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THE NUMBER OF CIRCULAR TRIADS IN A PAIRWISE COMPARISON MATRIX AND A CONSISTENCY TEST IN THE AHP

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Abstract A pairwise comparison matrix in the Analytic Hierarchy Process (AHP), which was proposed by Saaty in 1970s, consists of elements expressed on a numerical scale. The purpose of this paper is to propose a consistency test for ordinality of items in the pairwise comparison matrix. The original of this test is in a sensory test. In a sensory test we use a pick-the-winner ordinal scale to obtain the table of preferences for objects. In 1940 Kendall and Babington Smith proposed a consistency test for the preference table, using the number of circular triads in it. In this paper we show how to apply their test to a pairwise comparison matrix in the binary AHP and to one without a tie for up to nine items in the AHP. This is to test, using a pairwise comparison matrix, whether or not we can accept that items which are factors or alternatives are sufficiently ranked linearly before calculating weights of these items.

Keywords: AHP, binary AHP, circular triads, consistency test, sensory test

1. Introduction

The Analytic Hierarchy Process (AHP), which was proposed by Saaty in 1970s, consists of roughly three steps. The first is structuring a hierarchy to clear structure of a given problem. The second is calculating weights of all items which are factors or alternatives. The final step is calculating weights which are overall evaluations of alternatives for the main objective. Pairwise comparisons are used in the second step and in this paper we deal with pairwise comparison matrices by them.

In many literatures methods of checking consistency of a pairwise comparison matrix have been discussed. In general these are called consistency tests. As Monsuur pointed out in [9], one of the advantages of the AHP is that it is equipped with such measures. For instance, C.I. and C.R. are well-known as the reference values for consistency tests. The purpose of this paper is to propose a new consistency test for the ordinality of items in a pairwise comparison matrix in the binary AHP and to one without a tie for up to nine items in the AHP with slight modification of a consistency test for a preference table in a sensory test.

Now let O_i $(1 \leq i \leq n,\ n \geq 3)$ be objects. If O_i is preferred to O_j, we describe $O_i \to O_j$ according to [7]. Then a directed graph G is made from these arrows between each two objects. When we have $O_i \to O_{i+1}$ for any integer i $(1 \leq i \leq n-1)$ and $O_n \to O_1$ in G, the pair of these objects $O_1 O_2 \cdots O_n$, which is a pair of vertices in G, is called a circuit of length n or a circular n-ad. In particular a circuit of length 3 is called a circular triad. Kendall and Babington Smith in [7] showed that a circular n-ad contains at least $n-2$ circular triads and drew attention to the number of circular triads included in G on the ranking problem by pairwise comparisons.

We construct a directed graph M_G from a pairwise comparison matrix M in the AHP and circular triads of items in M is defined as ones of items in M_G. Then, as is well-known, some circular triads included in M cause inconsistency of itself. In this paper we show the maximum number of circular triads in M that we can accept in which items are sufficiently ranked linearly in the sense of ranking problems (cf. [6, 7]) and propose a test using this number. This is equal to a test whether M is consistent as measured by the number of circular triads and is also a consistency test for a pairwise comparison matrix in the AHP.

In Section 2 we review the coefficient of consistency ζ used in a sensory test to test consistency of a complete directed graph according to [7]. This coefficient in the AHP was mentioned in [12]. We don't deeply refer to it in this paper. In Section 3 we show the distributions of the number of circular triads for up to nine objects. Kendall and Babington Smith in [7] presented them for up to seven objects and Alway carried out for eight and nine objects in [1]. In Section 4 we review the test whether or not an observer is sufficiently capable of comparing objects pairwisely by using the number of circular triads included in a preference matrix according to [11], which is used in a sensory test.

