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NOTE ON THE CONTINUITY OF M-CONVEX AND L-CONVEX FUNCTIONS IN CONTINUOUS VARIABLES

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Abstract M-convex and L-convex functions in continuous variables constitute subclasses of convex functions with nice combinatorial properties. In this note we give proofs of the fundamental facts that closed proper M-convex and L-convex functions are continuous on their effective domains.

Keywords: Combinatorial optimization, convex function, continuity, submodular function, matroid.

1. Introduction

Two kinds of convexity concepts, called M-convexity and L-convexity, play primary roles in the theory of discrete convex analysis [6]. They are originally introduced for functions in integer variables by Murota [4, 5], and then for functions in continuous variables by Murota–Shioura [8, 10].

M-convex and L-convex functions in continuous variables constitute subclasses of convex functions with additional combinatorial properties such as submodularity and diagonal dominance (see, e.g., [6–11]). Fundamental properties of M-convex and L-convex functions are investigated in [9], such as equivalent axioms, subgradients, directional derivatives, etc. Conjugacy relationship between M-convex and L-convex functions under the Legendre-Fenchel transformation is shown in [10]. Subclasses of M-convex and L-convex functions are investigated in [8] (polyhedral M-convex and L-convex functions) and in [11] (quadratic M-convex and L-convex functions). As variants of M-convex and L-convex functions, the concepts of $M^\natural$-convex and $L^\natural$-convex functions are also introduced by Murota–Shioura [8, 10], where "$M^\natural$" and "$L^\natural$" should be read "M-natural" and "L-natural," respectively.

M-convex and L-convex functions in continuous variables appear naturally in various research areas. In inventory theory, a recent paper of Zipkin [13] sheds a new light on some classical results of Karlin–Scarf [2] and Morton [3] by pointing out that the optimal-cost function possesses $L^\natural$-convexity. Quadratic $L^\natural$-convex functions are exactly the same as the (finite dimensional case of) Dirichlet forms used in probability theory [1]. It is shown in [7, Section 14.8] that for (the finite dimensional distribution of) stochastic processes such as Gaussian processes and additive processes, cumulant generating functions and rate functions are $M^\natural$-convex and $L^\natural$-convex, respectively. The energy consumed in a nonlinear electrical network is an $L^\natural$-convex function when expressed as a function in terminal voltages, and is an $M^\natural$-convex function as a function in terminal currents [6, Section 2.2].

In this note, we discuss continuity issues of M-convex and L-convex functions in continuous variables. Although continuity is one of the most fundamental properties of functions, discussion on continuity is missing in the literature of M-convex and L-convex functions.

The aim of this note is to give proofs of the facts that closed proper M-convex and L-convex functions are continuous on their effective domains. The main results of this note are summarized as follows, where the precise definitions of closed proper M-convex and L-convex functions are given in Section 2.1.

Theorem 1.1. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.
(i) *If* f *is closed proper M-convex, then it is continuous on* $\mathrm{dom}\, f$.
(ii) *If* f *is closed proper* $M^\natural$*-convex, then it is continuous on* $\mathrm{dom}\, f$.

Theorem 1.2. *Let* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.
(i) *If* g *is closed proper L-convex, then it is continuous on* $\mathrm{dom}\, g$.
(ii) *If* g *is closed proper* $L^\natural$*-convex, then it is continuous on* $\mathrm{dom}\, g$.

It may be mentioned that our proof of Theorem 1.2 shows that an L-convex ($L^\natural$-convex) function is upper semi-continuous even if it is not closed.

2. Preliminaries

2.1. M-convex and L-convex functions

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function. A function f is said to be *convex* if its epigraph $\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\}$ is a convex set. A convex function f is said to be *proper* if the effective domain $\mathrm{dom}\, f$ of f given by $\mathrm{dom}\, f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$ is nonempty, and *closed* if its epigraph is a closed set.

