table 9: The result of Example 5 by ML method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.3163103	0.3163342	0.3170668	0.3216603
u_2	0.0388663	0.0388984	0.0398924	0.0465214
u_3	0.0035377	0.0035436	0.0037280	0.0050840
u_4	0.6377479	0.6376804	0.6355910	0.6217116
u_5	0.0035377	0.0035434	0.0037219	0.0050227
L	0.0042356	0.0042356	0.0042325	0.0040771
$L(\theta)$	0.0042356	0.0042172	0.0036866	0.0015287

Table 10: Data of Example 6

	P1	P2	P3	P4	P5	P6	P7
P1	\	○	○			○	●
P2	●	\	○	○			○
P3	●	●	\	○	●		
P4		●	●	\	○	●	
P5			○	●	\	○	○
P6	●			○	●	\	○
P7	○	●			●	●	\

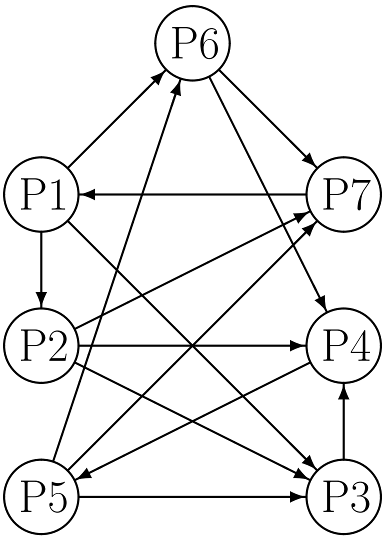

Figure 4: Example 6

Example 6 complex network structure

Next example is matches between 7 players (P1, $\cdots$, P7), and is incomplete comparison case. The results of matches are shown in Table 10, where symbol "○" represents win and "●" represents loss. Blanks on this table show no matches.

The results by MM and ML method are shown in Table 11 and Table 12.

Generally the rankings by ML are stable, but those by MM are sensitive to θ. There are three players 1, 2 and 5 with scores of 3-1. But $u_1 > u_2 > u_5$. The reason is that player 1 defeats player 2 and player 5 is defeated by the rather weak player 4, etc. But by

Table 11: The result of Example 6 by MM method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.3070867	0.2609518	0.2171020	0.1772905
u_2	0.1968503	0.1936101	0.1828904	0.1656218
u_3	0.0236220	0.0546982	0.0859419	0.1158292
u_4	0.0708661	0.0788630	0.0933924	0.1147332
u_5	0.2125984	0.1967581	0.1792487	0.1608407
u_6	0.0866142	0.1093732	0.1273456	0.1388801
u_7	0.1023622	0.1057457	0.1140790	0.1268044
L	0.0003241	0.0003913	0.0002824	0.0001504
$L(\theta)$	0.0003241	0.0001811	0.0001013	0.0000694

Table 12: The result of Example 6 by ML method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.3119317	0.2572702	0.2133768	0.1759584
u_2	0.2619885	0.2250988	0.1952624	0.1683719
u_3	0.0429434	0.0682604	0.0931934	0.1179506
u_4	0.0360681	0.0597248	0.0852822	0.1128655
u_5	0.1776723	0.1806114	0.1724867	0.1591124
u_6	0.1060709	0.1239583	0.1348979	0.1409249
u_7	0.0633250	0.0850761	0.1055006	0.1248163
L	0.0005830	0.0004827	0.0003025	0.0001519
$L(\theta)$	0.0005830	0.0002060	0.0001033	0.0000695

another observation, player 1 is defeated by the rather weaker player 7, but player 2 is not defeated by such weaker players, which is objectionable to $u_1 > u_2$. In any case, the ranking of players is determined not only by the number of wins or losses, but also by complexity of the network (Figure 4) to defeat or to be defeated. These situations are well grasped by our method.□

Example 7 baseball exhibition game

Next example is the baseball exhibition game (spring 2004) among 12 teams (T1 $\sim$ T12, team names listed in Table 13).

The results of matches are shown in Table 14, where blanks show no matches, symbol "○" shows win, "●" shows loss, and "$*$" shows tie.

In general, ranking is determined by the percentage of victories. The percentage $((x_{ij} + \frac{1}{2}t_{ij})/r_{ij})$ of each team is shown in Table 15.

On the other hand, we have the results of Example 7 by MM and ML methods. The results of both methods are shown in Table 16 and Table 17. And the rankings based on our methods for $\theta = 2/8$ are shown in Table 18, with the percentage of victories.

In Table 18, T6 is evaluated at higher order by our methods instead of low percentage of victories. We consider this result to be reasonable, because T6 had a fine match result against T10, the champion team, as shown in Table 14.

