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PREFERENCE VOTING AND RANKING USING DEA GAME CROSS EFFICIENCY MODEL

Jie Wu Liang Liang
University of Science and Technology of China

Yingchun Zha
Anhui University

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Abstract Data envelopment analysis (DEA), a useful assessment tool, has been used to solve the problem of preference voting and aggregation which require the determination of the weights associated with different ranking places. Instead of applying the same externally imposed weighting scheme to all candidates, DEA models allow each candidate to choose his/her own weights in order to maximize his/her own overall ratings subject to certain conditions. It is evident that competition exists among the candidates in a preferential election, while there is no literature considering the factor of competition. This paper proposes an approach to rank candidates based on DEA game cross efficiency model, in which each candidate is viewed as a player who seeks to maximize its own efficiency, under the condition that the cross efficiencies of each of other DMU's does not deteriorate. The game cross efficiency score is obtained when the DMU's own maximized efficiencies are averaged. The obtained game cross efficiency scores constitute a Nash Equilibrium point. Therefore, the results and orders based upon game cross efficiency analysis are more reliable and will benefit the decision-maker.

Keywords: DEA, preference voting, preference aggregation, cross efficiency, game

1. Introduction

In a preferential voting system, each voter selects a subset of the candidates and places them in a ranked order. The key issue of the preference aggregation in a preferential voting system is how to determine the weights associated with different ranking places. To avoid the subjectivity in determining the weights, data envelopment analysis (DEA) is used in Cook and Kress [1] to determine the most favorable weights for each candidate. Different candidates utilize different sets of weights to calculate their total scores, which are referred to as the best relative total scores and are all restricted to be less than or equal to one. The candidate with the biggest relative total score of one is said to be DEA efficient and may be considered as a winner. This approach proves to be effective, but very often leads to more than one candidate to be DEA efficient. To choose a winner from among the DEA-efficient candidates, Cook and Kress [1] suggest maximizing the gap between the weights so that only one candidate is left as DEA efficient. Green, Doyle and Cook [2] suggest using the cross-efficiency evaluation method in DEA to choose the winner. Noguchi, Ogawa and Ishii [3] also utilize cross-efficiency evaluation technique to select the winner, but present a strong ordering constraint on the weights. Hashimoto [4] proposes the use of the DEA exclusion model (i.e. super-efficiency model) to identify the winner. Obata and Ishii [5] suggest excluding non-DEA-efficient candidates and using normalized weights to discriminate the DEA-efficient candidates. Their method is subsequently extended to rank non-DEA-efficient candidates by Foroughi and Tamiz [6] and Foroughi, Jones and Tamiz [7]. Recently, Wang, Chin and Yang [8] also propose three new models to assess the weights and rank the candidates. In fact, candidates in a preferential election setting can be viewed as DMUs, and competition is obviously present among the candidates. Unfortunately, there is no literature discussed above considering this topic. In this paper, DEA game cross efficiency model, a useful assessment tool for considering the factor of competition, will be used to rank the candidates in a preferential election setting. The rest of the paper unfolds as follows: in section 2, the original models for preferential voting and aggregation are presented, and the approach based on DEA game cross efficiency model is proposed. A numerical example is illustrated in Section 3 and concluding remarks are given in Section 4.

2. Model

In the preferential voting framework each candidates i=1,...,n receives some number v_{i1} of first place votes, v_{i2} of second place votes..., v_{im} of mth place votes. The problem is to use these votes in a reasonable manner to obtain an overall *desirability index* Z_i for each candidate. In fact, for any predetermined set of weights w_1,...,w_m on each candidate's standing, the composite score, Z_i, of candidate i would be defined by:

$$Z_i = \sum_{j=1}^{m} w_j v_{ij} \tag{2.1}$$

Cook and Kress [1] propose a DEA model that allowed all candidates to choose his/her own weights in order to maximize his/her own *desirability index* subject to certain reasonable constraints on the desirabilities of all candidates. For candidate i of n candidates, the model of Cook and Kress [1] can be written as:

