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FAIR ALLOCATION BASED ON TWO CRITERIA: A DEA GAME VIEW OF "ADD THEM UP AND DIVIDE BY TWO"

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Abstract The phrase "add them up and divide by two (*Tashite Ni De Waru*, in Japanese)" is an established maxim in Japanese society, and has its historical roots in Japanese culture. This means adopting a middle course between two competing proposals in order to avoid a conflict. In the spirit of Max Weber (1949), this paper establishes a systematically correct scientific proof of "add them up and divide by two" (Add-Div), by using conceptions of the two OR methodologies, data envelopment analysis (DEA) and cooperative game theory. More specifically, in the DEA game scheme of Nakabayashi and Tone (2006), we study an allocation problem with two criteria, in which game-theoretic solutions such as the Shapley value and the nucleolus prove coincident with the result of the Add-Div method. While today some brilliant minds may propose that people should abandon the Add-Div mentality, this paper enables one to re-consider the concept of Add-Div not only as a matter of mentality but also as an application of OR methods. We also illustrate the use of the Add-Div concept where the "assurance region" (AR) method of DEA is incorporated.

Keywords: DEA, cooperative game theory, Shapley value, nucleolus, assurance region method, allocation problem, social choice problem

1. Introduction

Let us begin with citing *The Doctrine of the Mean* by Confucius, who is the most famous ancient Chinese sage, philosopher and political theorist.

> "The superior man embodies the course of the Mean; the mean man acts contrary to the course of the Mean."
>
> "There was Shun [who is the legendary ideal emperor of ancient China]: – He indeed was greatly wise! . . . He took hold of their two extremes, determined the Mean, and employed it in his government of the people. It was by this that he was Shun! . . . Men all say, 'We are wise;' but being driven forward and taken in a net, a trap, or a pitfall, they know not how to escape. Men all say, 'We are wise;' but happening to choose the course of the Mean, they are not able to keep it for a round month." ([7], p.386 and p.388)

Here, we see that both the importance and difficulty of the political philosophy of the 'Golden Mean' have been recognized in China since the fifth century BC. Then, how can we practice the Doctrine of the Mean? Let us now consider arithmetically this matter: What would we do if we want to arithmetically determine the Mean between two extremes? One good method would be to add them up and divide by two. Interestingly, "add them up and divide by two (*Tashite Ni De Waru*, in Japanese)" is an established maxim in Japanese society. The phrase "*Tashite Ni De Waru*" has often been used for the mentality of the

Japanese people, who prefer to adopt a compromise proposal between competing interests in order to avoid a conflict. For example, Shinji Fukukawa [4], who is an executive advisor of the Dentsu Inc. and a former administrative vice-minister of International Trade and Industry, said:

> "...the Japanese value system may deter the honing of political leadership. Politics are the mirror of a nation. As a historical example, the 17-article constitution compiled by Prince Shotoku [who was a regent and a politician in ancient Japan] in the eighth [seventh] century taught 'harmony is the chief virtue and confrontation should be avoided.' Given Japan's roots as an agricultural society, harmony and co-operation among families and villagers were important to ensure rich harvests and enjoy a stable life....
>
> An 'add-them-up-and-divide-by-two' mentality often has prevailed over a logic-based method of resolving any societal confrontations. As long as people themselves tend to avoid clear-cut solutions, politicians also are likely to do so."

We see here that "add them up and divide by two" (hereafter cited as Add-Div) has its historical roots in Japanese culture. In the meantime, Fukukawa seems to have an unfavorable opinion about the Add-Div mentality of the Japanese people. Today, there are not a few cases where brilliant minds propose that people should abandon the Add-Div mentality, but on the other hand, there are some who hold a positive view of Add-Div as the application of social equality. Thus, different individuals can have totally different values of Add-Div.

Under the circumstances, we agree with Max Weber [19] in thinking that the confusion between the scientific discussion of social phenomena and their evaluation should be avoided in the social sciences. In this sense, it is a very important task to establish a scientific proof of Add-Div, which can be acknowledged as systematically correct by people who seek the truth all over the world. In this paper, therefore, we attempt to develop a scientific theory of Add-Div, by using conceptions of the two OR methodologies, data envelopment analysis (DEA) and cooperative game theory. This enables us to re-consider the concept of Add-Div not only as a matter of mentality but also as an application of OR methods.

The rest of this paper unfolds as follows. Section 2 explains the DEA game scheme of Nakabayashi and Tone [10], and in this scheme Section 3 formulates theorems of Add-Div. We here show that the Add-Div method can be rationalized as game-theoretic solutions such as the Shapley value and the nucleolus. In Section 4, we illustrate the use of the Add-Div concept where the "assurance region" (AR) method of DEA is incorporated. Finally, Section 5 concludes with some remarks.

2. Explanation of the DEA Game Scheme

This section explains the DEA game scheme of Nakabayashi and Tone [10], before we proceed to the formulation of theorems of Add-Div.