In Sections 5 and 6 we show how to apply this test to a pairwise comparison matrix in the binary AHP and to one without a tie in the AHP. The proposal in that case is to test whether or not we can accept that items are ranked linearly. We don't deal with a pairwise comparison matrix with a tie between different items in the AHP in this paper. Jensen and Hicks researched in [5] the relationship between the number of circular triads in a pairwise comparison matrix and inconsistency of the matrix in the AHP. Though the purpose of their paper is different from one of this paper, they dealt with a pairwise comparison matrix with a tie between different items in their paper.

2. Coefficient of consistency in pairwise comparisons

Let O be the set of n objects $O_1, O_2, \cdots, O_n$. If O_i is preferred to O_j $(i \neq j)$, we set $a_{ij} = 1$ and $a_{ji} = 0$. For the sake of convenience we set $a_{ii} = 1$. Thus the preference matrix $A = (a_{ij})$ with the unity diagonal elements for O is obtained.

Now when $a_{ij} = 1$ $(i \neq j)$ we describe $O_i \rightarrow O_j$. Thus the complex of preferences can be represented by directed arrows in the corresponding complete graph in which any two different points are connected by an arrow. For instance, the preference matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{2.1}$$

generates the following graph A_G;

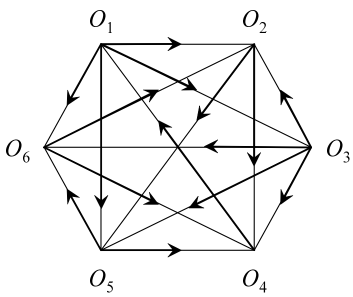

Figure 1: The directed graph A_G corresponding to A

In [7] Kendall and Babington Smith considered the number of circular triads included in a directed graph A_G, i.e., in a given preference matrix A. Let d be the number of circular triads in A_G and a_i $(1 \leq i \leq n)$ be the number of arrows which leave the vertex corresponding to O_i. Then it is clear that A_G has n vertices and $\binom{n}{2} = \frac{n(n-1)}{2}$ arrows. Furthermore, it holds that

$$\sum_{i=1}^{n} a_i = \frac{n(n-1)}{2} \tag{2.2}$$

and

$$d = \frac{n(n-1)(n-2)}{6} - \frac{1}{2}\sum_{i=1}^{n} a_i(a_i - 1) = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n} a_i^2. \tag{2.3}$$

Note 2.1 In [10] Nishizawa showed the method using the vertex matrix M_V corresponding to a comparison matrix $M = (m_{ij})$ in order to calculate the number of circular triads d, which is called cycles of length 3 there, included in M_G in the binary AHP. A vertex matrix $M_V = (v_{ij})$ is defined as $v_{ij} = 1$ if $m_{ij} > 1$, and $v_{ij} = 0$ otherwise. Then he showed that the trace of the three power of M_V is three times d (see Theorem 1 in [10]). We can use this in the replacement of Equation (2.3) when calculating d.

In [7] Kendall and Babington Smith showed the following useful theorem.

Theorem 2.1 Let G be a complete directed graph and n be the number of vertices in G. Then the maximum possible number of circular triads in G is

$$\begin{cases} \dfrac{n^3 - n}{24} & \text{if } n \text{ is odd,} \\ \dfrac{n^3 - 4n}{24} & \text{if } n \text{ is even,} \end{cases} \tag{2.4}$$

and the minimum number is zero. In particular there exist always complete directed graphs G with these limits. Moreover, for any integer k between the maximum and the minimum there exists at least one complete directed graph in which the number of circular triads is k.