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *M-convex* if it is convex and satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ satisfying

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \alpha_0]),$$

where $\chi_i \in \{0, 1\}^n$ denotes the characteristic vector of $i \in N = \{1, 2, \ldots, n\}$, and

$$\begin{aligned} \mathrm{supp}^+(x - y) &= \{i \in N \mid x(i) > y(i)\}, \\ \mathrm{supp}^-(x - y) &= \{i \in N \mid x(i) < y(i)\}. \end{aligned}$$

We call a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $M^\natural$*-convex* if the function $\widehat{f} : \mathbb{R}^{\widehat{N}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & ((x_0, x) \in \mathbb{R}^{\widehat{N}},\ x_0 = -x(N)), \\ +\infty & (\text{otherwise}) \end{cases} \tag{2.1}$$

is M-convex, where $\widehat{N} = \{0\} \cup N$ and $x(N) = \sum_{i \in N} x(i)$. An M-convex (resp., $M^\natural$-convex) function is said to be *closed proper M-convex* (resp., *closed proper* $M^\natural$*-convex*) if it is closed and proper, in addition.

A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *L-convex* if it is convex and satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q)$ $(\forall p, q \in \mathrm{dom}\, g)$,

(LF2) $\exists r \in \mathbb{R}$: $g(p + \alpha \mathbf{1}) = g(p) + \alpha r$ $(\forall p \in \mathrm{dom}\, g,\ \forall \alpha \in \mathbb{R})$,

where $p \wedge q, p \vee q \in \mathbb{R}^n$ are given by

$$(p \wedge q)(i) = \min\{p(i), q(i)\}, \quad (p \vee q)(i) = \max\{p(i), q(i)\} \quad (i \in N),$$

and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. We call a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $L^\natural$*-convex* if the function $\widehat{g} : \mathbb{R}^{\widehat{N}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p_0, p) = g(p - p_0 \mathbf{1}) \quad ((p_0, p) \in \mathbb{R}^{\widehat{N}})$$

is L-convex, where $\widehat{N} = \{0\} \cup N$. An L-convex (resp., L$^\natural$-convex) function is said to be *closed proper L-convex* (resp., *closed proper L$^\natural$-convex*) if it is closed and proper, in addition.

2.2. Basic facts from convex analysis

As technical preliminaries we describe some facts known in convex analysis. This also serves to illustrate the present issue.

Let S be a subset of $\mathbb{R}^n$. The *affine hull* $\mathrm{aff}(S)$ of S is given by

$$\mathrm{aff}(S) = \{\sum_{j=1}^{m} \lambda_j x_j \mid m : \text{positive integer},\ x_j \in S,\ \lambda_j \in \mathbb{R}\ (j = 1, 2, \ldots, m),\ \sum_{j=1}^{m} \lambda_j = 1\}.$$

We denote by $\mathrm{cl}(S)$ the closure of S, i.e., the smallest closed set containing S. The *relative interior* $\mathrm{ri}(S)$ of S is given as the set of vectors $x \in S$ such that there exists a sufficiently small $\varepsilon > 0$ satisfying

$$\{y \in \mathbb{R}^n \mid ||y - x|| \leq \varepsilon\} \cap \mathrm{aff}(S) \subseteq S.$$

The *relative boundary* of S is given by the set $\mathrm{cl}(S) \setminus \mathrm{ri}(S)$.

Theorem 2.1 ([12, Theorem 10.1]). *Any convex function is continuous on the relative interior of the effective domain.*

Theorem 2.1 implies, in particular, that a convex function is continuous on the effective domain if the effective domain is an open set.

On the other hand, a convex function is not necessarily continuous at relative boundary points of the effective domain, even if it is closed proper convex, as shown in the following example.

Example 2.2 ([12, Section 10]). Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be a function defined by

$$f(x, y) = \begin{cases} \dfrac{y^2}{2x} & (x > 0), \\ 0 & (x = y = 0), \\ +\infty & (\text{otherwise}), \end{cases}$$

which is closed proper convex since its epigraph $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq f(x, y)\}$ is a closed convex set. It is easy to see that f is continuous at every point of $\mathrm{dom}\, f$, except at the origin $(x, y) = (0, 0)$. For any positive number α, we have

$$\lim_{y \downarrow 0} f\bigl(\frac{y^2}{2\alpha}, y\bigr) = \lim_{y \downarrow 0} \alpha = \alpha \neq 0 = f(0, 0),$$