2.1. Problem

In the DEA game scheme, we deal with the problem of fairly allocating a certain amount of divisible goods or burdens among individuals or organizations. The so-called allocation problem has many economic and social examples. Also, "political science be described as the study of the authoritative allocation of values for a society," says Easton [3] (p.129). He adds:

> "...from time immemorial men have been asking questions that lead them to seek an understanding of the way values are authoritatively allocated, this has not been

just a matter of accident. A minimum condition for the existence of any society is the establishment of some mechanisms, however crude or inchoate, for arriving at authoritative social decisions about how goods, both spiritual and material, are to be distributed, ..." ([3], p.135)

Here, we see that the problem of how to determine the (authoritative) allocation is classical in nature, and has long been seen to be common to all societies. In the literature on cooperative game theory, there have been many applications to allocation problems. In the DEA game scheme, we can deal with these problems, especially under multi-criteria environment. It is not generally a difficult task to solve the allocation problem if only one criterion is applied; but the society is often faced with multiple criteria for determining the allocation. Three examples will suffice to show this.

1. **Market arcade problem**
   One shopping mall association in Japan made the following agreement on the arcade maintenance fee: "Every shop facing the arcade street has to pay a monthly fee. The method for arriving at this fee for each shop was discussed and approved at the general meeting. Share of cost was determined based on parameters such as category of business, the size of the shop, the number of employees, and so on."
2. **Local tax grants system**
   Japan's local tax grants have been allocated mostly based on the shortfalls in local government finances. An advisory committee established by a Minister for Public Management, Home Affairs, Posts and Telecommunications in 2006 proposed a new tax grants system, under which grants should be allocated basically based on each prefecture's population and area. The proposal is still controversial among the concerned people.
3. **Apportionment of the expenses of the United Nations (UN)**
   The member states of the UN have to pay an annual membership fee to maintain international security. The scale of assessments for the apportionment of the expenses is based on parameters such as gross national product (GNP), external debt, per capita income, and so on. In the UN General Assembly, the representatives of the member states discussed, based on the aforementioned parameters, various ideas about how to estimate the expenses in order to arrive at a single comprehensive scheme. Collectively they are framed within a set of potential parameters, whereas individually they express different opinions on the importance of various parameters. Tadokoro [15] summed up the UN discussion as follows: "The decision-making process of the General Assembly on the methodology for estimating share of expense often appears to be a political battle, in that each member state supports its favorite measures so as to minimize its own share." "The member states may attempt to minimize their own costs, but there are restraints on what they can claim because they have a discussion of an acceptable scheme commonly applied to all of them." ([15], p.131 and p.132)

In the above examples, the societies have taken the trouble to apply two or more criteria to determine the allocation. Probably, one reason is that it is difficult to make a fair judgment based on just one criterion, and another reason is that the concerned people have different opinions on the methods for estimating an efficient/equitable sharing. This issue is essentially related to *priority*, which is a class of conceptions of equity, as mentioned by Young [21].

"More often *priority* is based on a mixture of criteria.... *Priority* in these situations is not one-dimensional, it involves trade-offs among various principles."

"In this case fairness reduces to a procedural question of how to strike an *equitable balance* between diverse points of view. This brings us to a classical problem in group decision making, namely, how to design a process that fairly aggregates individual opinions into a collective decision." ([21], p.14 and p.15)

The problem we treat in the DEA game scheme belongs to what Young called *priority*. "This is," says Young [21] (pp.35-36), "the *opinion aggregation* or *social choice problem*, and it is not a simple one."

2.2. Mathematical modeling

Let us assume n players who have m criteria (or m policy proposals) for deciding about how goods are to be distributed. Their society is faced with a choice among m criteria. Criterion i $(i = 1, \cdots, m)$ assigns the share $x_{ij} \in (0, 1)$ to player j $(j = 1, \cdots, n)$. The vector $(x_{i1}, x_{i2}, \cdots, x_{in})$ denotes the apportionment assigned by Criterion i, and therefore, it holds that $\sum_{j=1}^{n} x_{ij} = 1$ $(i = 1, \cdots, m)$. The score matrix can be arranged in Table 1.

Table 1: The score matrix

	Player				
	1	2	$\cdots$	n	Sum
Criterion (Proposal) 1	x_{11}	x_{12}	$\cdots$	x_{1n}	1
Criterion (Proposal) 2	x_{21}	x_{22}	$\cdots$	x_{2n}	1
.	.	.	$\cdots$	.	.
.	.	.	$\cdots$	.	.
Criterion (Proposal) m	x_{m1}	x_{m2}	$\cdots$	x_{mn}	1

We now consider the following two mathematical programming problems for player j:

$$\begin{aligned} c(\{j\}) &= \max \left\{ \sum_{i=1}^{m} w_i x_{ij} \middle| \sum_{i=1}^{m} w_i = 1, w_i \geq 0 \ (i = 1, \cdots, m) \right\} \\ &= \max \{x_{1j}, x_{2j}, \cdots, x_{mj}\}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} d(\{j\}) &= \min \left\{ \sum_{i=1}^{m} w_i x_{ij} \middle| \sum_{i=1}^{m} w_i = 1, w_i \geq 0 \ (i = 1, \cdots, m) \right\} \\ &= \min \{x_{1j}, x_{2j}, \cdots, x_{mj}\}, \end{aligned} \tag{2.2}$$

where $w = (w_1, \cdots, w_m)$ is a vector of weights assigned to the criteria.