They defined the coefficient of consistency ζ for a given complete directed graph G of n vertices with d circular triads based on the theorem as follows;

$$\zeta = \begin{cases} 1 - \dfrac{24d}{n^3 - n}, & \text{if } n \text{ is odd,} \\ 1 - \dfrac{24d}{n^3 - 4n}, & \text{if } n \text{ is even.} \end{cases} \tag{2.5}$$

It is easy to see from Theorem 2.1 that $0 \leq \zeta \leq 1$ and $\zeta = 1$ if and only if there is no circular triad in G. As ζ decreases to zero, the inconsistency, which is measured by the number of circular triads, increases. So ζ can be used as measure of consistency in a preference matrix A. Clearly the objects may be completely ranked linearly by A if and only if $\zeta = 1$. And when ζ is not unity, Kendall and Babington Smith considered the following possibilities:

(a) The observer may be a bad judge.
(b) Some of the objects may differ by amounts which fall below the threshold of distinguishability for the observer.
(c) The property under judgment may not be a linear variate at all.
(d) Several of the effects may be operating simultaneously.

For these possibilities they noted that if we have no prior knowledge of the observer's capability, it is not in general possible to apportion his or her inconsistency among these causes except for the case that the inconsistency is of a marked and peculiar kind. So in order to test the significance of a value of ζ they researched the distribution of the number of circular triads it would have if all the preferences were allotted at random, and proposed a method to test whether an observer is capable of consistent ranking or not under the hypothesis that objects are ranked linearly. For example, there is an application of this method to a sensory test on pp.351–353 in [11].

In the next section the distributions are showed and in Section 4 their test is explained. In Sections 5 and 6 we modify the test to the case of testing whether or not we can accept that items are ranked linearly by the pairwise comparison matrix under the hypothesis that a decision maker has ability to rank items linearly in the binary AHP and the AHP, respectively.

3. Distributions of the number of circular triads

Let n ($n \geq 2$) be the number of objects and d be the number of circular triads in an observed configuration of preferences, which is represented by a complete directed graph. The maximum value of d for each n is obtained by Theorem 2.1. In this section we consider the number of the graphs with a given pair of n and d according to [7] and [1].

Table 1: Frequency f of d and probability P that won't exceed d for n objects

Values	$n=2$		$n=3$		$n=4$		$n=5$		$n=6$		$n=7$	
of d	f	P	f	P	f	P	f	P	f	P	f	P
0	2	1.000	6	0.750	24	0.375	120	0.117	720	0.022	5040	0.002
1			2	1.000	16	0.625	120	0.234	960	0.051	8400	0.006
2					24	1.000	240	0.469	2240	0.120	21840	0.017
3							240	0.703	2880	0.208	33600	0.033
4							280	0.977	6240	0.398	75600	0.069
5							24	1.000	3648	0.509	90384	0.112
6									8640	0.773	179760	0.198
7									4800	0.919	188160	0.287
8									2640	1.000	277200	0.420
9											280560	0.553
10											384048	0.737
11											244160	0.853
12											233520	0.964
13											72240	0.999
14											2640	1.000
Total	2	–	8	–	64	–	1024	–	32768	–	2097152	–

The distributions of d have been given by Kendall and Babington Smith when $2 \leq n \leq 7$

and by Alway when $8 \leq n \leq 10$. According to their results we have Tables 1 and 2 with the frequencies f of d and probabilities P that won't exceed values of d, which is $\Pr[\,0 \leq x \leq d\,]$, for n objects ($2 \leq n \leq 9$). Because we need the distributions of d for $n = 3$ to 9 when dealing with the AHP (see Section 6), we didn't make the table for 10 objects.

For example it is seen from Table 1 that if we make a complete directed graph for 5 objects at random, then the probability P that it has 2 or less circular triads is about $0.469(= (120 + 120 + 240)/1024)$.

