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A SWITCHING MODEL OF DYNAMIC ASSET SELLING PROBLEM

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Abstract This paper proposes an asset selling problem with a new selling strategy called the *switching strategy* where multiple homogeneous assets on hand must be sold up to a specified deadline. At each point in time the seller is permitted to decide between 1) proposing a selling price up front to an appearing buyer and 2) concealing the price and letting the buyer come up with an offer. Our analysis indicates that under certain conditions there emerges a time threshold after which the seller switches from concealing his idea for the selling price to proposing this price, and vice versa.

Keywords: Dynamic programming, posted price mechanism, reservation price mechanism

1. Introduction

According to Arnold and Lippman [1], four major mechanisms exist for selling an asset: auction, bargaining, posted price, and reservation price (sequential search). The first two mechanisms are self-explanatory. Here we shall provide the standard definition for the second two [3]. The posted price mechanism means that a seller proposes a selling price to each appearing buyer, who, judging from the price proposed, decides whether or not to purchase the asset. This selling mechanism is commonly found in the traditional brick-and-mortar business and e-business. The reservation price mechanism on the other hand assumes that the seller conceals the selling price and the appearing buyer offers a price. The seller then decides whether or not to sell the asset after comparing the offered price to his reservation price. A pricing system resembles this mechanism is the one supplier version of the Name-Your-Price system, which is an Internet pricing system that does not display the selling price of goods and services sold but enables buyers to name the price for the product they wish to purchase.

The literature on these four mechanisms is extensive and well established. A common feature in the work undertaken so far implicitly assumes that a single mechanism is employed *throughout the entire planning horizon*. In reality, when selling an asset, a seller is not restricted to use only one mechanism throughout the entire planning horizon. He may switch the mechanism used from one type to one of the others after certain points in time within the planning horizon so long as a higher profit can be obtained by doing so. Switching among mechanisms has been practically difficult in the past because changing a mechanism incurs huge cost in many cases. However, the recent advent of the Internet and electronic commerce has greatly reduced this cost since the task of switching among the mechanisms can be executed automatically even without the seller's presence. In addition, as competition among businesses gets keener, many businesses become willing to adopt new selling strategies that may have been considered unrealistic up to now, this being so long as there are opportunities for yielding higher profit. Motivated by these observations, we propose a new selling strategy called the *switching strategy* where the seller is permitted to switch between the mechanisms at every point in time over the planning horizon. Although we can consider a model with switching among all the four mechanisms stated above, the analysis of such a model can be expected to become rather complicated. For this reason, as a first attempt, in this paper we consider a model with switching between only the two simpler mechanisms: the posted price and reservation price mechanisms. Switching strategy involving other types of mechanisms such as auction and bargaining, which have more complicated characteristics, are left for future research.

The reservation price mechanism is closely related to the works on sequential search which aim to maximize the expected value of an offer that will be accepted from the search process [9][10][11][12]. In addition, the posted price mechanism considered in our model is similar to the dynamic pricing policy which assumes that selling prices can be dynamically adjusted at each point in time over the entire planning horizon [4][7][8]. Moreover, our paper also relates to the articles on the selling mechanism selection problem [1][2][13][14]. In these articles, separate models are developed for the different selling mechanisms stated earlier and they are compared against each other to determine which mechanism to choose. In this case, once a mechanism is committed to, no mechanism change is allowed so long as the selling process proceeds. To the best of our knowledge, none of the work done on the selling mechanisms so far considers the possibility of switching between these mechanisms within the planning horizon. The objective of this paper is to formulate a basic model of an asset selling problem with the switching strategy stated earlier, examine the properties of its optimal decision rules, and evaluate the economic effectiveness from adopting the switching strategy.

In this paper we assume that both seller and buyer do not play strategically; the decision to sell or purchase is assumed to depend solely on the seller's reservation price or buyer's maximum willingness to pay. Therefore, in our asset selling problem three possible strategies may exist: 1) To employ the posted price mechanism throughout the planning horizon, 2) to employ the reservation price mechanism throughout the planning horizon, and 3) to switch between employing the posted price (propose a price) and employing the reservation price (conceal the price) mechanisms at certain points in time during the planning horizon. The seller then has to make a decision as to which one to choose out of the above three options. For convenience, let us refer to the last strategy as the *switching strategy* and to the first and second ones as the *non-switching strategy*. Thus, we shall call the model with switching strategy the *switching model*, and the one with non-switching strategy the *non-switching model*.

In order to demonstrate that the switching strategy might prove beneficial, in this paper we assess the economic effectiveness of adopting the switching strategy by numerically comparing the total expected profits for the switching model to that for the non-switching model. Our numerical results show that the proposed switching model may yield a substantial improvement over the non-switching models. The switching model might be thought to be an imaginary model; however, if the relative difference in the total expected profits between the switching and non-switching models is large enough to be no longer negligible but significant, the seller would incur an opportunity loss for not adopting the switching strategy. In fact, we demonstrate by numerical examples in Section 5.2 that the relative difference may be greater than 20 percent. This very fact justifies the adoption of the switching strategy in the process of selling assets.

In Section 2 that follows, we provide a strict definition of our model and derive its optimality equations. In Section 3 we define several functions and examine their properties, by use of which we clarify in Section 4 the properties of the optimal decision rules. In Section 5 we provide numerical examples that ascertain the occurrence of switching and evaluate the economic effectiveness of adopting switching strategy. Finally, in Section 6 we present the overall conclusions of our research and suggest some further work which could be done.

2. Model Formulation

Consider a seller who faces a problem of selling i units of a homogeneous asset within a finite planning horizon. First, let the points in time t be numbered backward from the final point in time of the planning horizon, time 0 (the deadline) as $0, 1, \cdots$ and so on. In addition, let the time interval between times t and $t-1$ be called the period t, which is small enough that no more than one buyer may appear. Next, assume that a buyer who requests a unit arrives with a probability $\lambda \in (0, 1)$. When a buyer appears at a point in time, the seller has to make a decision between two alternatives: A1 $\langle$employ the posted price mechanism$\rangle$ and A2 $\langle$employ the reservation price mechanism$\rangle$.

1. If the seller chooses the alternative A1, he proposes a selling price to an appearing buyer. The buyer then decides whether or not to purchase the asset based on the price offered by the seller. By ξ let us denote the maximum permissible buying price for the buyer, implying that the buyer is willing to purchase if and only if the selling price z proposed by the seller is lower than or equal to ξ, i.e., $z \le \xi$. Here, assume that subsequent buyers' maximum permissible buying prices, $\boldsymbol{\xi}, \boldsymbol{\xi}', \cdots$* are independent identically distributed random variables having a known continuous distribution function $F_{\boldsymbol{\xi}}(\xi)$ with a finite expectation μ_ξ. Let $f_{\boldsymbol{\xi}}(\xi)$ denote its probability density function, which is truncated on both sides. More precisely, $F_{\boldsymbol{\xi}}(\xi)$ and $f_{\boldsymbol{\xi}}(\xi)$ are defined as follows; For certain given numbers a and b such that $0 < a < b < \infty$

$$F_{\boldsymbol{\xi}}(\xi) = 0, \quad \xi \le a, \qquad 0 < F_{\boldsymbol{\xi}}(\xi) < 1, \quad a < \xi < b, \qquad F_{\boldsymbol{\xi}}(\xi) = 1, \quad b \le \xi, \tag{2.1}$$
$$f_{\boldsymbol{\xi}}(\xi) = 0, \quad \xi < a, \qquad f_{\boldsymbol{\xi}}(\xi) > 0, \quad a \le \xi \le b, \qquad f_{\boldsymbol{\xi}}(\xi) = 0, \quad b < \xi. \tag{2.2}$$

Thus, the probability of an appearing buyer purchasing the asset, provided that a price z is offered by the seller, is given by $\Pr\{z \le \boldsymbol{\xi}\} = 1 - \Pr\{\boldsymbol{\xi} < z\} = 1 - \Pr\{\boldsymbol{\xi} \le z\} = 1 - F_{\boldsymbol{\xi}}(z)$ since $F_{\boldsymbol{\xi}}(\xi)$ is assumed to be continuous. Furthermore, let us define

$$\underline{f} = \inf\{f_{\boldsymbol{\xi}}(\xi) \mid \xi \in [a, b]\} > 0, \tag{2.3}$$

which will become inevitably necessary to successfully prove Lemma 3.2(a).

2. If the seller chooses the alternative A2, he does not propose a price. In this case, it is assumed that the buyer will definitely offers a price $w = \alpha\xi$ $(0 < \alpha \le 1)$ where ξ is the buyer's maximum permissible buying price. Here, let us call α, the *price offering ratio.* This measures the degree of a buyer's desirability for the asset, i.e., the greater

*Throughout this paper, random variables are represented by bold faced symbols.