Player j's share is not more than $c(\{j\})$ and not less than $d(\{j\})$, irrespective of what proposal is authoritatively selected in the society.

Toward the "opinion aggregation problem," we now think about an "aggregation" of players. Let a coalition S be a subset of the player set $N = \{1, \cdots, n\}$. The coalition S's total share is calculated by

$$x_i(S) = \sum_{j \in S} x_{ij} \quad (i = 1, \cdots, m). \tag{2.3}$$

Similar to an individual player, the following two problems can be considered:

$$\begin{aligned} c(S) &= \max \left\{ \sum_{i=1}^{m} w_i x_i(S) \middle| \sum_{i=1}^{m} w_i = 1, w_i \geq 0 \ (i = 1, \cdots, m) \right\} \\ &= \max \{x_1(S), x_2(S), \cdots, x_m(S)\}, \end{aligned} \tag{2.4}$$

and

$$d(S) = \min\left\{\sum_{i=1}^{m} w_i x_i(S) \middle| \sum_{i=1}^{m} w_i = 1, w_i \geq 0 \ (i = 1, \cdots, m)\right\} \quad (2.5)$$
$$= \min\{x_1(S), x_2(S), \cdots, x_m(S)\}.$$

Coalition S's share is not more than $c(S)$ and not less than $d(S)$, irrespective of what proposal is authoritatively selected in the society.

Thus, we have two types of characteristic function games (N, c) and (N, d). Nakabayashi and Tone [10] called them the DEA *max* and *min* games, because the DEA's fundamental concept of "variable weights" is the absolute basis of the mathematical programming problems (2.1), (2.2), (2.4) and (2.5). Refer to Cooper *et al.* [1] (pp.12-13) for a detailed explanation on this variable weights issue.

2.3. Possible cooperation among players in the game

We here discuss possible cooperation among players in the DEA game. If a specific player has the authority to determine the allocation, his/her favorite proposal will be adopted in the society. But we consider a case where the authority is not clearly vested in a specific player. All the three examples given in Section 2.1 can illustrate such a case. For a player without the authority the policy is indeterminate, i.e., the player is unsure what policy proposal is authoritatively selected in the society. In the case of benefit-sharing DEA game, player j's optimistic calculation predicts that he/she obtains a benefit of at most $c(\{j\})$, whereas his/her pessimistic calculation predicts that he/she obtains a benefit of at least $d(\{j\})$. (The functions c, d are reversed in the case of cost-sharing.) We now consider the following proposition.

Proposition 2.1 (Nakabayashi and Tone [10])

$$d(S) + c(N - S) = 1, \ \forall S \subset N. \quad (2.6)$$

This proposition implies that $d(S)$ and $c(N - S)$ originate from the same zero-sum (or fixed-sum) two-person game. This may embody the concept conceived by von Neumann and Morgenstern [11] (p.238): The game is played by the coalition S against the coalition $N - S$ for a share of the pie. The most favorable decision for the coalition $N - S$ coincides with the pessimistic expectation of the coalition S. It offers an analogy for a polarization between political friend and foe.

Let us now assume two players i and j $(i, j \in N)$ who calculate their payoffs based on their pessimistic criteria: – "This *criterion of pessimism* or *Wald criterion*, named for Abraham Wald who suggested it, minimizes the risk involved in making a decision." ([18], p.26) Player i expects to obtain (from the coalition $N - \{i\}$) a benefit of at least $d(\{i\})$, and player j expects to obtain (from the coalition $N - \{j\}$) a benefit of at least $d(\{j\})$. Then we consider the following proposition.

Proposition 2.2 (Nakabayashi and Tone [10]) *The characteristic function d is superadditive, i.e., for any $S \subset N$ and $T \subset N$ with $S \cap T = \phi$, we have*

$$d(S \cup T) \geq d(S) + d(T). \quad (2.7)$$

If two players i and j combine together, the sum of their guaranteed payoffs can be increased, i.e., $d(\{i, j\}) \geq d(\{i\}) + d(\{j\})$. Thus, cooperation among players can be expected when we assume that the players in the game follow their pessimistic inclinations and attempt to minimize their risks.

2.4. The DEA game scheme as an OR tool

We here think about a grand coalition and have the following proposition.

Proposition 2.3 (Nakabayashi and Tone [10])

$$c(N) = d(N) = 1. \tag{2.8}$$

The Shapley value [14] and the nucleolus [13] as the representative imputations of cooperative game satisfy grand coalition rationality, and hence, in these game-theoretic solutions of the games (N, c) and (N, d), the sum of imputed payoffs of all players is equal to 1. We can apply them to the problem of how to determine the allocation.