$$\max Z_{ii} = \sum_{j=1}^{m} \omega_{ij}\nu_{ij}$$

$$\begin{aligned} \text{s.t. } Z_{iq} = \sum_{j=1}^{m} \omega_{ij}\nu_{qj} \le 1, q = 1, \ldots, n, \\ \omega_{ij} - \omega_{i(j+1)} \ge d(j,\varepsilon), j = 1, \ldots, m-1, \\ \omega_{im} \ge d(m,\varepsilon), \\ d(\cdot,\varepsilon), \varepsilon \ge 0, d(\cdot,0) = 0, \end{aligned} \tag{2.2}$$

where $d(\cdot,\varepsilon)$ is called *the discrimination intensity function*, monotonic increasing in ε, and the parameter ε is called the *discriminating factor*. This model is solved for each candidate $i(i = 1, \ldots, n)$, and the resulting score Z_{ii}^* represents the preference score of the candidate i.

In model (2.2), the choice of form for $d(\cdot,\varepsilon)$ and the value of ε are two existing issues. For the *discrimination intensity function* $d(\cdot,\varepsilon)$, Cook and Kress [1] investigate three special cases of $d(\cdot,\varepsilon) : d(\cdot,\varepsilon) = \varepsilon, d(\cdot,\varepsilon) = \varepsilon/j$ and $d(\cdot,\varepsilon) = \varepsilon/j!$. Each of them leads to a different winner. Noguchi et al. [3] examine the six special cases of the *discriminating factor* $\varepsilon : \varepsilon = 0, 0.01, 0.05, 0.055, 0.06, 0.07$. These cases also resulted in different winners. To avoid the difficulties in determining the *discrimination intensity function* $d(\cdot,\varepsilon)$ and the *discrimination intensity factor* ε, Noguchi et al. [3] suggest a strong ordering DEA model as follows:

$$\max Z_{ii} = \sum_{j=1}^{m} \omega_{ij}\nu_{ij}$$

$$\text{s.t. } Z_{iq} = \sum_{j=1}^{m} \omega_{ij}\nu_{qj} \leq 1, q = 1, \ldots, n, \tag{2.3}$$
$$\omega_{i1} \geq 2\omega_{i2} \geq \ldots \geq m\omega_{im},$$
$$\omega_{im} \geq \varepsilon = \frac{2}{Nm(m+1)}$$

where N is the number of voters. In our view, the strong ordering constraint $\omega_{i1} \geq 2\omega_{i2} \geq \ldots \geq m\omega_{im}$ makes sense because it satisfies $\omega_{i1} > \omega_{i2} > \ldots > m\omega_{im}$ and $\omega_{i1} - \omega_{i2} > \omega_{i2} - \omega_{i3} > \ldots > \omega_{i(m-1)} - \omega_{im}$. It also makes the choice of the discrimination intensity function $d(\cdot, \varepsilon)$ unnecessary. Especially, as indicated in Noguchi et al. [3], the weights of each rank are determined in allowable region, and from the result of votes, the above strong ordering in Noguchi et al. [3] is superior to the method by Green et al. [2]. So, this strong ordering constraint will be adopted in the new models to be developed.

For each $DMU_i(i = 1, \ldots, n)$ under evaluation of model (2.3), we obtain a set of optimal weights $(\omega_{i1}^*, \omega_{i2}^*, \ldots, \omega_{im}^*)$. Using this set, the i-cross efficiency for any $DMU_p(p = 1, \ldots, n)$, is then calculated as:

$$E_{ip} = \sum_{j=1}^{m} \omega_{ij}^* \nu_{pj}, i, p = 1, 2, \ldots n \tag{2.4}$$

For $DMU_p(p = 1, \ldots, n)$, the average of all $E_{ip}(i = 1, \ldots, n)$, namely

$$\overline{E_p} = \frac{1}{n} \sum_{i=1}^{n} E_{ip} \tag{2.5}$$

can be used to determine the *cross efficiency score* for DMU_p.