(lower) the buyer's desirability may be, the closer the α may be to 1 (0). In this paper, we assume that α and ξ are stochastically independent and that subsequent buyers' price offering ratios, $\boldsymbol{\alpha}, \boldsymbol{\alpha'}, \cdots$, are independent identically distributed random variables having a known distribution function $F_{\boldsymbol{\alpha}}(\alpha)$ with a finite expectation $\mu_\alpha > 0$. Let us denote the distribution function of $\boldsymbol{w}$ by $F_{\boldsymbol{w}}(w)$ and its probability density function by $f_{\boldsymbol{w}}(w)$. Then the expectation of $\boldsymbol{w}$ is given by[†]

$$\mu_w = \mathbf{E}[\boldsymbol{w}] = \mathbf{E}_{\boldsymbol{\alpha},\boldsymbol{\xi}}[\boldsymbol{\alpha}\boldsymbol{\xi}] = \mu_\alpha \mu_\xi > 0. \tag{2.4}$$

Hence we have $F_{\boldsymbol{w}}(x) = \Pr\{\boldsymbol{w} \le x\} = \Pr\{\boldsymbol{\alpha}\boldsymbol{\xi} \le x\} = \Pr\{\boldsymbol{\xi} \le x/\boldsymbol{\alpha}\} = \mathbf{E}[F_{\boldsymbol{\xi}}(x/\boldsymbol{\alpha})]$. Accordingly, the probability density function $f_{\boldsymbol{w}}(x)$ is given by

$$f_{\boldsymbol{w}}(x) = \mathbf{E}[1/\boldsymbol{\alpha} f_{\boldsymbol{\xi}}(x/\boldsymbol{\alpha})]. \tag{2.5}$$

Then the decision rules of the model consist of:

1. The *switching rule* as to when to switch between alternatives A1 and A2.
2. The *pricing rule* as to what price to offer to an arriving buyer when the seller takes alternative A1.
3. The *selling rule* as to whether to sell the asset or not when the seller takes alternative A2.

Furthermore, we assume that assets remaining at time 0 must be sold at a per unit salvage price $\rho \in (-\infty, \infty)$. Here, $\rho < 0$ implies the disposal cost per unit to discard an unsold asset. Now, let $u_t(i,0)$ and $u_t(i,1)$ be the maximum total expected profits, respectively, with no buyer and with a buyer. Then, clearly

$$u_0(i,0) = \rho i, \qquad u_t(0,0) = u_t(0,1) = 0, \quad t \ge 0, \quad i \ge 0, \tag{2.6}$$

and

$$u_t(i,0) = \lambda u_{t-1}(i,1) + (1-\lambda) u_{t-1}(i,0), \qquad t \ge 1, \quad i \ge 0, \tag{2.7}$$

$$u_t(i,1) = \max \left\{ \begin{array}{ll} \mathrm{A1} : \max\limits_z \{(1-F_{\boldsymbol{\xi}}(z))\big(z + u_t(i-1,0)\big) + F_{\boldsymbol{\xi}}(z) u_t(i,0)\} & \cdots (1), \\ \mathrm{A2} : \displaystyle\int_0^b \max\{w + u_t(i-1,0), u_t(i,0)\} f_{\boldsymbol{w}}(w) dw & \cdots (2) \end{array} \right\}, \quad t \ge 0, \quad i \ge 1. \tag{2.8}$$

Here note that $f_{\boldsymbol{w}}(w)$ takes on value 0 on $(-\infty, 0] \cup (b, \infty)$ due to the fact that $f_{\boldsymbol{\alpha}}(\alpha)$ and $f_{\boldsymbol{\xi}}(\xi)$ take on value 0 on, respectively, $(-\infty, 0] \cup (1, \infty)$ and $(-\infty, a) \cup (b, \infty)$. The objective is to find the optimal decision rules so as to maximize the total expected profit over the planning horizon, i.e., the total expected revenue gained from selling the assets to appearing buyers *plus* the total expected salvage value of the assets remaining unsold at the deadline.

[†]For simplicity, we use the symbol $\mathbf{E}$ when we take the expectation with respect to a random variables. Whenever an expectation is taken with respect to two random variables, we write the random variables in the subscript to $\mathbf{E}$ such as $\mathbf{E}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$.

3. Preliminaries

This section defines the functions that will be used to transform the model's optimality equations given by Eqs. (2.6) to (2.8). First, for any x let us define the following two functions:

$$T_\xi(x) = \int_a^b \max\{\xi - x, 0\} f_{\boldsymbol{\xi}}(\xi) d\xi, \tag{3.1}$$

$$T_w(x) = \int_0^b \max\{w - x, 0\} f_{\boldsymbol{w}}(w) dw \tag{3.2}$$

where $f_{\boldsymbol{\xi}}(\xi)$ takes on value 0 on $(-\infty, a) \cup (b, \infty)$. Note that both $T_\xi(x)$ and $T_w(x)$ resemble the function T_F defined in [5]. Rearranging Eq. (3.2) by substituting Eq. (2.5) yields $T_w(x) = \mathbf{E}[1/\boldsymbol{\alpha} \int_0^b \max\{w - x, 0\} f_{\boldsymbol{\xi}}(w/\boldsymbol{\alpha}) dw]$. Since $w = \alpha\xi$ by definition, noting Eq. (3.1), we get

$$T_w(x) = \mathbf{E}\left[\boldsymbol{\alpha} \int_0^b \max\{\xi - x/\boldsymbol{\alpha}, 0\} f_{\boldsymbol{\xi}}(\xi) d\xi\right] = \mathbf{E}[\boldsymbol{\alpha} T_\xi(x/\boldsymbol{\alpha})]. \tag{3.3}$$

Next, for any x let us define

$$S_\xi(x) = \max_z (1 - F_{\boldsymbol{\xi}}(z))(z - x), \tag{3.4}$$

and by $z(x)$ let us designate the smallest z attaining the maximum of the right-hand side of Eq. (3.4) if it exists, i.e.,

$$S_\xi(x) = \big(1 - F_{\boldsymbol{\xi}}(z(x))\big)(z(x) - x). \tag{3.5}$$

The function $S_\xi(x)$ is also defined in [15]. Here, let us introduce two properties of $z(x)$ whose proofs can be found in [15].

Lemma 3.1 (You [15]) *$z(x)$ is nondecreasing in $x \in (-\infty, \infty)$ with $z(x) \geq a$ for any x and if $x \geq (<)\ b$, then $z(x) = b$ $(x < z(x) < b)$.*

Moreover, for convenience in the later discussions, let us define

$$J(x) = T_w(x) - S_\xi(x), \tag{3.6}$$

$$H(x) = \max\{S_\xi(x), T_w(x)\} = \max\{0, J(x)\} + S_\xi(x). \tag{3.7}$$

In addition, we define

$$x^\star = \inf\{x \,|\, z(x) > a\}, \quad a^\circ = \max\{x \,|\, T_w(x) = \mu_w - x\}, \quad b^\circ = \sup\{x \,|\, T_w(x) > 0\}, \tag{3.8}$$

if they exist. Below let us investigate some properties of $J(x)$.

Lemma 3.2

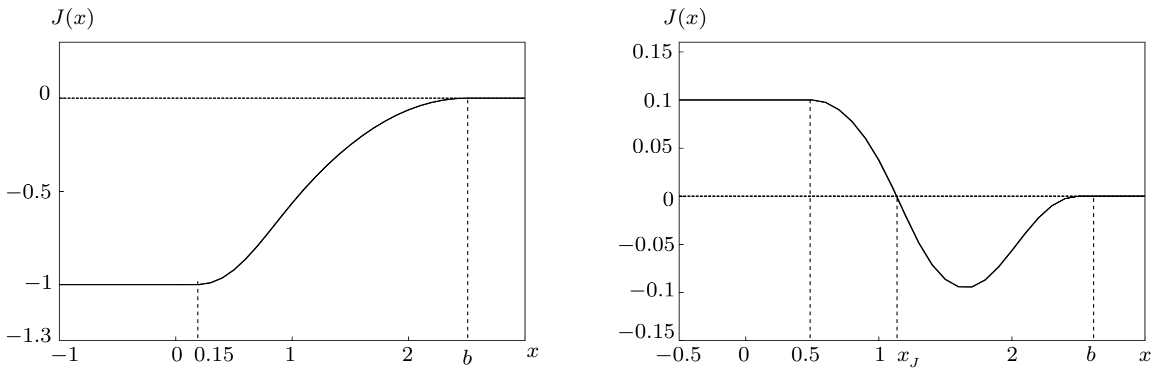
(a) $x^\star \le \inf\{x \mid S_\xi(x) > a - x\} < a$ *and* $\min\{x^\star, a^\circ\} \le b^\circ \le b$.
(b) $J(x) = \mu_w - a$ *on* $(-\infty, \min\{x^\star, a^\circ\}]$ *and* $J(x) = 0$ *on* $[b, \infty)$.
(c) *If* $b^\circ < b$, *then* $J(x)$ *is strictly increasing and negative* (*i.e.*, $J(x) < 0$) *on* $[b^\circ, b)$.