As long as players follow their optimistic inclinations and assert their claims based on their favorite criteria, cooperation may not be expected. One might think that the solution concept of cooperative game is inapplicable to the case. We think, however, the DEA game solution is potentially useful even if there is practically a lack of cooperation. The main reason is that the non-cooperation condition may be only a temporary one. Our view is that a person's inclination is practically variable. Even if a player shows an optimistic and self-centered attitude, it is difficult for the player without the authority to be completely outside his/her pessimistic frame of mind. An evaluator is permitted to assume a pessimistic and rational player in the game-theoretic model to reach a rational conclusion. We also think a person's inclination is controllable to some extent in a society. For example, once it is authoritatively decided that player's share is estimated based on unfavorable criteria for the player, the decision raises the player's pessimistic expectation.

The classical definition of Operations Research by Morse and Kimball [9] (p.1) is "*a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.*" Suppose an executive department has the problem of how to determine a reasonable allocation based on multiple criteria. Then, an OR worker may analyze the problem, by using the DEA game scheme as an OR tool, on the assumption that players in the game follow their pessimistic inclinations that can provide a quantitative basis for the practical decision.

3. Formulation of Theorems of Add-Div

This section deals with a case in which a society is faced with a choice between two criteria for deciding about how goods are to be distributed. In the DEA game scheme we analyze the two-criterion case, and prove that game-theoretic solutions such as the Shapley value and the nucleolus are coincident with the result of the Add-Div method.

3.1. On dealing with the two-criterion case

There are infinite possibilities for estimating share of goods, and therefore, one might think that the two-criterion case, which we are treating here, is not so important. On the contrary, we think the two-criterion case is important because this case is not a rare occurrence, which we now explain below.

Assume a society of M members where each proposes a most desirable way of allocating the goods. Then, the total number of proposals is M, i.e., a finite number. There will be overlap among their proposals, and these proposals are usually further screened in the decision-making process. Even if one of the members tries to push forward an excessive claim, the society (or the other members with some authority of society) can exclude it from the consideration. Thus, the agenda usually focuses on a few alternatives that they consider to be important in consensus-making. Through such a screening and selection process, a two-criterion case can emerge.

Another process is possible. Whenever a counterproposal is placed on the agenda, the society is faced with a choice between two alternatives, i.e., the current policy and the proposed one.

A formulation of the above processes is the task for a separate study, i.e., we here study the case in which there are two criteria. We think this problem is important because our experiences in OR practice (and everyday life) show that a set of two alternatives in decision or evaluation problems is actually frequent rather than rare.

3.2. The Shapley value of bicriteria DEA game

The Shapley value [14] is a representative solution concept of cooperative game. We here demonstrate that the Shapley solution of bicriteria DEA game is the same as the result of the Add-Div method.

The Shapley value for the game (N, d) is defined by $\phi_j(d)$ as

$$\phi_j(d) = \sum_{S:j \in S \subset N} \frac{(s-1)!(n-s)!}{n!} \{d(S) - d(S - \{j\})\}, \tag{3.1}$$

where s is the number of members of coalition S. We first introduce the following theorem.

Theorem 3.1 (Nakabayashi and Tone [10]) *The Shapley values of the DEA max and min games (N, c) and (N, d) are the same.*

In a two-criterion case, we have the following lemma.

Lemma 3.1 *In a two-criterion case, the game (N, c+d) is additive, i.e., for every $S \subset N$, it holds that*

$$c(S) + d(S) = \sum_{j \in S} c(\{j\}) + \sum_{j \in S} d(\{j\}). \tag{3.2}$$

Proof : Since $c(\{j\}) + d(\{j\}) = x_{1j} + x_{2j}$ $(\forall j \in N)$ and $c(S) + d(S) = x_1(S) + x_2(S)$ $(\forall S \subset N)$, we have

$$\begin{aligned} c(S) + d(S) &= x_1(S) + x_2(S) \\ &= \sum_{j \in S}(x_{1j} + x_{2j}) \\ &= \sum_{j \in S} (c(\{j\}) + d(\{j\})) \\ &= \sum_{j \in S} c(\{j\}) + \sum_{j \in S} d(\{j\}). \qquad \square \end{aligned}$$

Note that Lemma 3.1 does not hold for cases involving three or more criteria.

We now have the following theorem.

Theorem 3.2 (Add-Div Theorem I) *In a two-criterion case, for any player k, the Shapley values $\phi_k(c)$ and $\phi_k(d)$ are given by*

$$\phi_k(c) = \phi_k(d) = \frac{c(\{k\}) + d(\{k\})}{2}. \tag{3.3}$$

Proof : By the Shapley's additivity axiom, it holds that $\phi_k(c + d) = \phi_k(c) + \phi_k(d)$, and it follows from Theorem 3.1 that $\phi_k(c) = \phi_k(d)$. Hence,

$$\phi_k(c) = \phi_k(d) = \frac{1}{2}\phi_k(c + d). \tag{3.4}$$

From Lemma 3.1 we have $c(S) + d(S) - c(S - \{k\}) - d(S - \{k\}) = c(\{k\}) + d(\{k\})$ $(\forall S \subset N)$, and then it holds that

$$\phi_k(c + d) = c(\{k\}) + d(\{k\}). \tag{3.5}$$

We have, by (3.4) and (3.5),

$$\phi_k(c) = \phi_k(d) = \frac{c(\{k\}) + d(\{k\})}{2}. \qquad \square$$

Thus, we see that the Shapley value for two-criterion cases can be obtained by applying the simple decision rule – namely, "add them up and divide by two."