Table 2: Frequency f of d and probability P that won't exceed d

Values	$n = 8$		$n = 9$	
of d	f	P	f	P
0	40320	0.00015	362880	0.00001
1	80640	0.00045	846720	0.00002
2	228480	0.00130	2580480	0.00006
3	403200	0.00280	5093760	0.00013
4	954240	0.00636	12579840	0.00031
5	1304576	0.01122	19958400	0.00060
6	3042816	0.02255	44698752	0.00125
7	3870720	0.03697	70785792	0.00228
8	6926080	0.06278	130032000	0.00418
9	8332800	0.09382	190834560	0.00695
10	15821568	0.15276	361525248	0.01221
11	14755328	0.20773	443931264	0.01867
12	24487680	0.29895	779950080	0.03002
13	24514560	0.39027	1043763840	0.04521
14	34762240	0.51977	1529101440	0.06746
15	29288448	0.62888	1916619264	0.09535
16	37188480	0.76742	2912257152	0.13773
17	24487680	0.85864	3078407808	0.18253
18	24312960	0.94921	4506485760	0.24811
19	10402560	0.98797	4946417280	0.32009
20	3230080	1.00000	6068256768	0.40839
21			6160876416	0.49804
22			7730384256	0.61054
23			6292581120	0.70211
24			6900969600	0.80253
25			5479802496	0.88227
26			4327787520	0.94525
27			2399241600	0.98016
28			1197020160	0.99758
29			163094400	0.99995
30			3230080	1.00000
Total	268435456	–	68719476736	–

An algorithm to obtain the distributions of d with a computer was described in [1] in detail. This time we calculated those for up to $n = 9$ using other algorithm with a computer and obtained the same results with them. Indeed we made all cases of preference tables $A = (a_{ij})$ by setting $a_{ij} = 0$ or 1 for a preference between different objects O_i and O_j $(i < j)$. And using the property that a triad $O_iO_jO_k$ $(i < j < k)$ is circular if and only if $a_{ij} = a_{jk}$ and $a_{ij} \neq a_{ik}$, we judged whether or not the triad $O_iO_jO_k$ for all pairs of i, j and k $(i < j < k)$ is circular.

Note 3.1 Kendall in [6] established the χ^2-approximation to the distribution of the number of circular triads d for any n objects $(n \geq 8)$. Alway in [1] showed an algorithm using Equation (2.3) to obtain the distribution of d for any n objects and presented concretely that of d for $n = 10$.

4. Consistency test of a pairwise comparison matrix in a sensory test

We can use Tables 1 and 2 in order to check whether the number of circular triads could have arisen by chance if the observer were completely incompetent, or, alternatively, whether there is some degree of consistency in the observer's preferences notwithstanding a lack of perfection as in [7]. Kendall and Babington Smith showed the following example in [7]. The chances that if the preferences are made at random there will be more than two circular triads are 983 in 1000 for $n = 7$ by Table 1, so if we find such two or less triads, it is improbable that the observer is completely incapable of judgment. Then we might be led to suppose the observer's small deviation from internal consistency is due to fluctuation of attention, very close resemblance to the objects giving rise to the inconsistencies, or both.

Thus Tables 1 and 2 are used to check whether or not an observer who pairwisely compares n objects is sufficiently capable of making judgments. At the rest of this section we review the consistency test for a preference table in a sensory test using Tables 1 and 2 according to [11]. For this test we need Table 3, which has the maximum values $d_{0.05,n}$ of the number of circular triads d for each n $(3 \leq n \leq 10)$ satisfying that probability $\Pr[\ 0 \leq x \leq d\ ] < \alpha = 0.05$, which is gotten from Tables 1 and 2. Though Kendall and Babington Smith in [7] seemed to have adopted the significant level $\alpha = 0.01$, we adopt $\alpha = 0.05$ due to practicalities in use. Indeed Table 3 without $n = 8$ to 10 is in Appendix 24 on p.870 in [11].

Table 3: The values of $d_{0.05,n}$ in a sensory test

The number of objects n	3	4	5	6	7	8	9	10
The value of $d_{0.05,n}$	-	-	-	1	3	7	13	21
The total number of triads	1	4	10	20	35	56	84	120

Remark 4.1 (1) There exists no value of d for $n = 3$ to 5 satisfying that $\Pr[\ 0 \leq x \leq d\ ] < 0.05$. So symbol "-"is filled in each cells of $d_{0.05,n}$ for $n = 3$ to 5 in Table 3.
(2) Though $\Pr[\ 0 \leq x \leq 1\ ] \fallingdotseq 0.051$ for $n = 6$ by Table 1, we set $d_{0.05,6} = 1$ in consideration of the distribution of d being discrete as in [11].
(3) It follows from the table in [1] that $\Pr[\ 0 \leq x \leq 21\ ] \fallingdotseq 0.052$ for $n = 10$.