Proof: See Appendix A.1. ∎

From Lemma 3.2(b) we see that $J(x)$ is constant on $(-\infty, \min\{x^\star, a^\circ\}]$ and $[b, \infty)$. However, its shape on $(\min\{x^\star, a^\circ\}, b)$ cannot be easily determined. It will be seen later that the shape of this function decisively influences whether or not the optimal decision rules exhibit the occurrence of switching at some points in time during the planning horizon.

Now, in general by $x_{_J}$ let us denote the solutions of the equation $J(x) = 0$ if they exist, i.e., $J(x_{_J}) = 0$. We will see later on that only the solutions on the interval $(\min\{x^\star, a^\circ\}, b)$ characterize the properties of the optimal decision rule with respect to switching. Noting these facts, let us provide a more precise definition of the solution as follows. For certain δ and δ' such that $\min\{x^\star, a^\circ\} < \delta \le \delta' < b$, if $J(x) = 0$ on $[\delta, \delta']$ with $J(\delta - \varepsilon) \ne 0$ and $J(\delta' + \varepsilon') \ne 0$ for any infinitesimal $\varepsilon > 0$ and $\varepsilon' > 0$, then let $x_{_J} = \delta$. If $\delta = \delta'$, the solution is an *isolated* solution. Note that the solution $x_{_J}$ defined in this matter may be multiple, such that $\min\{x^\star, a^\circ\} < x_{_J}^1 < \cdots < x_{_J}^k < b$ for $1 \le k \le N$. Below, let us give some examples showing that $J(x) = 0$ may or may not have an isolated solution on the interval $(\min\{x^\star, a^\circ\}, b)$ depending on the given distribution functions $F_{\boldsymbol{\xi}}(\xi)$ and $F_{\boldsymbol{\alpha}}(\alpha)$.

Example 1 *Let* $F_{\boldsymbol{\xi}}(\xi)$ *be the uniform distribution function on* $[1.5, 2.5]$, *i.e.*, $a = 1.5$ *and* $b = 2.5$. *If* $F_{\boldsymbol{\alpha}}(\alpha)$ *is a uniform distribution function on* $[0.1, 0.4]$, *then* $J(x) = 0$ *has no solution on the interval* $(\min\{x^\star, a^\circ\}, b)$ *with* $\min\{x^\star, a^\circ\} \approx 0.1500$ (*see Figure 1(I)*). *If* $F_{\boldsymbol{\alpha}}(\alpha)$ *is a uniform distribution function on* $[0.7, 0.9]$, *then* $J(x) = 0$ *has an isolated solution,* $x_{_J} \approx 1.1339$, *with* $\min\{x^\star, a^\circ\} \approx 0.50$ (*see Figure 1(II)*).



(I) $\alpha \in [0.1, 0.4]$ (II) $\alpha \in [0.7, 0.9]$

Figure 1: The shapes of $J(x)$

Example 2 *Let* $F_{\boldsymbol{\xi}}(\xi)$ *be the distribution function on* $[0.1, 3.0]$, *i.e.*, $a = 0.1$ *and* $b = 3.0$ *such that* $f_{\boldsymbol{\xi}}(\xi) \approx 0.05701$ *on* $[0.1, 0.599]$, $f_{\boldsymbol{\xi}}(\xi)$ *is a triangle on* $[0.599, 0.7]$ *with its maximum at* $\xi = 0.6$, *and* $f_{\boldsymbol{\xi}}(\xi) \approx 0.06982$ *on* $[0.7, 3.0]$. *Let* $F_{\boldsymbol{\alpha}}(\alpha)$ *be a uniform distribution on* $[0.64, 0.74]$. *Then* $J(x) = 0$ *has three isolated solutions,* $x_{_J}^1 \approx -0.5566$, $x_{_J}^2 \approx 0.4630$, *and* $x_{_J}^3 \approx 0.7471$, *on the interval* $(\min\{x^\star, a^\circ\}, b)$ *with* $\min\{x^\star, a^\circ\} \approx -17.8272$ *as shown in Figure 2(II)*.

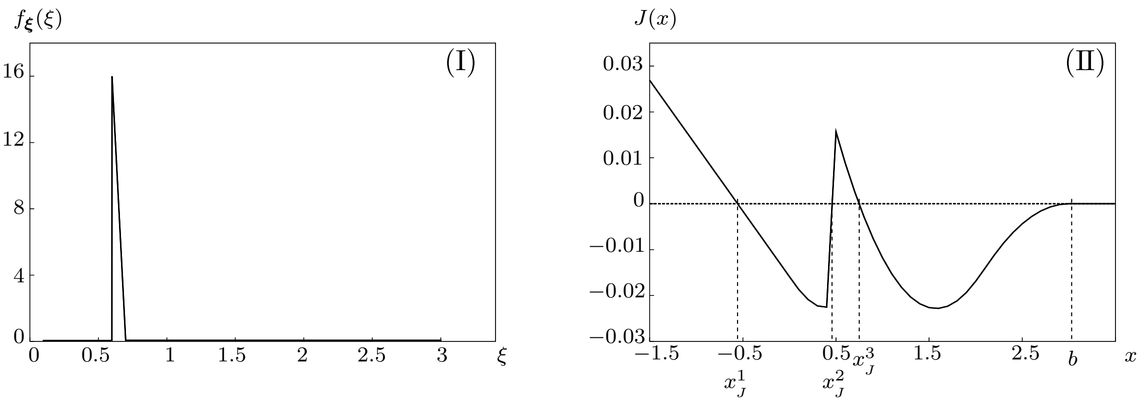


Figure 2: $f_{\boldsymbol{\xi}}(\xi)$ and $J(x)$ where $\min\{x^\star, a^\circ\} \approx -17.8272$

4. Properties of the Optimal Decision Rules

This section describes the optimal decision rules of our model and examines their properties.

4.1. Selling and pricing rules

In this subsection, we clarify the properties of the optimal selling and pricing rules. For convenience in the analysis, Let us define

$$U_t(i) = u_t(i,0) - u_t(i-1,0), \qquad t \geq 0, \quad i \geq 1 \tag{4.1}$$

and let $U(i) = \lim_{t\to\infty} U_t(i)$ for any given $i \geq 1$ if it exists. Then, from Eq. (2.6) we have

$$U_0(i) = \rho, \qquad i \geq 1. \tag{4.2}$$

Noting Eqs. (4.1), (3.2), (3.4), and (3.7), we can rewrite Eq. (2.8) as follows.

$$\begin{aligned} u_t(i,1) &= \max\{S_\xi(U_t(i)) + u_t(i,0), T_w(U_t(i)) + u_t(i,0)\} \\ &= H(U_t(i)) + u_t(i,0) \qquad (4.3) \\ &= \max\{0, J(U_t(i))\} + S_\xi(U_t(i)) + u_t(i,0), \qquad t \geq 0, \quad i \geq 1. \qquad (4.4) \end{aligned}$$

From Eq. (4.4) the optimal pricing and selling rules for $t \geq 0$ and $i \geq 1$ can be prescribed as:

(a) If $J(U_t(i)) \geq 0$, employ the reservation price mechanism; in other words, let an appearing buyer offer a price. Then, for a price w offered by the buyer appearing at time t, if $w \geq U_t(i)$, sell the asset, or else do not, so $U_t(i)$ becomes the seller's *minimum permissible selling price*[‡].

(b) If $J(U_t(i)) \leq 0$, employ the posted price mechanism; in other words, propose a price to an appearing buyer. The optimal selling price is given by the smallest z attaining the maximum of Eq.(2.8 (1)) if it exists, denoted by $z_t(i)$. Since Eq.(2.8 (1)) can be expressed as $S_\xi(U_t(i)) + u_t(i,0) = \max_z(1 - F_{\boldsymbol{\xi}}(z))(z - U_t(i)) + u_t(i,0)$, we have $z_t(i) = z(U_t(i))$ (see Eq. (3.5)).