Let us now review what Littlechild and Owen [8] in 1973 mentioned: "In the twenty years since its proposal, the Shapley value has received surprisingly few applications. Perhaps, one reason is the difficulty of computing it for large games." If there was another bottleneck in the implementation of the solution concept for cooperative game, it would be the difficulty of constructing the characteristic function in real-life problems. Littlechild and Owen [8] constructed the function in the real-life problem, which yields "a simple expression for the Shapley value." Their study has significantly contributed to the application of game theory. In the DEA game scheme the problems are given computationally implementable form, and also the analysis here has shown a real-world situation where we have "a simple expression for the Shapley value." This should be said with some emphasis.

We show in the following example that the Shapley value for the three-criterion case cannot be obtained by applying the simple decision rule.

[Example 3.1]

Table 2 exhibits a DEA game of three players with three criteria. Also, the bottom line of Table 2 shows the average of the vectors on the three individual criteria. Table 3 exhibits the characteristic function values of the individual players for the *max* and *min* games, and the average of them. We obtain the Shapley value of this game, as shown in Table 4. The Shapley value is very different from the average vectors in Tables 2 and 3.

3.3. The nucleolus of bicriteria DEA game

The nucleolus [13] as well as the Shapley value is a well-known solution concept of cooperative game. Kohlberg [5] introduced, in his Theorem 2, a necessary and sufficient condition for a payoff vector x to be the nucleolus of a characteristic function v. By using his results, we here demonstrate that the nucleolus solution of bicriteria DEA game is the same as the result of the Add-Div method.

Let us introduce a couple of definitions established in [5].

Table 2: A three-person and three-criterion game

	Player			
	A	B	C	Sum
Criterion 1	0.25	0.3	0.45	1
Criterion 2	0.5	0.25	0.25	1
Criterion 3	0.125	0.5	0.375	1
Average	0.292	0.35	0.358	1

Table 3: Max and min values of the individual players

	Player			
	A	B	C	Sum
Max values	0.5	0.5	0.45	1.45
Min values	0.125	0.25	0.25	0.625
Average	0.3125	0.375	0.35	1.0375

Table 4: The Shapley value

	Player			
	A	B	C	Sum
Shapley value	0.3	0.3625	0.3375	1

Definition 3.1 (Kohlberg [5]) *Let* $b_0, b_1, \cdots, b_p$ *be a sequence of sets whose elements are coalitions of* N. *This sequence is a* coalition array *whenever:*

(i) *every coalition of* N *is contained in exactly one of the sets* $b_1, \cdots, b_p$,

(ii) b_0 *contains only one-element coalitions.*

For every game (N, v) and every payoff vector $x = (x_1, \cdots, x_n)$ with $x(N) = v(N)$, let $b_1(x, v)$ be the set of those $S \subseteq N$ for which

$$\max\{v(S) - x(S) : S \subseteq N\}$$

is attained. Similarly, $b_2(x, v)$ is the set of those $S \subseteq N$ where

$$\max\{v(S) - x(S) : S \notin b_1(x, v)\}$$

is attained, and so on. Finally, let $b_0(x) = \{\{i\} : x_i = 0\}$. It is obvious that $b_0(x)$, $b_1(x, v)$, $\cdots$, $b_p(x, v)$ is a coalition array. We shall say that it is the array that *belongs* to (v, x).

Definition 3.2 (Kohlberg [5]) *A coalition array* $b_0, \cdots, b_p$ *has* property I *if for all* $k = 1, 2, \cdots, p$ *and any* $y = (y_1, \cdots, y_n)$,

(1) $y(S) \geq 0$ *for all* $S \in b_0$,

(2) $y(S) \geq 0$ *for all* $S \in b_1 \cup \cdots \cup b_k$, *and*

(3) $y(N) = 0$

imply

(4) $y(S) = 0$ *for all* $S \in b_1 \cup \cdots \cup b_k$.

Now we are ready to consider our main problem. In DEA games with two criteria, define a payoff vector $z = (z_1, \cdots, z_n)$ such that

$$z_k = \frac{c(\{k\}) + d(\{k\})}{2} \quad (k = 1, \cdots, n). \tag{3.6}$$

Then, we have the following lemma.