which shows that f is not continuous at the origin. □

A sufficient condition for a closed proper convex function to be continuous on the effective domain is given in terms of "locally simplicial" sets. A subset S of $\mathbb{R}^n$ is said to be *locally simplicial* if for each $x \in S$ there exists a finite collection of simplices $T_1, T_2, \ldots, T_m$ contained in S such that

$$U \cap (T_1 \cup T_2 \cup \cdots \cup T_m) = U \cap S$$

for some neighborhood U of x. The class of locally simplicial sets includes line segments, polyhedra, and relatively open convex sets.

Theorem 2.3 ([12, Theorem 10.2]). *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function. For a locally simplicial set $S \subseteq \mathrm{dom}\, f$, the function f is continuous on S. In particular, f is continuous on $\mathrm{dom}\, f$ if $\mathrm{dom}\, f$ is locally simplicial.*

3. Continuity of Closed Proper M-/L-convex Functions

We now consider the continuity of closed proper M-/L-convex functions.

The effective domains of closed proper M-/L-convex functions are "essentially polyhedral" in the sense that the closure of the effective domains are polyhedra (see Theorems 3.2 and 3.3 below). Hence, the continuity of closed proper M-/L-convex functions follows from Theorem 2.3 when the effective domains are closed sets. The effective domains of closed proper M-/L-convex functions, however, are not necessarily closed, as shown in the following example.

Example 3.1. Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function defined by

$$\varphi(x) = \begin{cases} \dfrac{1}{x} & (0 < x \leq 1), \\ +\infty & (\text{otherwise}). \end{cases}$$

Then, φ is a closed proper convex function such that the effective domain $\mathrm{dom}\,\varphi$ is an interval $\{x \in \mathbb{R} \mid 0 < x \leq 1\}$, which is neither a closed set nor a relatively open set.

Using φ we define functions $f, g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$\begin{aligned} f(x,y) &= \begin{cases} \varphi(x) & (x+y=0), \\ +\infty & (x+y \neq 0), \end{cases} \qquad ((x,y) \in \mathbb{R}^2), \\ g(x,y) &= \varphi(x-y) \qquad ((x,y) \in \mathbb{R}^2). \end{aligned}$$

Then, f and g are closed proper M-convex and L-convex functions, respectively. Neither $\mathrm{dom}\,f$ nor $\mathrm{dom}\,g$ is a closed set. □

Although the effective domains are not always closed, they are well-behaved and almost polyhedral, as follows.

A polyhedron $S \subseteq \mathbb{R}^n$ is said to be *M-convex* (resp., *M$^\natural$-convex, L-convex, L$^\natural$-convex*) if the indicator function $\delta_S : \mathbb{R}^n \to \{0, +\infty\}$ defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S) \end{cases}$$

is M-convex (resp., M$^\natural$-convex, L-convex, L$^\natural$-convex).

Theorem 3.2. *For any closed proper M-convex (resp., M$^\natural$-convex) function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the set $\mathrm{cl}(\mathrm{dom}\,f)$ is an M-convex (resp., M$^\natural$-convex) polyhedron.*

Proof. The proof is given in Section 4.1. □

Theorem 3.3. *For any closed proper L-convex (resp., L$^\natural$-convex) function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the set $\mathrm{cl}(\mathrm{dom}\,g)$ is an L-convex (resp., L$^\natural$-convex) polyhedron.*

Proof. The proof is given in Section 4.2. □

Theorem 3.4. *The effective domain of a closed proper M-convex (resp., M$^\natural$-convex) function is a locally simplicial set.*

Proof. The proof is given in Section 4.3. □

Theorem 3.5. *The effective domain of a closed proper L-convex (resp., L$^\natural$-convex) function is a locally simplicial set.*

Proof. The proof is given in Section 4.4. □

The continuity of closed proper M-/L-convex functions, as claimed in Theorems 1.1 and 1.2, follows from Theorems 2.3, 3.4, and 3.5.