[‡]Minimum permissible selling price means a seller's reservation price for selling an asset; he is willing to sell it if and only if the price offered by a buyer is greater than his reservation price.

From the above we see that the optimal pricing and selling rules closely relate to $U_t(i)$. Below we discuss some properties of $U_t(i)$.

Lemma 4.1

(a) $U_t(i)$ *is nonincreasing in* $i \geq 0$ *for* $t \geq 0$ *and nondecreasing in* $t \geq 0$ *for* $i \geq 0$.

(b) $U_t(i)$ *converges to* $U(i)$ *as* $t \to \infty$ *for* $i \geq 1$ *with* $U(i) \geq b$.

Proof: See Appendix A.2. ❚

The properties of $z_t(i)$ given by the theorem below follows immediately from the facts that $z_t(i) = z(U_t(i))$ by definition and that the monotonicity of $U_t(i)$ in i and t is inherited to $z_t(i)$ due to Lemmas 3.1 and 4.1(a).

Theorem 4.1 *The optimal selling price* $z_t(i)$ *is nonincreasing in* $i \geq 0$ *for* $t \geq 0$ *and nondecreasing in* $t \geq 0$ *for* $i \geq 0$ *with* $a \leq z_t(i) \leq b$.

Intuition suggests the following. If the seller has substantial assets remaining unsold at a point in time or if the deadline is approaching, in order to avoid leftover assets at the deadline, he may become more compelled to sell, implying that he will lower the selling price (if employing the posted price mechanism is optimal) or his minimum permissible selling price (if employing the reservation price mechanism is optimal) as the number of assets remaining unsold i increases or as the remaining time periods up to the deadline t decreases. Therefore, it can be conjectured that $z_t(i)$ and $U_t(i)$ are both nonincreasing in i and nondecreasing in t. Lemma 4.1(a) and Theorem 4.1 affirm our conjecture; these results are consistent with those in [1], [3], [8], and [15]. Furthermore, another immediate consequence of Theorem 4.1 is that if the alternative A1 is optimal, the seller will charge a price which lies in between a and b, the upper and lower bounds of the distribution function of the buyer's maximum permissible buying price.

4.2. Switching property

In this subsection, we shall investigate the conditions under which switching occurs. First we provide the strict definitions of the switching property.

Definition 4.1 *Let* $i \geq 1$.

(a) *When* t *moves from* 0 *to* ∞, *if* $J(U_t(i))$ *changes from "*< 0*" to "*> 0*" or from "*> 0*" to "*< 0*", let the sign of* $J(U_t(i))$ *be said to change. If the sign change occurs, let the optimal decision rules be said to possess a* switching property, *or else do not.*

(b) *If a sign change occurs* $k \geq 1$ *times, let the optimal decision rules be said to possess the* k-switching property.

(c) *Let us refer to the point in time when the sign change of* $J(U_t(i))$ *occurs as the* switching time threshold, *denoted by* $t^*(i)$.

If $k = 1$ ($k \geq 2$), it is said to possess a *single* (*multiple*) switching property. As time t moves from 0 to ∞, $U_t(i)$ starting with $U_0(i)$ increases and converges to $U(i) \geq b$ (see Lemma 4.1(b)). Paying attention to this fact, we obtain the following theorem which prescribes whether or not switching occurs as the process proceeds.

Theorem 4.2

(a) *Let* $b \leq \rho$. *Then the optimal decision rules have no switching property.*

(b) *Let $\rho < b$.*

1. *Let $J(x) = 0$ have no solution $x_{_J}$ on $(\min\{x^\star, a^\circ\}, b)$. Then the optimal decision rules have no switching property.*
2. *Let $J(x) = 0$ have the solutions $x_{_J}^1, x_{_J}^2, \dots, x_{_J}^k$ with $k \geq 1$ on $(\min\{x^\star, a^\circ\}, b)$; let $x_{_J}^1 < x_{_J}^2 < \dots < x_{_J}^k$ without loss of generality. Further, if $x_{_J}^\ell \leq \rho < x_{_J}^{\ell+1}$ for a given ℓ such that $0 \leq \ell \leq k$ where $x_{_J}^0 = -\infty$ and $x_{_J}^{k+1} = \infty$, then the optimal decision rules have $(k - \ell)$-switching property.*

Proof: Note that $U_0(i) = \rho$ from Eq. (4.2) and that $U_t(i)$ is monotone in t from Lemma 4.1(a).

(a) Let $b \leq \rho$. Then since $b \leq U_0(i) \leq U_t(i)$ for $t \geq 0$, we have $J(U_t(i)) = 0$ for $t \geq 0$ from Lemma 3.2(b). Hence in this case, sign change of $J(U_t(i))$ does not occur on $t \geq 0$, so the optimal decision rules have no switching property.

(b) Let $\rho < b$.

(b1) Immediate from the fact that $J(x)$ is constant on $(-\infty, \min\{x^\star, a^\circ\}]$ and $[b, \infty)$ from Lemma 3.2(b).

(b2) If $x_{_J}^{\ell+1}, x_{_J}^{\ell+2}, \dots, x_{_J}^k$ are isolated solutions, the sign of $J(U_t(i))$ changes at these solutions as t moves from 0 to ∞. Even if all of $x_{_J}^{\ell+1}, x_{_J}^{\ell+2}, \dots, x_{_J}^k$ are not isolated solutions, clearly sign change of $J(U_t(i))$ also occurs $k - \ell$ times as t moves from 0 to ∞. ∎

5. Numerical Experiments

The objective of the numerical experiments is twofold: To exemplify the occurrence of switching and to assess the economic effectiveness of adopting the switching strategy.

5.1. Switching property

Below Let us provide some examples where the optimal decision rules have no, 1-, or 3-switching property. Consider $f_{\boldsymbol{\xi}}(\xi)$ and $F_{\boldsymbol{\alpha}}(\alpha)$ defined in Example 2 with $b = 3.0$. Then $J(x)$ can be depicted as in Figure 3. Here, let $\lambda = 0.5$ and $i = 5$. Then by calculation we obtain $\min\{x^\star, a^\circ\} \approx -17.8272$ and $U(5) = 3.0$, and know that $J(x) = 0$ has three isolated solutions: $x_{_J}^1 \approx -0.5566$, $x_{_J}^2 \approx 0.4630$, and $x_{_J}^3 \approx 0.7471$, i.e., $k = 3$. Let $\rho = -3.0$. Then $\rho = -3.0 < -0.5566 \approx x_{_J}^1 < x_{_J}^2 < x_{_J}^3 < b = 3.0$, implying that the conditions in Theorem 4.2(b,b2) with $\ell = 0$ are satisfied. Consequently, the optimal decision rules have a 3-switching property for $t \geq 0$. Similarly, if $\rho = 0.01$, 0.5, and 1.5, then it can be easily confirmed that the optimal decision rules have, respectively, a 2-, 1-, and no switching property.

We should note that the optimal decision rules with respect to switching in Theorem 4.2 are prescribed on the assumption that none of the i units of the asset on hand is sold throughout the entire planning horizon. In reality, the assumption stated above may fail to hold since it is possible that some units are sold to appearing buyers as the process proceeds. Taken this fact into consideration, we need to interpret the switching property as illustrated in the following scenario. For simplicity of discussion, let $F_{\boldsymbol{\xi}}(\xi)$ and $F_{\boldsymbol{\alpha}}(\alpha)$ be the uniform distribution functions, respectively, on $[1.5, 15.5]$ and $[0.58, 0.9]$; and let $\lambda = 0.54$ and $\rho = 0.7$. From the calculation we see that a switching time threshold exists for each of $i = 1$ and 2 where $t^*(1) = 2$ and $t^*(2) = 5$. Here note that if $t \leq t^*(i)$, then for each of $i = 1$ and 2 the seller should employ the reservation price mechanism (conceal the price), or

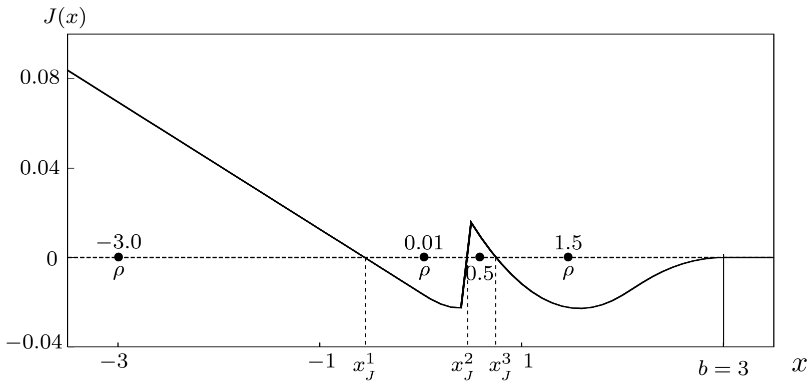


Figure 3: Switching property where $\min\{x^\star, a^\circ\} \approx -17.8272$

else posted price mechanism (propose the price). In the scenario below let the process start from time $t = 6$ when a seller has $i = 2$ units on hand (see Figure 4).