In what follows, we will present our new model, which is based on the DEA game cross efficiency model presented in Liang, Wu, Cook and Zhu [9]. The new model is given as follows:

$$\max Z_{ii}^d = \sum_{j=1}^{m} \omega_{ij}^d \nu_{ij}$$
$$\text{s.t. } Z_{iq} = \sum_{j=1}^{m} \omega_{ij}^d \nu_{qj} \leq 1, q = 1, \ldots, n$$
$$Z_{id} = \sum_{j=1}^{m} \omega_{ij}^d \nu_{dj} \geq \alpha_d \tag{2.6}$$
$$\omega_{i1}^d \geq 2\omega_{i2}^d \geq \ldots \geq m\omega_{im}^d$$
$$\omega_{im}^d \geq \varepsilon = \frac{2}{Nm(m+1)}$$

where $\alpha_d \leq 1$ is a parameter. In the algorithm to be developed, this α_d initially takes the value given by the average original cross efficiency of DMU_d. We refer to model (2.6) as the DEA game d-cross efficiency model. Note that model (2.6) maximizes the efficiency of DMU_i under the condition that the efficiency of a given DMU_d, namely $\sum_{j=1}^{m} \omega_{ij}^d \nu_{dj}$, is not less than a given value (α_d). Thus, the efficiency of DMU_i is further constrained by the requirement that the ratio efficiency of DMU_d is not less than its original average cross efficiency.

Model (2.6) is solved n times for each DMU_i, one for each $d = 1, \ldots, n$. the optimal value to model (2.6) actually represents a game cross efficiency with respect to DMU_d (d-game cross efficiency). We have:

Definition: Let $\omega_{ij}^{d*}(\alpha_d)$ be an optimal solution to model (2.6). For each DMU_i, $\alpha_i = \frac{1}{n}\sum_{d=1}^{n}\sum_{j=1}^{m}\omega_{ij}^{d*}(\alpha_d)\nu_{ij}$ is called the average game cross efficiency for that DMU.

Note that the average game cross efficiency no longer represents a regular DEA cross-efficiency value. Liang, Wu, Cook and Zhu [9] presented a procedure for determining the final game cross efficiency for DMU_i.

Step 1: Solve model (2.3) and obtain a set of original average DEA cross efficiency scores defined in (2.5). Let $t = 1$ and $\alpha_d = \alpha_d^1 = E_d$.

Step 2: Solve model (2.6). Let $\alpha_i^2 = \frac{1}{n}\sum_{d=1}^{n}\sum_{j=1}^{m}\omega_{ij}^{d*}(\alpha_d^1)\nu_{ij}$ or in a general format,

$$\alpha_i^{t+1} = \frac{1}{n}\sum_{d=1}^{n}\sum_{j=1}^{m}\omega_{ij}^{d*}(\alpha_d^t)\nu_{ij} \tag{2.7}$$

where $\omega_{ij}^{d*}(\alpha_d^t)$ represents the optimal value of ω_{ij}^{d*} in model (2.6) when $\alpha_d = \alpha_d^t$.

Step 3: If $|\alpha_i^{t+1} - \alpha_i^t| \geq \delta$ for some i, where δ is a specified small positive value, then let $\alpha_d = \alpha_d^{t+1}$ and go to step 2. If $|\alpha_i^{t+1} - \alpha_i^t| < \delta$ for some i, then stop. α_i^{t+1} is the final game cross efficiency given to DMU_i.

As for the above algorithm, some remarks should be indicated as follows: In Step 1, the $\overline{E_d}$ represent traditional (average) cross-efficiency scores for $DMU_d, d = 1, 2, \ldots, n$, and are the initial values for α_d(denoted as α_d^1 in model (2.6). Also, the notation $\alpha_d = \alpha_d^t, t \geq 1$, given in Step 2, means that in model (2.6), α_d is replaced with α_d^t. Step 3 is used to indicate when to terminate the process of executing model (2.6).