Lemma 3.2

$$d(S) - z(S) = d(N - S) - z(N - S).$$

Proof : For any $S \subseteq N$, using Lemma 3.1 we have

$$\begin{aligned} d(S) - z(S) &= d(S) - \sum_{j \in S} \frac{c(\{j\}) + d(\{j\})}{2} \\ &= d(S) - \frac{c(S) + d(S)}{2} = \frac{d(S) - c(S)}{2}, \end{aligned}$$

and

$$d(N-S) - z(N-S) = d(N-S) - \sum_{j \in N-S} \frac{c(\{j\}) + d(\{j\})}{2} = \frac{d(N-S) - c(N-S)}{2}.$$

From Proposition 2.1, we have $d(S) - c(S) = d(N-S) - c(N-S)$. Hence, it holds that $d(S) - z(S) = d(N-S) - z(N-S)$. □

We also have the following lemma.

Lemma 3.3 *The coalition array that belongs to* (d, z) *has property I.*

Proof: The proof is by contradiction. Assume the assertion were false. Then we could have $\exists y = (y_1, \cdots, y_n)$ such that for $\exists k$, y satisfies (1), (2) and (3) of Definition 3.2, and for $\exists S \in b_1 \cup \cdots \cup b_k$, $y(S) > 0$.

From Lemma 3.2, $N - S$ as well as S is contained in $b_1 \cup \cdots \cup b_k$, and the condition (2) follows that $y(N-S) \geq 0$. Hence, it holds that $y(S) + y(N-S) > 0$. This contradicts the condition (3), and the proof is completed. □

In a similar manner, we can prove the following lemma.

Lemma 3.4 *The coalition array that belongs to* (c, z) *has property I.*

Kohlberg [5] demonstrated that a payoff vector x is the nucleolus of v if and only if the coalition array that belongs to (v, x) has property I. Hence, from Lemmas 3.3 and 3.4, z is the nucleolus of both c and d, and we have the following theorem.

Theorem 3.3 (Add-Div Theorem II) *In a two-criterion case, the nucleolus solutions of games* (N, c) *and* (N, d) *for player* k *are given by*

$$\frac{c(\{k\}) + d(\{k\})}{2}.$$

In Theorems 3.2 and 3.3, we can rationalize the Add-Div method as the solution concepts of cooperative game.

We have the following example that the nucleolus for the three-criterion case cannot be obtained by applying the simple decision rule.

[Example 3.2]

The nucleolus allocations of both the *max* game (N, c) and the *min* game (N, d) for the data set in Table 2 are (0.3125, 0.36875, 0.31875) and (0.275, 0.375, 0.35), respectively. Both nucleolus allocations are different from the average vectors in Tables 2 and 3, and from the Shapley value in Table 4.

As shown in Examples 3.1 and 3.2, the "add them up and divide by *three*" method cannot be rationalized as a solution concept of cooperative game.

4. Incorporating the AR Method

In the preceding section the theorems of Add-Div were formulated and proved, although the Add-Div method has recently been questioned by many people. This section begins with looking at a case in which the Add-Div method will be criticized. We see why the Add-Div method is so problematic to many people and illustrate the use of the Add-Div concept where the "assurance region" (AR) method of DEA is incorporated.

4.1. The need for the thoughtful and careful use

Let us consider the following quotation from Kyogoku [6] (p.120):

> "In many cases, the mass media will criticize the monetary amounts that have been set on the ground that they are ad hoc unprincipled (add up and divide by two) decisions that are not based on long-term considerations."

In many political cases, decisions should certainly be based on long-term considerations. Hence, the thoughtless and indiscriminate use of the Add-Div method must be avoided.

Meanwhile, the serious problem is that in taking long-term considerations into account, there will be various criteria for evaluating a policy. When members of a democratic society have different criteria for the evaluation, the principle of social equality becomes necessary in some cases where the DEA game scheme is potentially useful. We think it is important to try to find out the thoughtful and careful use of the DEA game scheme. When considering the use of it as a means of supporting group decision-making, the group members should talk long over the direction of their common policy and choose proper criteria. Inappropriate criteria must be excluded from the consideration.

They may wish to find only one proper criterion, but there are two or more criteria in some cases. In order to build a consensus, they will sometimes need to try to find a common acceptable range for weights of the criteria. In this connection, Takamura and Tone [16] attempted decision analyses for a national project in Japan, by using the "assurance region" (AR) model of DEA. Their study shows that the AR method is so useful in the process of reaching a consensus. Nakabayashi and Tone [10] presented the DEA game where the AR method was incorporated, which is also potentially useful in the process of reaching a consensus. The following subsection illustrates the use of the AR DEA game with two criteria and demonstrates that Add-Div Theorems hold in the AR case, too.

4.2. Illustration of the use of the AR DEA game

Let us assume three players who assert their claims to limited goods. Concerned people discuss how to estimate share of the goods and finally choose the two criteria that define the data matrix exhibited in Table 5. From Theorems 3.2 and 3.3, we have the solution of this DEA game in the bottom line of Table 5, as $(0.375, 0.275, 0.35)$.