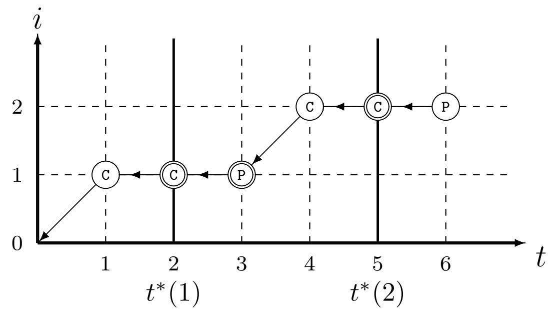


Figure 4: Scenarios of selling process (The symbols Ⓒ and Ⓟ represent the decisions of, respectively, concealing and proposing the selling price, and the symbol ◎ indicates that switching occurs at that time point)

Since $t = 6 > 5 = t^*(2)$, the seller should propose a price. If no asset is sold, the process proceeds to $t = 5$. Since $t = 5 = t^*(2)$, he should *switch* to concealing the price. If no asset is sold, the process proceeds to $t = 4$ and he should conceal the price at that time since $t = 4 < 5 = t^*(2)$. Assume that one unit is sold at that time. Then the number of assets at $t = 3$ is reduced to $i = 1$, hence he should *switch* to proposing the price at that time since $t = 3 > 2 = t^*(1)$. If no asset is sold at $t = 3$, the process proceeds to $t = 2$ and he should *switch* to concealing the price since $t = t^*(1) = 2$. Accordingly, in this scenario, switching occurs three times throughout the entire planning horizon.

5.2. Economic effectiveness of adopting switching strategy

First, let $\tilde{u}_t(i, 0)$ be the maximum total expected profit of the non-switching model (employing either posted price or reservation price mechanism throughout the entire planning horizon) starting from time $t \geq 0$ with $i \geq 0$ assets, provided that no buyer exists. Next, let $\varphi_t(i)$ be the *relative difference* in the maximum total expected profits between the switching and non-switching models, provided that $u_t(i, 0) \neq 0$, i.e., $\varphi_t(i) = (u_t(i, 0) - \tilde{u}_t(i, 0))/u_t(i, 0)$. For convenience, let $\varphi_t(i)$ for posted price and reservation price mechanisms be denoted by, respectively, $\varphi^p_t(i)$ and $\varphi^c_t(i)$. For all the numerical examples below let $F_{\boldsymbol{\xi}}(\xi)$ and $F_{\boldsymbol{\alpha}}(\alpha)$ be the uniform distribution functions, respectively, on $[1.5, 15.5]$ and $[0.58, 0.9]$, and let $\lambda = 0.54$ and $\rho = 0.7$. Our main observations are:

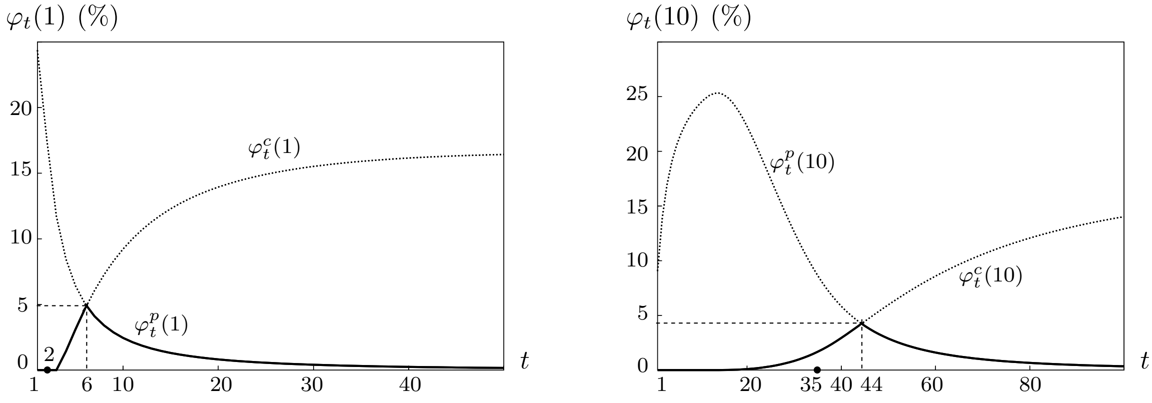


Figure 5: Relationship of $\varphi_t^c(i)$ and $\varphi_t^p(i)$ with t where the symbol • indicates the switching time threshold $t^*(i)$

1. Figure 5(I,II) depicts the relationship between, respectively, $\varphi_t(1)$ and $\varphi_t(10)$ with t. In this case we have the switching time threshold $t^*(1) = 2$ and $t^*(10) = 35$. When $i = 1$ ($i = 10$), it is optimal to conceal a price if $t \leq 2$ ($t \leq 35$), or else propose it. In addition, from Figure 5(I) we see that $\varphi_t^p(1)$ and $\varphi_t^c(1)$ can be as high as 24% and 16%, respectively. This implies that the seller may bear the risk of having a large difference in the maximum total expected profits by using non-switching strategy throughout the entire planning horizon, instead of adopting the switching strategy.
2. As illustrated in Section 5.1, the quantity of assets on hand plays a key role in determining whether or not a switch of mechanism occurs as the selling process proceeds. Thus, a proper adoption of switching strategy in business requires the monitoring of the quantity on hand. If the relative difference between the switching and non-switching models is not large enough, the sellers may confront the situation where the cost of frequent monitoring exceeds the benefit for adopting the switching strategy. In such a case, from a practical viewpoint, the seller should not adopt the switching strategy. Suppose for this reason, the seller determines to adopt the non-switching strategy. Then the bold curves of Figure 5(I,II) tell us the following. When $i = 1$ ($i = 10$) it is optimal to adopt the reservation price mechanism throughout the entire planning horizon if the process starts from time $t \leq 6$ ($t \leq 44$), or else the posted price mechanism. In addition, it should be noted that when $i = 1$ ($i = 10$) the maximum relative difference for adopting non-switching strategy is approximately 5% (4.3%), which occurs at time $t = 6$ ($t = 44$), and that the relative difference decreases gradually as the planning horizon becomes greater or less than the time $t = 6$ ($t = 44$). Furthermore, we obtain another numerical example demonstrating that the maximum relative difference for adopting non-switching strategy may increase by more than twofold. For instance, by letting $i = 1$, $\rho = -5.0$, $F_{\boldsymbol{\alpha}}(\alpha)$ be uniform on $[0.3, 0.9]$, and both λ and $F_{\boldsymbol{\xi}}(\xi)$ be unchanged, it becomes 11%.

6. Conclusions and Suggested Future Studies

In this paper we have proposed a basic model for an asset selling problem where the seller can switch between employing the posted price mechanism and employing the reservation price mechanism over the planning horizon. From our analysis, we obtained some conditions that guarantee the occurrence of switching. Below, we shall reemphasize the two distinctive points derived from our analysis.

1. Through the analysis, we showed that a switch of mechanism between the posted price and reservation price mechanisms may occur. A numerical experiment also demonstrated that a multiple switching property may exist.
2. The numerical results we obtained in Section 5 demonstrated that the adoption of the switching strategy may be effective in increasing the seller's profit. We showed that the relative difference between the maximum total expected profits for the switching and non-switching models can be as high as 24 percent. This implies that a seller may incur an opportunity loss if the switching strategy is not adopted.

Below, let us mention some interesting directions for extending our model that could make it more practical. Possible extensions include: 1) switching among more than two types of selling mechanisms such as among auction, posted price, and bargaining, 2) consideration of the strategic interaction between the seller and buyer by introducing the game theoretical concept, 3) future availability of the buyer who leaves the selling process without purchase, and 4) introduction of a cost of attracting buyers, called the search cost.

Appendix

A. Proof of Lemmas

Before proving Lemmas 3.2 and 4.1, we must introduce the following four additional lemmas. The proof of Lemma A.1 can be found in [15].