We now address the issue of whether an observer is sufficiently capable of making judgments by using the number of circular triads d included in a preference matrix A according to [11]. It is easy to see that in this test we need more than 5 objects from Remark 4.1 (1). This method does not require a directed graph A_G corresponding to A.

We set the hypothesis that the observer pairwisely compares n objects ($6 \leq n \leq 10$) which could be completely ranked linearly, but cannot construct a pairwise comparison matrix which is consistent as measured by the number of circular triads. On this assumption the observer compares n objects pairwisely in practice. We set $a_{ij} = 1$ and $a_{ji} = 0$ if the observer prefers the object O_i to the object O_j ($i \neq j$), and $a_{ii} = 0$ to get the preference matrix $A = (a_{ij})$.

(S_1) Count the cardinal number of integers j such that $a_{ij} = 1$ for each i-th row, which is denoted by a_i.

(S_2) Substitute these a_i for a_i in Equation (2.3) to calculate d.

(S_3) If $d \leq d_{0.05,n}$ from Table 3, then we reject the hypothesis and think that the observer is sufficiently capable of making judgments.

If the preference matrix A doesn't pass this test, then we may refer the coefficient of consistency ζ for A using Equation (2.5). If ζ is near enough to 0 when it is compared with 1, then we reject the hypothesis and think that the observer is sufficiently capable of making judgments. We note whether it is accepted with the numerical value of ζ of which extent is entrusted to the observer's judgment in the end. In fact the reference value to ζ isn't showed in [11] and [6].

5. Consistency test of a pairwise comparison matrix in the binary AHP

As a special case of the AHP we know the case where elements m_{ij} ($i \neq j$) of a pairwise comparison matrix M take one of only two intensity scale of importance values. In fact let θ be an integer such that $\theta > 1$. If a decision maker prefers the item O_i to the item O_j ($i \neq j$), then we set $m_{ij} = \theta$ and $m_{ji} = 1/\theta$, and $m_{ii} = 1$ as is usual. We use items and pairwise comparison matrices in the AHP as the technical terms, while we do objects and preference matrices in a sensory test. Thus we obtain a pairwise comparison matrix $M = (m_{ij})$ that is called a binary comparison matrix. This kind of the AHP is called the binary AHP and researched in [2–4, 10, 13] and so on.

When we describe $O_i \to O_j$ for $m_{ij} = \theta$, we have the complete directed graph M_G for M (see [13]). Thus a circular triad in a binary comparison matrix M in the binary AHP is defined. Here we recall the definition of consistency for a binary comparison matrix M according to [13]. If the following condition for M in the binary AHP holds;

$$m_{ij} > 1 \text{ and } m_{jk} > 1 \quad \text{imply} \quad m_{ik} > 1 \quad \text{for any } i,\ j \text{ and } k, \tag{5.1}$$

then M is called logically consistent. Logically consistent is regarded as consistent in the binary AHP. Then it is easily seen that a test of consistency is useful for a decision maker in the binary AHP. We note that the definition of consistency in the binary AHP is different from one in the AHP. Indeed the condition of the consistency of a pairwise comparison matrix in the AHP is that $m_{ij} \times m_{jk} = m_{ik}$ for any i, j and k, which is insignificant in the binary AHP.

In this section we show how to apply the consistency test in a sensory test, which is reviewed in Section 4, to a pairwise comparison matrix in the binary AHP. In order to do so we consider the consistency test in a sensory test from a different point of view.