In Liang, Wu, Cook and Zhu [9], this algorithm has been proven to be convergent and the game cross efficiency determined by the solution from the proposed algorithm above is a Nash Equilibrium point to the DEA game in which DMU is viewed as player and the game cross efficiencies are considered as the payoffs. The game cross efficiency score is a Nash equilibrium solution and therefore is a stable solution. Thus, the results and decisions based upon game cross efficiency analysis are reliable.

3. Numerical Example

In this section, we examine a numerical example using the proposed models to illustrate their use and show their capabilities of choosing the winner and ranking candidates.

We consider the example discussed in Cook and Kress [1], in which 20 voters are asked to rank four out of six candidates A-F on a ballot. For example, candidate "a" receives 3 first, 3 second, 4 third and 3 fourth-placed votes. The votes each candidate receives are shown in Table 1.

For this example, n = 6, m = 4 and N = 20, so the parameter of ε in model (2.6) is equal to 0.005. For the value of δ in the algorithm, we set $\delta = 0.0001$ and we use the regular cross efficiency defined in (2.4) as the starting point for our game cross efficiency scores. Cross efficiency is not unique and can be calculated by imposing a secondary goal. For example, we can use an aggressive strategy which not only obtains the maximum DEA efficiency for a DMU as the primary goal, but also as a secondary goal, minimizes the other DMUs' cross efficiencies (Sexton, Silkman and Hogan [10]). We can also use a benevolent

Table 1: Votes received by six candidates

Candidate	First place	Second place	Third place	Fourth place
A	3	4	4	3
B	4	5	6	2
C	6	2	3	2
D	6	2	2	6
E	0	4	3	4
F	1	4	3	3

strategy which not only obtains the maximum DEA efficiency but also maximizes the other DMUs' cross efficiencies (Doyle and Green [11]). The cross-efficiency calculated without imposing the secondary goal is referred to as an arbitrary strategy, as defined in (2.4). The

Table 2: Scores and rankings of the six candidates by different models

	Efficiency in Model(2.3)		Cross Efficiency in (2.5)						Game cross efficiency	
Candidate	Score	Rank	Aggressive	Rank	Arbitrary	Rank	Benevolent	Rank	Score	Rank
A	0.7376	4	0.6785	4	0.7229	4	0.7332	4	0.7359	4
B	1.0000	1	0.9172	3	0.9749	2	0.9909	2	0.9964	2
C	1.0000	1	0.9658	2	0.9692	3	0.9813	3	0.9876	3
D	1.0000	1	0.9901	1	0.9995	1	1	1	1	1
E	0.4364	6	0.3196	6	0.4034	6	0.4175	6	0.4241	6
F	0.5198	5	0.4366	5	0.5055	5	0.5162	5	0.5185	5

results of model (2.3) in Noguchi et al. [3] are reported in the second column of Table 2. The results of the cross-efficiency under three strategies are listed in the third column of Table 2. The game cross efficiency is shown in the last column. All these cross-efficiency scores lead to the same game cross efficiency scores. Figure 1 shows the solution process for