Lemma A.1 (You [15])

(a) *$S_\xi(x)$ is continuous**, nonincreasing on $(-\infty,\infty)$, and strictly decreasing on $(-\infty,b)$.*
(b) *$S_\xi(x) > 0$ on $(-\infty,b)$ and $S_\xi(x)=0$ on $[b,\infty)$.*
(c) *$S_\xi(x)+x$ is nondecreasing on $(-\infty,\infty)$ and strictly increasing on (a,∞).*
(d) *If $x\le y$, then $S_\xi(x)-S_\xi(y)\le y-x$††.*

Lemma A.2

(a) *$T_\xi(x) > 0$ for $x\in(-\infty,b)$ and $T_\xi(x)=0$ for $x\in[b,\infty)$.*
(b) *$T_\xi(x)=\mu_\xi - x$ for $x\in(-\infty,a]$ and $T_\xi(x) > \mu_\xi - x$ for $x\in(a,\infty)$.*

Proof: Since $T_\xi(x) = \int_x^b(\xi-x)dF_{\boldsymbol{\xi}}(\xi) \ge \int_y^b(\xi-x)dF_{\boldsymbol{\xi}}(\xi)$ for any x and y, we have $T_\xi(x)-T_\xi(y) \ge \int_y^b(\xi-x)dF_{\boldsymbol{\xi}}(\xi) - \int_y^b(\xi-y)dF_{\boldsymbol{\xi}}(\xi) = -(x-y)(1-F_{\boldsymbol{\xi}}(y))$. Similarly we get $T_\xi(x)-T_\xi(y)\le -(x-y)(1-F_{\boldsymbol{\xi}}(x))$. Hence for any x and y we have

$$-(x-y)(1-F_{\boldsymbol{\xi}}(y)) \le T_\xi(x)-T_\xi(y) \le -(x-y)(1-F_{\boldsymbol{\xi}}(x)), \qquad \text{(A.1)}$$

**The assertion that $S_\xi(x)$ is continuous on $(-\infty,\infty)$ is not provided in [15]. However, this assertion is obvious from the fact that $(1-F_{\boldsymbol{\xi}}(z))(z-x)$ is continuous on $(-\infty,\infty)$ for any x.

††The assertion is equivalent to the one of Lemma 3.1(e) in [15] since $S_\xi(x)\ge 0$ from (b).

from which we get

$$(x-y)F_{\boldsymbol{\xi}}(y) \le T_\xi(x) + x - T_\xi(y) - y \le (x-y)F_{\boldsymbol{\xi}}(x). \qquad \text{(A.2)}$$

(a) The assertion can be proven in the same way as Lemma 3.1(c) in [6].

(b) Let $y < x$. Then since $(x-y)F_{\boldsymbol{\xi}}(y) \ge 0$, we have $T_\xi(y) + y \le T_\xi(x) + x$ from Eq. (A.2), hence $T_\xi(x)+x$ is nondecreasing on $(-\infty, \infty)$. Let $a < y < x$. Then $(x-y)F_{\boldsymbol{\xi}}(y) > 0$ due to Eq. (2.1). Thus $T_\xi(y) + y < T_\xi(x) + x$ from Eq. (A.2). That is, $T_\xi(x) + x$ is strictly increasing on (a, ∞), hence on $[a, \infty)$. Let $x \le a$. Then since $\xi - x \ge 0$ for $a \le \xi \le b$, we have $T_\xi(x) = \int_a^b (\xi - x) f_{\boldsymbol{\xi}}(\xi) d\xi = \mu_\xi - x$, so $T_\xi(a) = \mu_\xi - a$. Let $x > a$. Then $T_\xi(x) + x > T_\xi(a) + a = \mu_\xi - a + a = \mu_\xi$, so $T_\xi(x) > \mu_\xi - x$. ∎

Lemma A.3

(a) $T_w(x)$ *is continuous and nonincreasing on* $(-\infty, \infty)$.
(b) $T_w(x) > 0$ *on* $(-\infty, b^\circ)$ *and* $T_w(x) = 0$ *on* $[b^\circ, \infty)$ *where* $b^\circ \le b$.
(c) $T_w(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(d) $T_w(x) = \mu_w - x$ *on* $(-\infty, a^\circ]$ *and* $T_w(x) > \mu_w - x$ *on* (a°, ∞).

Proof: (a) Immediate from the the definition of $T_w(x)$.

(b) Since $T_\xi(0) = \int_a^b \xi f_{\boldsymbol{\xi}}(\xi) d\xi = \mu_\xi$, from Eqs. (3.3) and (2.4) we have $T_w(0) = \mathbf{E}[\boldsymbol{\alpha} T_\xi(0)] = \mu_\alpha \mu_\xi > 0 \cdots (1^*)$. If $x \ge b$, then $x/\alpha \ge b$ due to $0 < \alpha \le 1$, hence $T_\xi(x/\alpha) = 0$ from Lemma A.2(a), so $T_w(x) = 0$ due to Eq. (3.3). From this result, (a), and (1^*) there exists a supremum $b^\circ \le b$ of x such that $T_w(x) > 0$ (see Eq. (3.8)), so that $T_w(x) > 0$ for $x < b^\circ$ and $T_w(x) = 0$ for $x \ge b^\circ$.

(c) Let $x > y$. Then noting Eqs. (3.3) and (A.2), we get $T_w(x) + x - T_w(y) - y = \mathbf{E}[\boldsymbol{\alpha}\big(T_\xi(x/\boldsymbol{\alpha}) + x/\boldsymbol{\alpha} - T_\xi(y/\boldsymbol{\alpha}) - y/\boldsymbol{\alpha}\big)] \ge \mathbf{E}[\boldsymbol{\alpha}(x/\boldsymbol{\alpha} - y/\boldsymbol{\alpha}) F_{\boldsymbol{\xi}}(y/\boldsymbol{\alpha})] = (x-y)\,\mathbf{E}[F_{\boldsymbol{\xi}}(y/\boldsymbol{\alpha})] \ge 0$, so that $T_w(x) + x \ge T_w(y) + y$. Thus the assertion holds.

(d) Define $G(x) = T_w(x) + x - \mu_w$. Let $x \le 0$. Then since $0 < a$, we get $x/\alpha \le 0 < a$ due to $\alpha > 0$, so $\alpha T_\xi(x/\alpha) = \alpha(\mu_\xi - x/\alpha) = \alpha\mu_\xi - x$ from Lemma A.2(b). Hence, from Eqs. (3.3) and (2.4) we get $T_w(x) = \mathbf{E}[\boldsymbol{\alpha}\mu_\xi - x] = \mu_\alpha\mu_\xi - x = \mu_w - x$, so $G(x) = 0 \cdots (2^*)$ for $x \le 0$. Since $\mu_\xi > a > 0$, we have $\mu_\xi/\alpha > a$ due to $0 < \alpha \le 1$, so $\alpha T_\xi(\mu_\xi/\alpha) > \alpha(\mu_\xi - \mu_\xi/\alpha) = \alpha\mu_\xi - \mu_\xi$ from Lemma A.2(b). Thus $T_w(\mu_\xi) = \mathbf{E}[\boldsymbol{\alpha} T_\xi(\mu_\xi/\boldsymbol{\alpha})] > \mathbf{E}[\boldsymbol{\alpha}\mu_\xi - \mu_\xi] = \mu_\alpha\mu_\xi - \mu_\xi = \mu_w - \mu_\xi$, so $G(\mu_\xi) > 0 \cdots (3^*)$. In addition, since $G(x)$ is nondecreasing on $(-\infty, \infty)$ from (c), noting (2^*) and (3^*), we see that there exists a maximum a° of x such that $G(x) = 0$, i.e., $T_w(x) = \mu_w - x$ (see Eq. (3.8)). Thus, $G(x) > 0$ for $x > a^\circ$, so $T_w(x) > \mu_w - x$ for $x > a^\circ$ and $G(x) = 0$ for $x \le a^\circ$, so $T_w(x) = \mu_w - x$ for $x \le a^\circ$. ∎

Lemma A.4

(a) $H(x)$ *is continuous and nonincreasing on* $(-\infty, \infty)$.
(b) $H(x) > 0$ *on* $(-\infty, b)$ *and* $H(x) = 0$ *on* $[b, \infty)$.
(c) $\lambda H(x) + x$ *is strictly increasing on* $(-\infty, \infty)$.
(d) *If* $x \le y$, *then* $H(x) - H(y) \le y - x$.

Proof: (a) Immediate from Eq. (3.7), Lemmas A.1(a), and A.3(a).

(b) Let us pay attention to Eq. (3.7), Lemmas A.1(b), A.3(b), and the fact that $b^\circ \le b$ from Lemma A.3(b). If $x < b$, then $S_\xi(x) > 0$, hence $H(x) \ge S_\xi(x) > 0$, and if $x \ge b$, then $x \ge b > b^\circ$, hence $H(x) = 0$.