In fact we suppose that if n items are ranked linearly, the decision maker can make a consistent pairwise comparison matrix as measured by the number of circular triads by comparing items pairwisely. On this assumption we test whether or not we can accept that items are sufficiently ranked linearly. This test is useful because one purpose of pairwise comparisons in the binary AHP is to rank items linearly.

For this consistency test in the binary AHP we replace any symbols "-" in Table 3 with 0 and have Table 4. It is natural that M without a circular triad is consistent in the binary AHP and it is very useful for a decision maker that there is a standard for the number of circular triads that are able to be disregarded in M.

Table 4: The values of $d_{0.05,n}$ in the binary AHP

The number of objects n	3	4	5	6	7	8	9	10
The value of $d_{0.05,n}$	0	0	0	1	3	7	13	21
The total number of triads	1	4	10	20	35	56	84	120

Now we show how to apply the consistency test in Section 4 to the binary AHP in order to test whether we can accept that items are sufficiently ranked linearly. It is easy to do so because a binary comparison matrix has no tie.

We set the hypothesis that n items aren't ranked linearly for some n ($3 \leq n \leq 10$). On this assumption the decision maker compares n items pairwisely to make a binary comparison matrix $M = (m_{ij})$ in practice.

(B_1) Count the cardinal number of integers j such that $m_{ij} > 1$ for each i-th row, which is denoted by m_i.

(B_2) Substitute these m_i for a_i in Equation (2.3) to calculate d.

(B_3) If $d \leq d_{0.05,n}$ from Table 4, then we reject the hypothesis and think that the items are sufficiently ranked linearly. It is noted that we have supposed that the decision maker can make the binary comparison matrix which is consistent as measured by the number of circular triads.

If the binary comparison matrix M doesn't pass this test, then we refer the coefficient of consistency ζ for M using Equation (2.5). If ζ is near enough to 0 when it is compared with 1, then we reject the hypothesis and think that items are sufficiently ranked linearly. We note that the extent to which it is accepted by the numerical value of ζ depends finally on the decision maker's judgment.

Essentially we supposed that the decision maker is capable of comparing items pairwisely, because we shouldn't use the binary AHP if it is not so. If we have $d \geq d_{0.05,n} + 1$ and the hypothesis isn't rejected by the value of ζ, the followings are considered according to [7].

(a) Some of the items may differ by amounts which fall below the threshold of distinguishability for the decision maker.

(b) The property under pairwise comparisons may not be a linear variate at all.

(c) Several of the effects may be operating simultaneously.

So in this case we accept one of these and use other decision making support system, otherwise we need to change some values of elements in M in order to pass that consistency test. For instance, see [10] for a method of searching such elements. Nishizawa proposed a consistency improving method in the binary AHP and applied it to the AHP.

Note 5.1 If we have n items ($n \geq 11$) in the binary AHP, then we need to calculate $d_{0.05n}$ by a computer as in [1] .

Example 5.1 We use the following table in [10] to explain our method in the binary AHP.

Table 5: Results of matches (Application 2 in [10])

	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	θ	θ	$1/\theta$	θ	θ
O_2	$1/\theta$	1	θ	θ	θ	θ
O_3	$1/\theta$	$1/\theta$	1	θ	$1/\theta$	θ
O_4	θ	$1/\theta$	$1/\theta$	1	$1/\theta$	θ
O_5	$1/\theta$	$1/\theta$	θ	θ	1	θ
O_6	$1/\theta$	$1/\theta$	$1/\theta$	$1/\theta$	$1/\theta$	1

Now we use the procedure (B_1)–(B_3) in this section in order to see whether or not items O_i $(1 \le i \le 6)$ are ranked linearly. We have Table 6 about m_i in (B_1).