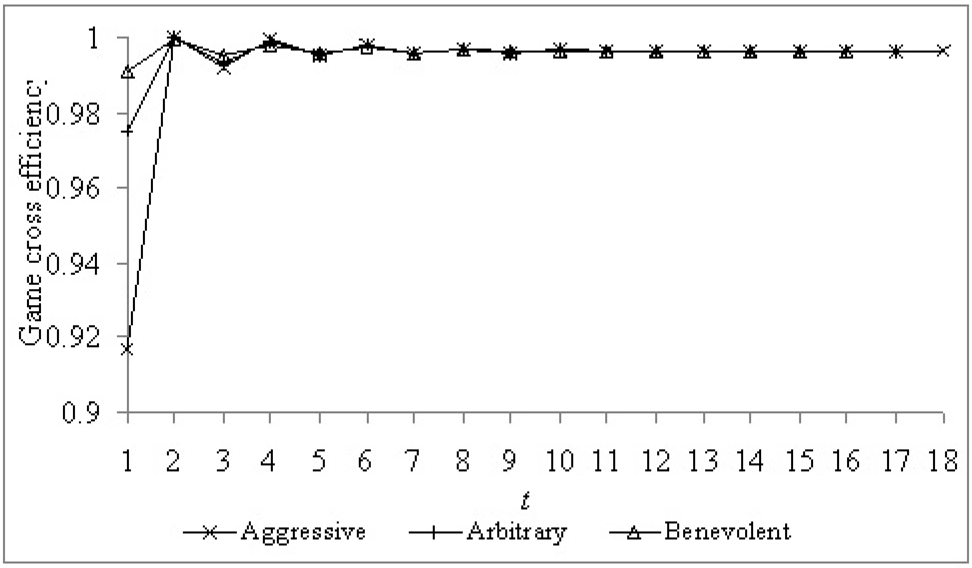

Figure 1: Achieving the game cross efficiency for candidate B

candidate B. If one views the candidates as competitive, it is noted that in a cooperative sense each "player" has an improved score over that which it received under the usual cross efficiency models (except in the case of the 100% efficient candidate D).

Figure 2 shows after 17 iterations, the proposed algorithm finds the game cross efficiency scores for the six candidates and it can be seen that the game cross efficiency score increases when t becomes an even number and decreases when t becomes an odd number.

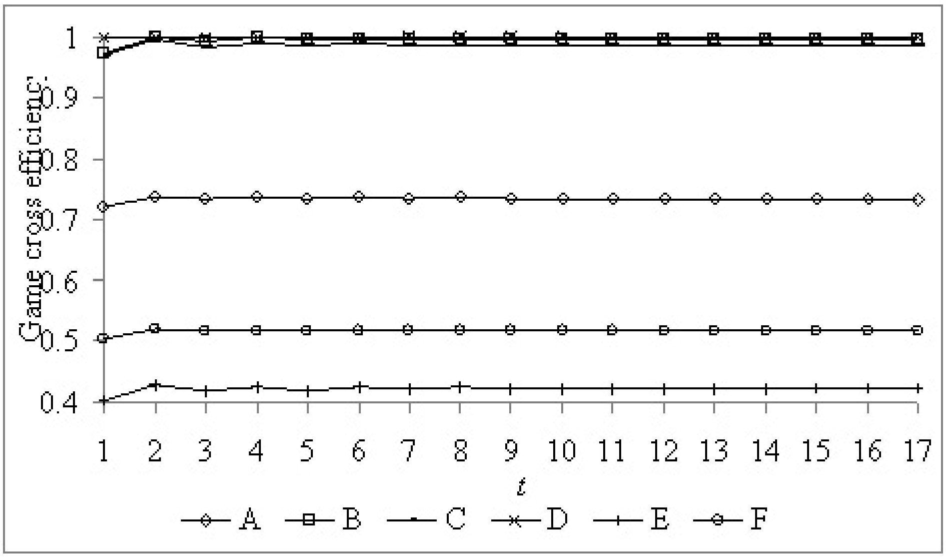

Figure 2: Game cross efficiency calculation for six candidates

We should point out that this example is chosen only for illustrative purposes and for better understanding of the principles of the proposed approach. The contribution of our approach lies in the fact that we find a solution that is a Nash Equilibrium and the solution is not affected by the multiple optimal solutions in the DEA models.

4. Conclusions

This paper has considered a preferential voting system using DEA game cross efficiency model, in which each candidate is viewed as a player that seeks to maximize its own efficiency, under the condition that the cross efficiencies of each of other DMU's does not deteriorate. In DEA game cross efficiency model, the game cross efficiency scores obtained are fixed and constitute a Nash Equilibrium point. Therefore, it is considered that the approach proposed provides an alternative method for determining an ordering of candidates in the preferential voting system.

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Jie Wu
School of Management
University of Science and Technology of China
Hefei, Anhui Province, P. R. China, 230026.
E-mail: `jacky012@mail.ustc.edu.cn`