(c) Since $\lambda S_\xi(x)+x = \lambda(S_\xi(x)+x)+(1-\lambda)x$ and $\lambda T_w(x)+x = \lambda(T_w(x)+x)+(1-\lambda)x$, it follows from Lemmas A.1(c), A.3(c), and the assumption of $\lambda < 1$ that $\lambda S_\xi(x)+x$ and $\lambda T_w(x)+x$ are both strictly increasing on $(-\infty,\infty)$. Therefore, the assertion is evident from the fact that $\lambda H(x)+x = \max\{\lambda S_\xi(x)+x, \lambda T_w(x)+x\}$.

(d) Let $x \le y$. Then from Eq. (A.1) we get $T_\xi(x) - T_\xi(y) \le y-x$ due to $-(x-y) \ge 0$. Since $x/\alpha \le y/\alpha$, from Eq. (3.3) we obtain $T_w(x)-T_w(y) = \mathbf{E}[\boldsymbol{\alpha}(T_\xi(x/\boldsymbol{\alpha}) - T_\xi(y/\boldsymbol{\alpha}))] \le \mathbf{E}[\boldsymbol{\alpha}(y/\boldsymbol{\alpha} - x/\boldsymbol{\alpha})] = \mathbf{E}[y-x] = y-x$. Therefore, from Eq. (3.7) and Lemma A.1(d) we obtain $H(x)-H(y) \le \max\{S_\xi(x)-S_\xi(y), T_w(x)-T_w(y)\} \le \max\{y-x, y-x\} = y-x$. ∎

A.1. Proof of Lemma 3.2

(a) The assertion is proven by the subsequent three steps below.

S1. Let us prove that $x^\star$ is finite and that $x^\star < a$.

1. Let $h(z) = (1-F_{\boldsymbol{\xi}}(z))(z-a)/F_{\boldsymbol{\xi}}(z)$ for $z > a$. Further let $h^\star = \sup_{z>a} h(z)$. Then for an infinitesimal $\varepsilon > 0$ such that $b > b-\varepsilon > a$, we get $h(b-\varepsilon) = (1-F_{\boldsymbol{\xi}}(b-\varepsilon))(b-\varepsilon-a)/F_{\boldsymbol{\xi}}(b-\varepsilon) > 0$ due to Eq. (2.1), so $h^\star \ge h(b-\varepsilon) > 0 \cdots (1^*)$. Assume that $h^\star = \infty$. Since $h(z) = 0$ for $z \ge b$ due to Eq. (2.1), we have $h^\star = \sup_{b>z>a} h(z)$. The assumption of $h^\star = \infty$ implies that there exists at least one z' with $b > z' > a$ such that $h(z') = (1-F_{\boldsymbol{\xi}}(z'))(z'-a)/F_{\boldsymbol{\xi}}(z') \ge R/\underline{f}$ for any given sufficiently large $R > 1$, from which $(1-F_{\boldsymbol{\xi}}(z'))(z'-a) = F_{\boldsymbol{\xi}}(z')h(z') \ge F_{\boldsymbol{\xi}}(z')R/\underline{f} = \Pr\{\boldsymbol{\xi} \le z'\}R/\underline{f} \cdots (2^*)$. Since $f_{\boldsymbol{\xi}}(\xi) \ge \underline{f}$ for $a \le \xi \le z' < b$ due to Eq. (2.3), we have $\Pr\{\boldsymbol{\xi} \le z'\} = \int_a^{z'} f_{\boldsymbol{\xi}}(\xi)d\xi \ge \underline{f}\int_a^{z'} d\xi = (z'-a)\underline{f}$. Thus, from (2^*) and Eq. (2.3) we have $(1-F_{\boldsymbol{\xi}}(z'))(z'-a) \ge (z'-a)\underline{f}R/\underline{f} = (z'-a)R$, leading to the contradiction of $1-F_{\boldsymbol{\xi}}(z') \ge R > 1$ due to $z' > a$. Hence it must be that $h^\star < \infty \cdots (3^*)$.
2. Using the above result, let us prove that $x^\star$ is finite. Note that $S_\xi(x) = \max_{z\ge a}(1-F_{\boldsymbol{\xi}}(z))(z-x) \cdots (4^*)$ due to Lemma 3.1. Then for any given x let us consider the four successive assertions: 1) $\mathcal{A}_1\langle z(x) > a\rangle$, 2) $\mathcal{A}_2\langle (1-F_{\boldsymbol{\xi}}(a))(a-x) < (1-F_{\boldsymbol{\xi}}(z'))(z'-x)$ for at least one $z' > a\rangle$, 3) $\mathcal{A}_3\langle a - h(z') < x$ for at least one $z' > a\rangle$, and 4) $\mathcal{A}_4\langle \inf_{z>a}\{a-h(z)\} < x\rangle$.

   a. Suppose $\mathcal{A}_1$ is true.

      i Let $x \ge b$, so $x \ge b > a$. Then $z(x) = b > a$ from Lemma 3.1, so $F_{\boldsymbol{\xi}}(z(x)) = 1$ due to Eq. (2.1). Here since $F_{\boldsymbol{\xi}}(a) = 0$, we have $(1-F_{\boldsymbol{\xi}}(a))(a-x) = a-x < 0 = (1-F_{\boldsymbol{\xi}}(z(x)))(z(x)-x)$, implying that $\mathcal{A}_2$ holds, i.e., $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$.

      ii Let $x < b$. Assume that $(1-F_{\boldsymbol{\xi}}(a))(a-x) \ge (1-F_{\boldsymbol{\xi}}(z'))(z'-x)$ for all $z' > a$, hence for all $z' \ge a$, implying that $z(x) = a$ from (4^*), which contradicts the assumption that $\mathcal{A}_1$ is true. Hence it must be that $(1-F_{\boldsymbol{\xi}}(a))(a-x) < (1-F_{\boldsymbol{\xi}}(z'))(z'-x)$ for at least one $z' > a$, thus $\mathcal{A}_2$ must be true, i.e., $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$.

   Hence $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ for any x. Let $\mathcal{A}_2$ be true. Then if $z(x) = a$, we have $(1-F_{\boldsymbol{\xi}}(a))(a-x) < (1-F_{\boldsymbol{\xi}}(z'))(z'-x) \le S_\xi(x) = (1-F_{\boldsymbol{\xi}}(z(x)))(z(x)-x) = (1-F_{\boldsymbol{\xi}}(a))(a-x)$, which is a contradiction, thus $z(x) > a$ due to Lemma 3.1, i.e., $\mathcal{A}_1$ is true, so that $\mathcal{A}_2 \Rightarrow \mathcal{A}_1$. From all the above we have $\mathcal{A}_1 \Leftrightarrow \mathcal{A}_2$.

   b. Consider z' such that $z' > a$. Then $F_{\boldsymbol{\xi}}(z') > 0$ due to Eq. (2.1). Since $F_{\boldsymbol{\xi}}(a) = 0$, we have

$$(1 - F_{\boldsymbol{\xi}}(a))(a - x) - (1 - F_{\boldsymbol{\xi}}(z'))(z' - x)$$
$$= a - x - (1 - F_{\boldsymbol{\xi}}(z'))((a - x) + (z' - a))$$
$$= F_{\boldsymbol{\xi}}(z')\big(a - x - (1 - F_{\boldsymbol{\xi}}(z'))(z' - a)/F_{\boldsymbol{\xi}}(z')\big) = F_{\boldsymbol{\xi}}(z')(a - h(z') - x),$$

hence it can be immediately seen that $\mathcal{A}_2 \Leftrightarrow \mathcal{A}_3$.

c. If $\mathcal{A}_3$ is true, then clearly so also is $\mathcal{A}_4$, i.e., $\mathcal{A}_3 \Rightarrow \mathcal{A}_4$. If $\mathcal{A}_4$ is true, then $a - h(z') < x$ for at least one $z' > a$, hence $\mathcal{A}_3$ is true, i.e., $\mathcal{A}_4 \Rightarrow \mathcal{A}_3$, so that $\mathcal{A}_3 \Leftrightarrow \mathcal{A}_4$.

Since $\mathcal{A}_1 \Leftrightarrow \mathcal{A}_4$ from all the above, noting Eq. (3.8), we eventually obtain $x^\star = \inf\{x \mid z(x) > a\} = \inf\{x \mid \inf_{z>a}\{a - h(z)\} < x\} = \inf_{z>a}\{a - h(z)\} = a - \sup_{z>a} h(z) = a - h^\star$, which is finite due to $0 < h^\star < \infty$ from $(1^*, 3^*)$. Hence $x^\star < a$.