4. Proofs

4.1. Proof of Theorem 3.2

For any closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define a function $f0^+ : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$(f0^+)(y) = \lim_{\lambda\to\infty} \frac{f(x + \lambda y) - f(x)}{\lambda} \quad (y \in \mathbb{R}^n),$$

where $x \in \mathbb{R}^n$ is any fixed vector in $\mathrm{dom}\, f$. The function $f0^+$ is called the *recession function* of f (see [12] for the original definition of the recession function). The recession function $f0^+$ is a positively homogeneous closed proper convex function, i.e., $f0^+$ is closed proper convex and satisfies $(f0^+)(\lambda x) = \lambda(f0^+)(x)$ for every $x \in \mathbb{R}^n$ and $\lambda > 0$. Our proof of Theorem 3.2 is based on the following fact, where for a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the *conjugate* $f^\bullet$ of f is given by

$$f^\bullet(p) = \sup\{p^{\mathrm{T}}x - f(x) \mid x \in \mathrm{dom}\, f\} \qquad (p \in \mathbb{R}^n).$$

Theorem 4.1 ([12, Theorem 13.3]). *For any closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the recession function $f0^+$ is the support function of* $\mathrm{dom}\, f^\bullet$*, i.e., it holds that*

$$(f0^+)(x) = \sup\{p^{\mathrm{T}}x \mid p \in \mathrm{dom}\, f^\bullet\} \qquad (x \in \mathbb{R}^n).$$

It suffices to consider a closed proper M-convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, its conjugate function $g = f^\bullet$ is a closed proper L-convex function [10, Theorem 1.1]. As shown below, the recession function $g0^+$ of g is L-convex. This implies that the support function of (the closure of) $\mathrm{dom}\, f^\bullet$ is a positively homogeneous L-convex function, which in turn implies that $\mathrm{cl}(\mathrm{dom}\, f^\bullet)$ is an M-convex polyhedron [8, Theorem 4.38].

We now show the L-convexity of the recession function $g0^+$. Namely, we prove that $g0^+$ satisfies (LF1) and (LF2).

Let $p_0 \in \mathrm{dom}\, g$ be any fixed vector. Then, the recession function $g0^+$ is given as

$$(g0^+)(p) = \lim_{\lambda\to\infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} \quad (p \in \mathbb{R}^n).$$

Since g satisfies (LF2), there exists $r \in \mathbb{R}$ such that

$$g(p + \alpha\mathbf{1}) = g(p) + \alpha r \quad (\forall p \in \mathrm{dom}\, g,\ \forall \alpha \in \mathbb{R}). \tag{4.1}$$

For any $p \in \mathrm{dom}\, g0^+$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}(g0^+)(p + \alpha\mathbf{1}) &= \lim_{\lambda\to\infty} \frac{g(p_0 + \lambda(p + \alpha\mathbf{1})) - g(p_0)}{\lambda} \\ &= \lim_{\lambda\to\infty} \frac{g(p_0 + \lambda p) + \lambda\alpha r - g(p_0)}{\lambda} \\ &= \lim_{\lambda\to\infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} + \alpha r \\ &= (g0^+)(p) + \alpha r,\end{aligned}$$

where the second equality is by (4.1). Hence, (LF2) holds for $g0^+$.

Let $p, q \in \mathrm{dom}\, g0^+$. For any $\lambda \in \mathbb{R}_+$, we have

$$g(p_0 + \lambda p) + g(p_0 + \lambda q) \geq g(p_0 + \lambda(p \wedge q)) + g(p_0 + \lambda(p \vee q))$$

by (LF1) for g. Hence, we have

$$\begin{aligned} & g0^+(p) + g0^+(q) \\ = \ & \lim_{\lambda\to\infty} \frac{g(p_0+\lambda p) - g(p_0)}{\lambda} + \lim_{\lambda\to\infty} \frac{g(p_0+\lambda q) - g(p_0)}{\lambda} \\ \geq \ & \lim_{\lambda\to\infty} \frac{g(p_0+\lambda(p\wedge q)) - g(p_0)}{\lambda} + \lim_{\lambda\to\infty} \frac{g(p_0+\lambda(p\vee q)) - g(p_0)}{\lambda} \\ = \ & g0^+(p\wedge q) + g0^+(p\vee q), \end{aligned}$$

i.e., (LF1) holds for $g0^+$.