Table 5: A three-person and two-criterion game

	Player			
	A	B	C	Sum
Criterion 1	0.25	0.3	0.45	1
Criterion 2	0.5	0.25	0.25	1
Average	0.375	0.275	0.35	1

We further assume all the concerned people agree that Criterion 1 is more important than Criterion 2 in consensus-making. (But, at the same time they cannot completely reject Criterion 2.) The consensus holds that the solution $(0.375, 0.275, 0.35)$ needs to be revised, because this solution means the two criteria are of equal value, in spite of the commonly agreed ranking of criteria. Every player in the game should not be free to choose his/her weights, and the region of weights must be limited to some special area. We here can apply the following method called "assurance region" (AR), originally developed in DEA literature, e.g., Thompson *et al.* [17].

Let $w = (w_1, w_2)$ be a vector of weights assigned to the criteria. We now set a constraint on the ratio w_2 over w_1 as follows:

$$u \geq \frac{w_2}{w_1} \geq l,$$

where u and l denote the upper and lower bounds of the ratio w_2/w_1. These bounds must be set in agreement among the concerned people. In this AR case, programs (2.4) and (2.5) are, respectively, modified as

$$\begin{aligned} c(S) = \max_{w} \quad & w_1 \sum_{j \in S} x_{1j} + w_2 \sum_{j \in S} x_{2j} \\ \text{subject to} \quad & w_1 + w_2 = 1 \\ & u \geq \frac{w_2}{w_1} \geq l \\ & w_1, w_2 \geq 0, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} d(S) = \min_{w} \quad & w_1 \sum_{j \in S} x_{1j} + w_2 \sum_{j \in S} x_{2j} \\ \text{subject to} \quad & w_1 + w_2 = 1 \\ & u \geq \frac{w_2}{w_1} \geq l \\ & w_1, w_2 \geq 0. \end{aligned} \tag{4.2}$$

For all $S \subset N$, the values of objective functions $c(S)$ and $d(S)$ are, respectively, computed by the following formulae:

$$c(S) = \max \left\{ \frac{\sum_{j \in S} x_{1j} + u \sum_{j \in S} x_{2j}}{1+u}, \frac{\sum_{j \in S} x_{1j} + l \sum_{j \in S} x_{2j}}{1+l} \right\}, \tag{4.3}$$

and

$$d(S) = \min \left\{ \frac{\sum_{j \in S} x_{1j} + u \sum_{j \in S} x_{2j}}{1+u}, \frac{\sum_{j \in S} x_{1j} + l \sum_{j \in S} x_{2j}}{1+l} \right\}. \tag{4.4}$$

We now have the following lemma.

Lemma 4.1 *In a bicriteria DEA game, Lemma 3.1 holds even if the AR method is incorporated into the game.*

Proof : We can demonstrate this lemma as follows:

$$\begin{aligned} c(S) + d(S) &= \frac{\sum_{j \in S} x_{1j} + u \sum_{j \in S} x_{2j}}{1+u} + \frac{\sum_{j \in S} x_{1j} + l \sum_{j \in S} x_{2j}}{1+l} \\ &= \sum_{j \in S} \left(\frac{x_{1j} + u x_{2j}}{1+u} + \frac{x_{1j} + l x_{2j}}{1+l} \right) \\ &= \sum_{j \in S} (c(\{j\}) + d(\{j\})) \\ &= \sum_{j \in S} c(\{j\}) + \sum_{j \in S} d(\{j\}). \qquad \square \end{aligned}$$

Since Lemma 3.1 is crucial for the proofs of both Theorems 3.2 and 3.3, we have now, using Lemma 4.1, the following theorem.

Theorem 4.1 (Add-Div Theorem III) *In a bicriteria DEA game, both Theorems 3.2 and 3.3 hold even if the AR method is incorporated into the game.*

This theorem demonstrates that even if the AR method is incorporated into the DEA games, both the Shapley value and the nucleolus for two-criterion cases can be obtained by applying the simple decision rule – namely, "add them up and divide by two."

In general the solution of bicriteria AR DEA game is obtained by

$$\frac{c(\{k\}) + d(\{k\})}{2} = \frac{1}{2}\left(\frac{x_{1k} + ux_{2k}}{1+u} + \frac{x_{1k} + lx_{2k}}{1+l}\right). \tag{4.5}$$

If we set

$$t = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1+l}\right), \tag{4.6}$$

then we have

$$\frac{c(\{k\}) + d(\{k\})}{2} = tx_{1k} + (1-t)x_{2k}. \tag{4.7}$$

Thus, the determination of the bounds u and l is reduced to the determination of a parameter t. The range of t is from 0 to 1, with $u, l \in [0, \infty)$.

4.3. A numerical example

We are now applying the AR method to the game exhibited in Table 5. When all the concerned people agree that Criterion 1 is more important than Criterion 2, we should set the constraint $1 \geq w_2/w_1$ (and do not require the lower bound). Substitute the values $u = 1$ and $l = 0$ into the equation (4.6), and then we have $t = 0.75$. By (4.7), we obtain the solution of this AR DEA game in the bottom line of Table 6. While Player A was ranked higher than Player C before the AR method is applied, Player C ranks first in this AR case because Criterion 1 is considered to be more important (Player C has the highest score on Criterion 1).