S2. Note that $S_\xi(x) \geq (1 - F_{\boldsymbol{\xi}}(a))(a - x) = a - x$ for any x from Eq. (2.1). If $x < x^\star$, then $z(x) = a$ from the definition of $x^\star$ and Lemma 3.1, so $F_{\boldsymbol{\xi}}(z(x)) = 0$ from Eq. (2.1). Thus $S_\xi(x) = (1 - F_{\boldsymbol{\xi}}(z(x)))(z(x) - x) = a - x$ for $x < x^\star$. Further $S_\xi(a) > 0 = a - a$ from Lemma A.1(b), so $S_\xi(x) > a - x$ for $x = a$. In addition, since $S_\xi(x) \geq 0$ for any x due to Lemma A.1(b), if $x > a$, then $S_\xi(x) \geq 0 > a - x$. From the above we can depict the graph of $S_\xi(x)$ as in Figure A.6, from which we immediately see that $x^\star \leq \inf\{x|S_\xi(x) > a - x\} < a$. Hence the former half of the assertion holds. From the above, we get $S_\xi(x) = a - x$ for $x \leq \inf\{x \mid S_\xi(x) > a - x\} \cdots (5^*)$.

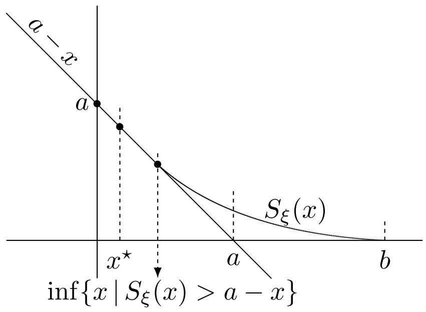


Figure A.6: $x^\star \leq \inf\{x \mid S_\xi(x) > a - x\} < a$

S3. Now since $T_w(a^\circ) = \mu_w - a^\circ$ from Lemma A.3(d), if $\mu_w < a^\circ$, then $T_w(a^\circ) < 0$, which contradicts Lemma A.3(b). Hence $\mu_w \geq a^\circ$. In addition, $0 = T_w(b^\circ) \geq \mu_w - b^\circ$ from Lemma A.3(b,d), hence $b^\circ \geq \mu_w$, so $a^\circ \leq b^\circ$. Therefore, $\min\{x^\star, a^\circ\} \leq a^\circ \leq b^\circ \leq b$ from Lemma A.3(b).

(b) Let $x \leq \min\{x^\star, a^\circ\}$, so $x \leq x^\star$ and $x \leq a^\circ$. Then since $x \leq x^\star \leq \inf\{x \mid S_\xi(x) > a - x\}$ due to (a), we have $S_\xi(x) = a - x$ from (5^*). In addition, owing to $x \leq a^\circ$, we get $T_w(x) = \mu_w - x$ from Lemma A.3(d). Therefore, from Eq. (3.6) we obtain $J(x) = \mu_w - x - (a - x) = \mu_w - a$ for $x \leq \min\{x^\star, a^\circ\}$. Let $x \geq b$. Then $J(x) = 0$ from Lemmas A.3(b) and A.1(b).

(c) Let $b^\circ < b$. If $b^\circ \leq x < b$, then $T_w(x) = 0$ from Lemma A.3(b) and $S_\xi(x) > 0$ from Lemma A.1(b), hence $J(x) = -S_\xi(x) < 0$ from Eq. (3.6). In addition, $J(x)$ is strictly increasing on $[b^\circ, b)$ due to Lemma A.1(a). ∎

A.2. Proof of Lemma 4.1

For convenience of analysis, let

$$U_t(0) = M, \qquad t \geq 0, \tag{A.3}$$

for a sufficiently large M such that $\rho \leq M$ and $b \leq M$. Then since $H(U_t(0)) + u_t(0,0) = H(M) = 0 = u_t(0,1)$ for $t \geq 0$ due to Lemma A.4(b) and Eq. (2.6), we see that Eq. (4.3) holds for $i \geq 0$ instead of $i \geq 1$. Accordingly, since $u_{t-1}(i,1) - u_{t-1}(i,0) = H(U_{t-1}(i))$ for $t \geq 1$ and $i \geq 0$ from Eq. (4.3), we can rewrite Eq. (2.7) as $u_t(i,0) = \lambda H(U_{t-1}(i)) + u_{t-1}(i,0)$ for $t \geq 1$ and $i \geq 0$, from which we get

$$U_t(i) = \lambda\big(H(U_{t-1}(i)) - H(U_{t-1}(i-1))\big) + U_{t-1}(i), \qquad t \geq 1, \quad i \geq 1. \tag{A.4}$$

Further, from Eqs. (4.2) and (A.3) we have $U_0(i) = \rho \leq M = U_0(0) \cdots (1^*)$ for $i \geq 1$. Suppose $U_{t-1}(i) \leq M$ for $i \geq 1$. Then from Lemma A.4(c,b) we have $\lambda H(U_{t-1}(i)) + U_{t-1}(i) \leq \lambda H(M) + M = M$ for $i \geq 1$. Since $H(U_{t-1}(i-1)) \geq 0 \cdots (2^*)$ for $i \geq 1$ due to Lemma A.4(b), from Eq. (A.4) we have $U_t(i) \leq M$ for $i \geq 1$. Thus, by induction we have $U_t(i) \leq M = U_t(0) \cdots (3^*)$ for $t \geq 0$ and $i \geq 1$.

(a) The former half of the assertion is clearly true for $t = 0$ due to (1^*). Suppose $U_{t-1}(i)$ is nonincreasing in $i \geq 0$. Noting that $U_t(1) \leq U_t(0)$ from (3^*), let us prove that $U_t(i) \leq U_t(i-1)$ for $i \geq 2$. Now from Eq. (A.4) we get

$$\begin{aligned} U_t(i) - U_t(i-1) = \lambda\big(&H(U_{t-1}(i)) - H(U_{t-1}(i-1)) + H(U_{t-1}(i-2)) - H(U_{t-1}(i-1))\big) \\ &+ U_{t-1}(i) - U_{t-1}(i-1) \cdots (5^*), \qquad i \geq 2. \end{aligned}$$

Since $U_{t-1}(i) \leq U_{t-1}(i-1) \leq U_{t-1}(i-2) \cdots (4^*)$ for $t \geq 1$, noting (4^*) and Lemma A.4(d,a), we see that $H(U_{t-1}(i)) - H(U_{t-1}(i-1)) \leq U_{t-1}(i-1) - U_{t-1}(i) \cdots (6^*)$ for $t \geq 1$ and that $H(U_{t-1}(i-2)) - H(U_{t-1}(i-1)) \leq 0 \cdots (7^*)$ for $t \geq 1$. Accordingly, from (5^*), (6^*), (7^*), and (4^*) we get

$$\begin{aligned} U_t(i) - U_t(i-1) &\leq \lambda(U_{t-1}(i-1) - U_{t-1}(i)) + U_{t-1}(i) - U_{t-1}(i-1) \\ &= (1-\lambda)(U_{t-1}(i) - U_{t-1}(i-1)) \leq 0, \end{aligned}$$

hence $U_t(i) \leq U_t(i-1)$ for $i \geq 2$. Thus by induction the former half of the assertion holds. Now from Eq. (A.3) we see that the later half of the assertion holds for $i = 0$. Let $i \geq 1$. Then $U_{t-1}(i) \leq U_{t-1}(i-1)$ for any $t \geq 1$, hence $H(U_{t-1}(i)) \geq H(U_{t-1}(i-1))$ for any $t \geq 1$ due to Lemma A.4(a). Thus $U_t(i) \geq U_{t-1}(i)$ for any $t \geq 1$ from Eq. (A.4), so the assertion holds for $i \geq 1$.

(b) Since $U_t(i)$ is upper bounded in i and t due to (3^*), it follows that $U_t(i)$ converges to a finite $U(i)$ for $i \geq 0$ as $t \to \infty$. Hence, from Eq. (A.4) we can easily show that $U(i) = \lambda\big(H(U(i)) - H(U(i-1))\big) + U(i)$, from which $H(U(i)) = H(U(i-1))$ for $i \geq 1$. Since $H(U(0)) = H(M) = 0$ from Eq. (A.3) and Lemma A.4(b), we have $H(U(i)) = 0$ for $i \geq 0$. Thus, from Lemma A.4(b) we obtain $U(i) \geq b$ for $i \geq 0$. ∎

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