4.2. Proof of Theorem 3.3

It suffices to consider a closed proper L-convex function $g : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. The properties (LF1) and (LF2) for g imply that $D = \mathrm{dom}\, g$ satisfies the following properties:

(LS1) $p,\ q \in D \Longrightarrow p\wedge q,\ p\vee q \in D$,

(LS2) $p \in D \Longrightarrow p + \lambda\mathbf{1} \in D$ $(\forall\lambda\in\mathbb{R})$.

Therefore, Theorem 3.3 follows immediately from the next theorem.

Theorem 4.2. *For any nonempty set $D \subseteq \mathbb{R}^n$, let*

$$\begin{aligned} \gamma_D(i,j) &= \sup\{p(j) - p(i) \mid p \in D\} \qquad (i,j\in N), \\ \widetilde{D} &= \{p\in\mathbb{R}^n \mid p(j) - p(i) \leq \gamma_D(i,j)\ (i,j\in N)\}. \end{aligned}$$

If D satisfies (LS1) *and* (LS2)*, then we have* $\mathrm{cl}(D) = \widetilde{D}$.

Proof. The inclusion $\mathrm{cl}(D) \subseteq \widetilde{D}$ is easy to see. To prove the reverse inclusion, we show that $q\in D$ holds for any vector q in the relative interior of $\widetilde{D}$.

We first show that for any $i,j\in N$ there exists $p_{ij}\in D$ such that

$$p_{ij}(j) - p_{ij}(i) \geq q(j) - q(i).$$

If $-\gamma_D(j,i) = \gamma_D(i,j)$, then any vector in D can be chosen as p_{ij} since for any $p\in D$ we have $p(j)-p(i) = \gamma_D(i,j) = q(j)-q(i)$. Hence, we suppose that $-\gamma_D(j,i) < \gamma_D(i,j)$ holds. Then, we have $q(j)-q(i) < \gamma_D(i,j)$ since q is in the relative interior of $\widetilde{D}$. By the definition of $\gamma_D(i,j)$, there exists some $p_{ij}\in D$ such that $q(j)-q(i) \leq p_{ij}(j)-p_{ij}(i) \leq \gamma_D(i,j)$.

By (LS2), we may assume that $p_{ij}(i) = q(i)$ and $p_{ij}(j) \geq q(j)$. For each $i\in N$, the vector $p_i = p_{i1}\vee p_{i2}\vee\cdots\vee p_{in}$ $(\in D)$ satisfies $p_i(i) = q(i)$, $p_i(j)\geq q(j)$ for all $j\in N$. Therefore, it holds that $q = p_1\wedge p_2\wedge\cdots\wedge p_n \in D$. □

4.3. Proof of Theorem 3.4

For any set $S\subseteq\mathbb{R}^n$ and a vector $x\in S$, we denote by $\mathrm{cone}(S,x)$ the conic hull of the vectors $\{y-x \mid y\in S\}$, i.e., $\mathrm{cone}(S,x)$ is the set of vectors $d\in\mathbb{R}^n$ such that $d=\sum_{k=1}^m \alpha_k(y_k-x)$ for some positive integer m and $y_k\in S$, $\alpha_k > 0$ $(k=1,2,\ldots,m)$. The following is immediate from the definition of locally simplicial sets.

Lemma 4.3. *A convex set $S\subseteq\mathbb{R}^n$ is locally simplicial if for each $x\in S$,* $\mathrm{cone}(S,x)$ *is a polyhedral cone.*

For the proof of Theorem 3.4 it suffices to consider an M-convex function. Then, Theorem 3.4 follows from Lemma 4.3 and the following lemma.

Lemma 4.4. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a closed proper M-convex function. For any* $x \in \mathrm{dom}\, f$*, it holds that*

$$\mathrm{cone}(\mathrm{dom}\, f, x) = \mathrm{cone}(R_x, x),$$

where $R_x \subseteq \mathbb{R}^n$ *is a polyhedral cone given by*

$$R_x = \{\chi_j - \chi_i \mid i, j \in N,\ i \neq j,\ x + \alpha(\chi_j - \chi_i) \in \mathrm{dom}\, f \text{ for some } \alpha > 0\}.$$

To prove Lemma 4.4 we use the following properties of M-convex functions.