Table 6: AR DEA game solution

	Player			
	A	B	C	Sum
Criterion 1 (C1)	0.25	0.3	0.45	1
Criterion 2 (C2)	0.5	0.25	0.25	1
$0.75 \times \text{C1} + 0.25 \times \text{C2}$	0.3125	0.2875	0.4	1

Figure 1 shows a simple sensitivity analysis. The choice of the bounds u, l or the parameter t can have an impact on the ranking of the players, and so the careful choice is needed.

5. Concluding Remarks

In this paper, we have fitted the notion described in the Japanese maxim "add them up and divide by two (*Tashite Ni De Waru*, in Japanese)" into an OR framework using two methodologies, DEA and cooperative game theory. Today some brilliant minds may propose that people should abandon the "add them up and divide by two" (Add-Div) mentality. Such a proposal forces people to choose whether or not to abandon it. However, this paper

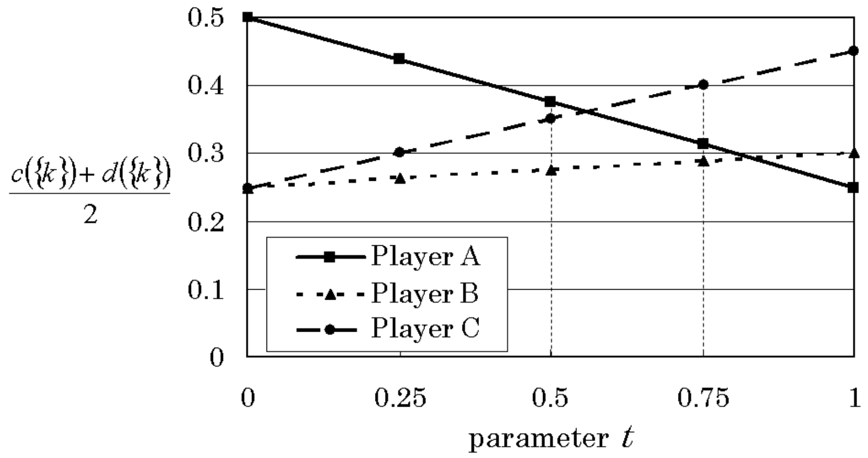

Figure 1: Sensitivity analysis

enables people to re-consider the concept of Add-Div not only as a matter of mentality but also as an application of OR methods. People can make a choice among various OR tools for decision-making, including the Add-Div method, depending on the nature of the case, and employ it sensibly.

To develop the theory of Add-Div, we used the two main concepts, DEA's fundamental concept of variable weights and the solution concept for cooperative game. More specifically, the Add-Div theory is based on the two major assumptions, (1) It does not judge the relative importance of the criteria in advance; (2) The principle of equal treatment for all the players and potential coalitions is embodied in the solution. The presence of these theoretical assumptions facilitates an explicit comparison of different conclusions that flow from different assumptions. (As an extreme example, a decision by a dictatorial player who has a one-sided viewpoint will be very different from the solution concept of the DEA game.) Furthermore, we need to study the possibility of other scientific proofs of Add-Div, e.g., by using the axiomatic theory of bargaining.

In this paper, we have also seen why the Add-Div method is so problematic to many people. One reason could be that the Add-Div decision may not be based on long-term considerations, and therefore, the thoughtless and indiscriminate use of the Add-Div method must be avoided. Concerned people have to talk among themselves to carefully choose proper criteria. They can easily arrive at a decision if they find only one proper criterion, but there are two or more criteria in some cases. In the case of three or more criteria no game-theoretic proof will be given for a simple decision rule, whereas as shown in our analysis, if their discussion focuses on just two criteria, they can make a rational and simple decision by the Add-Div method; however, limiting the number of criteria to two is not always easy. The difficulty of *the Doctrine of the Mean* may lie in 'taking hold of their two extremes' rather than in 'determining the Mean.'

In the spirit of Takamura and Tone [16], this paper has discussed the use of the AR DEA game that is potentially useful in the process of reaching a consensus. Takamura and Tone [16] developed a consensus-making scheme by using not only the AR DEA model but also a number of other methods such as Inverted DEA [20], analytic hierarchy process (AHP) [12] and the Delphi method. Tone actually applied this scheme step by step to the Japanese Council and finally formed a consensus among 19 Council members. When dealing with a multi-person, multi-criteria evaluation problem, we may need to employ a combination of several different methods, and in a practical way, even more importantly, to incorporate various ideas into a cooperative framework for the search of a consensus.

Finally, we would like to point out another possible reason why the Add-Div method is so problematic to people. The following is a quotation from [2]: "Without a clear philosophy, no one will follow suit. Because Japanese always add one plus one and then divide by two." Our study has applied both DEA and cooperative game theory to the problem, and has established an OR philosophy that lies at the base of Add-Div. We hope this study facilitates the re-consideration of the Add-Div concept and encourages its thoughtful and careful use to resolve social and political confrontations in real-life problems.

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