The results by MM and ML have almost the same tendency, except T4 whose evaluation by ML is rather higher than that by MM, and the ranking (by ML) of T4 is near to that of the percent of victory.

Table 13: Team names of Example 7

T1	Hanshin
T2	Hiroshima
T3	Kyojin
T4	Chunichi
T5	Yokohama
T6	Yakult
T7	Orix
T8	Seibu
T9	Lotte
T10	Nippon Ham
T11	Dai'ei
T12	Kintetsu

Table 14: Data of Example 7

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
T1	\	○○	○	●	○ ● ○	○∗	○ ○ ●○	∗	○		●	●●
T2	●●	\	○○	∗ ● ○	● ● ○	○	●○	○			∗ ○ ○	
T3	●	●●	\	●	○∗	○		○ ○ ●	○○	∗●	○ ● ●	○
T4	○	∗ ○ ●	○	\	∗○	●●	○ ○ ●	○ ● ○	●	●●	○●	●
T5	● ○ ●	○ ○ ●	●∗	∗●	\	●	○	●	●●	● ● ●	○	○
T6	●∗	●	●	○○	○	\	●		●	○ ∗ ●	∗ ○ ○	●
T7	● ● ○●	○●		● ● ○	●	○	\		●○	●		●○
T8	∗	●	● ● ○	● ○ ●	○			\	○●			●●
T9	●		●●	○	○○	○	○●	●○	\		●○	●
T10			∗○	○○	○ ○ ○	● ∗ ○	○			\		○●
T11	○	∗ ● ●	● ○ ○	●○	●	∗ ● ●			○●		\	○
T12	○○		●	○	●	○	○●	○○	○	●○	●	\

There is an anomaly about the ranking (by both MM and ML for $\theta \neq 0$) $u_6 > u_2$: the number of wins (losses) of T2 is larger (smaller) than that of T6 and T2 defeats T6. But this concerns only the number of wins or losses. If we consider the strengths of teams defeated by T6 or T2, the above anomaly is not suitable. In fact, the sum of strengths (by MM, $\theta = 2/8$) of teams defeated by T2 is
$2u_3 + (1 + 1/2)u_4 + u_5 + u_6 + u_7 + u_8 + (2 + 1/2)u_{11} = 0.623$
and that of T6 is
$\frac{1}{2}u_1 + 2u_4 + u_5 + (1 + 1/2)u_{10} + (2 + 1/2)u_{11} = 0.650$
(where weight of tie is 1/2), and that of T6 is higher than T2.

However for $\theta = 0$ we have $u_2 > u_6$ which does not induce such an anomaly, and the value of L for $\theta = 0$ is the highest. This suggests that for the larger number of data the smaller values (than 1/8) of θ might be desirable. But the above results might be due to the errors of the approximation formula (2.8). These questions are left to future researches. □

Table 15: The percentage of victories

team	games	wins	losses	ties	percentage
T1	18	10	6	2	0.6111
T2	17	9	6	2	0.5882
T3	18	8	8	2	0.5000
T4	21	9	10	2	0.4762
T5	20	6	12	2	0.3500
T6	16	6	7	3	0.4688
T7	16	6	10	0	0.3750
T8	13	4	8	1	0.3462
T9	14	7	7	0	0.5000
T10	13	9	2	2	0.7692
T11	16	6	8	2	0.4375
T12	14	9	5	0	0.6429

Table 16: The result of Example 7 by MM method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.0966629	0.0985090	0.1003173	0.1005262
u_2	0.0846077	0.0808138	0.0787858	0.0781405
u_3	0.0709965	0.0711839	0.0726398	0.0752133
u_4	0.0538891	0.0553887	0.0593660	0.0648668
u_5	0.0451872	0.0438569	0.0449157	0.0491795
u_6	0.0816908	0.0896627	0.0978236	0.1052270
u_7	0.0514164	0.0552164	0.0589105	0.0628883
u_8	0.0319636	0.0337803	0.0382307	0.0458150
u_9	0.0532860	0.0565402	0.0610176	0.0664344
u_{10}	0.2226044	0.2107818	0.1916684	0.1666446
u_{11}	0.0636676	0.0610858	0.0595097	0.0596019
u_{12}	0.1440279	0.1431805	0.1368149	0.1254624
L	1.112×10^{-17}	1.092×10^{-17}	9.920×10^{-18}	7.304×10^{-18}
$L(\theta)$	1.112×10^{-17}	8.889×10^{-18}	5.026×10^{-18}	1.944×10^{-18}

8. Conclusion

We proposed stochastic models for binary and ternary AHP (§1), and the specific approximation formula (2.6)(§2). In §3 the minimax estimation method through the basic matrix $\boldsymbol{X}$(3.6) was proposed. We proved that $\boldsymbol{X}$ has the maximum eigenvalue equal to 1 (Theorem 1). Least square method for binary AHP was proposed and we proved that the least square solution coincides with that of the minimax method. Semi-maximum likelihood method was proposed (§4). We analyzed ternary AHP and applied our method to this (§5). We introduced likelihood function for our models to evaluate the parameter θ (§6). Various examples, which were especially interesting, included example of actual baseball results, were solved by our methods (§7). Our proposed methods, when applied to actual problems, might encounter the contradictions which we cannot explain, as seen in Example 7. Such problems are due to lack of concept of errors in our methods. The error analysis is left to future researches.