Lemma 4.5 ([10, Proposition 2.2])**.** *If* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is closed proper M-convex, then* $x(N) = y(N)$ *for all* $x, y \in \mathrm{dom}\, f$*.*

Lemma 4.6 ([10, Theorem 3.11])**.** *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a closed proper convex function. Then,* f *satisfies* (M-EXC) *if and only if it satisfies* (M-EXC$_\mathrm{s}$)*:*

(M-EXC$_\mathbf{s}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \alpha_0(x, y, i)]),$$

where $\alpha_0(x, y, i)$ *is the number given by*

$$\alpha_0(x, y, i) = \frac{x(i) - y(i)}{2|\mathrm{supp}^-(x - y)|}$$

and satisfies $\alpha_0(x, y, i) \leq \{y(j) - x(j)\}/2$*.*

Proof of Lemma 4.4. It is easy to see that $\mathrm{cone}(R_x, x) \subseteq \mathrm{cone}(\mathrm{dom}\, f, x)$. To prove the reverse inclusion, it suffices to show that $y - x \in \mathrm{cone}(R_x, x)$ for any $y \in \mathrm{dom}\, f$.

We will show that there exists a sequence of vectors y_k $(k = 0, 1, 2, \ldots)$ such that $y_0 = y$ and

$$y_k \in \mathrm{dom}\, f,\ y_k \neq x,\ y - y_k \in \mathrm{cone}(R_x, x) \quad (k = 0, 1, 2, \ldots), \tag{4.2}$$

$$||y_{k+1} - x||_1 \leq (1 - \frac{1}{2n^2})||y_k - x||_1. \tag{4.3}$$

This implies that $y - x = \lim_{k\to\infty}(y - y_k) \in \mathrm{cone}(R_x, x)$, since $\mathrm{cone}(R_x, x)$ is a closed set.

We define the vectors y_k $(k = 0, 1, 2, \ldots)$ iteratively as follows. Suppose that y_k is already defined and satisfies the condition (4.2). Since $y_k \neq x$, we have $\mathrm{supp}^+(y_k - x) \neq \emptyset$. Let $i \in \mathrm{supp}^+(y_k - x)$ be such that

$$y_k(i) - x(i) = \max\{y_k(i') - x(i') \mid i' \in \mathrm{supp}^+(y_k - x)\}. \tag{4.4}$$

By Lemma 4.6, there exists $j \in \mathrm{supp}^-(y_k - x)$ such that

$$y_k - \alpha(\chi_i - \chi_j) \in \mathrm{dom}\, f,\ x + \alpha(\chi_i - \chi_j) \in \mathrm{dom}\, f,$$

where $\alpha = (y_k(i) - x(i))/2n$. Then, y_{k+1} is defined as $y_{k+1} = y_k - \alpha(\chi_i - \chi_j)$.

We now show that the vector y_{k+1} satisfies the conditions (4.2) and (4.3). Since $y_{k+1}(i) > x(i)$, we have $y_{k+1} \neq x$. Since $x + \alpha(\chi_i - \chi_j) \in \mathrm{dom}\, f$, we have $\chi_i - \chi_j \in R_x$, which, together with $y - y_k \in \mathrm{cone}(R_x, x)$, implies

$$y - y_{k+1} = (y - y_k) + \alpha(\chi_i - \chi_j) \in \mathrm{cone}(R_x, x).$$

Since $y_k(N) = x(N)$ by Lemma 4.5, it holds that

$$\begin{aligned} ||y_k - x||_1 &= 2\sum\{y_k(i') - x(i') \mid i' \in \mathrm{supp}^+(y_k - x)\} \\ &\leq 2n(y_k(i) - x(i)) \\ &= 4n^2\alpha, \end{aligned}$$

where the inequality is by (4.4). Hence, it holds that

$$||y_{k+1} - x||_1 = ||y_k - x||_1 - 2\alpha \leq (1 - \frac{1}{2n^2})||y_k - x||_1.$$

□

4.4. Proof of Theorem 3.5

Theorem 3.5 follows from Lemma 4.3 and Lemma 4.7 below. Note that it suffices to consider an L-convex function.