Table 17: The result of Example 7 by ML method

	$\theta = 0$	$\theta = 1/8$	$\theta = 2/8$	$\theta = 3/8$
u_1	0.0990557	0.1002844	0.1012809	0.1008274
u_2	0.0845238	0.0806964	0.0787401	0.0782328
u_3	0.0690365	0.0691333	0.0704946	0.0730626
u_4	0.0617139	0.0627771	0.0653129	0.0687439
u_5	0.0444267	0.0441532	0.0462186	0.0509619
u_6	0.0740756	0.0839412	0.0943468	0.1038023
u_7	0.0499570	0.0521744	0.0553401	0.0599205
u_8	0.0381914	0.0388396	0.0416314	0.0474450
u_9	0.0590765	0.0600909	0.0624644	0.0663344
u_{10}	0.2310818	0.2206158	0.2011777	0.1739164
u_{11}	0.0564146	0.0557830	0.0563685	0.0584278
u_{12}	0.1324465	0.1315107	0.1266241	0.1183250
L	1.321×10^{-17}	1.263×10^{-17}	1.085×10^{-17}	7.609×10^{-18}
$L(\theta)$	1.321×10^{-17}	1.006×10^{-17}	5.393×10^{-18}	2.005×10^{-18}

Table 18: Percentage vs. our methods results ($\theta = 2/8$) of Example 7

order	team	games	wins	losses	ties	percentage	MM	ML
1	T10	13	9	2	2	0.7692	0.1916684	0.2011777
2	T12	14	9	5	0	0.6429	0.1368149	0.1266241
3	T1	18	10	6	2	0.6111	0.1003173	0.1012809
4	T6	16	6	7	3	0.4688	0.0978236	0.0943468
5	T2	17	9	6	2	0.5882	0.0787858	0.0787401
6	T3	18	8	8	2	0.5000	0.0726398	0.0704946
7	T9	14	7	7	0	0.5000	0.0610176	0.0624644
8	T11	16	6	8	2	0.4375	0.0595097	0.0563685
9	T4	21	9	10	2	0.4762	0.0593660	0.0653129
10	T7	16	6	10	0	0.3750	0.0589105	0.0553401
11	T5	20	6	12	2	0.3500	0.0449157	0.0462186
12	T8	13	4	8	1	0.3462	0.0382307	0.0416314

Furthermore, the accuracy $(\sum(\dot{x}_{ij}(\theta) - u_i/(u_i + u_j))^2$ of the approximation formula (2.8) can be considered to be the criterion of goodness of various methods. The research of the consistency of this and the likelihood value is also left to future researches.

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A. Appendix

Proof of $\dot{x}_{ij}(\theta) > 1/2$ (see (2.6));

For simplicity we omit the suffix i, j of x_{ij} and r_{ij}, then we have only to prove that if $0 < \theta < 1/2$, $x/r > 1/2$ then

$$(x/r)(1-\theta^r) > 1/2 \tag{A.1}$$

(for positive integers r and x).

It suffices for us to prove this for $x = \frac{r}{2} + 1$ (for even r), $= (r+1)/2$ (for odd r) and positive $\theta < 1/2$.

So (A.1) results in

$$(\frac{r}{2} + 1)\theta^r < 1 \text{ (for even } r), \tag{A.2}$$

$$(r+1)\theta^r < 1 \text{ (for odd } r). \tag{A.3}$$

The inequality (A.3) includes (A.2), so we have only to prove (A.3).

For $r = 1$ we have (A.3). As for $r \geqq 3$ if (A.3) is valid for $\theta = 1/2$, then (A.3) is also valid for $\theta < 1/2$, so after all if

$$(r+1) < 2^r \tag{A.4}$$

is valid then we have the desired inequality. And we have (A.4) for any positive (not necessarily odd) integers $r (\geqq 2)$. For example, the formula

$$2^r = (1+1)^r = 1 + {}_rC_1 + {}_rC_2 + \cdots = 1 + r + {}_rC_2 + \cdots$$

whose 3rd, 4th $\cdots$ terms are positive, proves (A.4).

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