Lemma 4.7. *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an L-convex function. For any $p \in \mathrm{dom}\, g$, it holds that*

$$\mathrm{cone}(\mathrm{dom}\, g, p) = \mathrm{cone}(R_p, p),$$

where $R_p \subseteq \mathbb{R}^n$ is a polyhedral cone given by

$$R_p = \{\chi_X \mid X \subset N,\ p + \alpha\chi_X \in \mathrm{dom}\, g \text{ for some } \alpha > 0\} \cup \{+\mathbf{1}, -\mathbf{1}\}.$$

To prove Lemma 4.7, we use the following property of L-convex functions.

Lemma 4.8 ([9, Proposition 3.10])**.** *If $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is L-convex, then we have*

$$g(p) + g(q) \geq g(p + \lambda\chi_X) + g(q - \lambda\chi_X)$$

for all $p, q \in \mathrm{dom}\, g$ and $\lambda \in [0, \lambda_1 - \lambda_2]$, where $\chi_X \in \{0, 1\}^n$ denotes the characteristic vector of $X \subseteq N$, and

$$\begin{aligned} \lambda_1 &= \max\{q(i) - p(i) \mid i \in N\}, \\ X &= \{i \in N \mid q(i) - p(i) = \lambda_1\}, \\ \lambda_2 &= \max\{q(i) - p(i) \mid i \in N \setminus X\}. \end{aligned}$$

Proof of Lemma 4.7. It is easy to see that $\mathrm{cone}(R_p, p) \subseteq \mathrm{cone}(\mathrm{dom}\, g, p)$, where it is noted that $p + \alpha\mathbf{1} \in \mathrm{dom}\, g$ for all $\alpha \in \mathbb{R}$. To show the reverse inclusion, it suffices to show that $q - p \in \mathrm{cone}(R_p, p)$ for any $q \in \mathrm{dom}\, g$.

Since both of the sets $\mathrm{dom}\, g$ and $\mathrm{cone}(R_p, p)$ satisfy the property (LS2) (see Section 4.2 for the definition of (LS2)), we may assume that $p \leq q$ and $p(i_0) = q(i_0)$ for some $i_0 \in N$. We prove $q - p \in \mathrm{cone}(R_p, p)$ by induction on the number m of distinct values in $\{q(i) - p(i) \mid i \in N\}$.

If $m = 0$, then we have $q - p = \mathbf{0} \in \mathrm{cone}(R_p, p)$. Hence, we assume $m > 0$, which implies $q(i_1) > p(i_1)$ for some $i_1 \in N$. By Lemma 4.8, we have

$$p + (\lambda_1 - \lambda_2)\chi_X \in \mathrm{dom}\, g,\ q - (\lambda_1 - \lambda_2)\chi_X \in \mathrm{dom}\, g,$$

where

$$\begin{aligned} \lambda_1 &= \max\{q(i) - p(i) \mid i \in N\}, \\ X &= \{i \in N \mid q(i) - p(i) = \lambda_1\}, \\ \lambda_2 &= \max\{q(i) - p(i) \mid i \in N \setminus X\}. \end{aligned}$$

We note that λ_1 and λ_2 are finite values and X is a nonempty proper subset of N. Put $\widetilde{q} = q - (\lambda_1 - \lambda_2)\chi_X$. Then, the number of distinct values in $\{\widetilde{q}(i) - p(i) \mid i \in N\}$ is equal to $m - 1$. Therefore, the induction hypothesis implies $\widetilde{q} - p \in \mathrm{cone}(R_p, p)$. We also have $\chi_X \in R_p$ since $p + (\lambda_1 - \lambda_2)\chi_X \in \mathrm{dom}\, g$. Hence, it holds that

$$q - p = (\widetilde{q} - p) + (\lambda_1 - \lambda_2)\chi_X \in \mathrm{cone}(R_p, p).$$

□

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