Table 6: The cardinal number m_i of integers j such that $m_{ij} > 1$

	O_1	O_2	O_3	O_4	O_5	O_6
m_i	4	4	2	2	3	0

In (B_2) the number of the circular triads d in Table 5 is calculated from Table 6 as follows;

$$d = \frac{1}{6} \times 6 \times 5 \times 4 - \frac{1}{2} \sum_{i=1}^{6} m_i(m_i - 1) = 3. \tag{5.2}$$

In (B_3) because $d = 3 > d_{0.05,6} = 1$, we cannot think that these items are ranked linearly. We calculate the coefficient of consistency ζ as follows;

$$\zeta = 1 - \frac{24 \times 3}{6^3 - 4 \times 6} = 0.625. \tag{5.3}$$

It is following from $d \ge d_{0.05,6} + 1$ and the value of ζ that it is difficult to rank items O_i $(1 \le i \le 6)$ in Table 5 linearly by the binary AHP. Certainly it is no problem to apply the binary AHP to the table under a rule of ranking by the binary AHP (see Remark 5.1).

Remark 5.1 The consistency test in this section should be used in order to rank some items subjectively by a decision maker. For instance, the binary AHP is also used for sport games or matches among n teams (see [10, 13]). In such case the information taken from a match between team O_i and O_j is only "a victory" or "defeat", without a tie. Let θ (> 1) be a fixed. We set $m_{ij} = \theta$ and $m_{ji} = 1/\theta$ when O_i wins O_j $(i \ne j)$, and $m_{ii} = 1$ to obtain the binary comparison matrix $M = (m_{ij})$. If $M = (m_{ij})$ is not completely logically consistent according to the above test, we think that these teams cannot be ranked linearly basically, but we don't need to completely reject the ranking by the binary AHP. In fact we can use it as a ranking by the binary AHP, though it is difficult for a team to plan a strategy when using this ranking.

6. Consistency test of a pairwise comparison matrix without a tie in the AHP

According to Section 5 we apply the consistency test in a sensory test, which is reviewed in Section 4, to the AHP. We use the scales $\{1/k,\ k \mid 1 \le k \le 9,\ k \text{ an integer}\}$ to compare items pairwisely in the AHP. So we deal with up to 9 items for this test. Furthermore, we deal only with a pairwise comparison matrix without a tie between different items.

When $m_{ij} > 1$, we describe $O_i \to O_j$. Thus we construct a complete directed graph M_G which is identified with M when discussing circular triads in it. We consider a test using M whether or not we can accept that items are ranked linearly as mentioned at the beginning of Section 5. This is a part of consistency test in the AHP and a test of logically consistency on a term of the binary AHP.

When a pairwise comparison matrix $M = (m_{ij})$ has no tie between each different items, then we can consider that the probability of $O_i \to O_j$ is equal to that of $O_j \to O_i$ when we decide a direction of the arrow between O_i and O_j on the scale $\{1/k,\ k \mid 1 \le k \le 9,\ k \text{ an integer}\}$ at random. So we can use the procedure (B_1)–(B_3) in Section 5 in this case for replacing Table 4 with Table 7 on the same hypothesis in the binary AHP.

Table 7: The values of $d_{0.05,n}$ in the AHP

The number of objects n	3	4	5	6	7	8	9
The value of $d_{0.05,n}$	0	0	0	1	3	7	13
The total number of triads	1	4	10	20	35	56	84

Moreover, we have the following theorem to easily test whether or not items are completely ranked linearly without the consistency test in the AHP.

Theorem 6.1 Let n be an integer such that $n \ge 3$, $M = (m_{ij})$ be a pairwise comparison matrix for n items in the AHP. For any integer i $(1 \le i \le n)$, we set m_i the cardinal number of $\{m_{ij} \mid 1 \le j \le n,\ j \ne i, m_{ij} > 1\}$. Let $S = \{m_i \mid 1 \le i \le n\}$. Then $S = \{0,\ 1,\ \ldots,\ n-1\}$ if and only if there exists no tie and no circuit in M.

Proof. Let M_G be the complete directed graph corresponding to M. When $S = \{0,\ 1,\ \ldots,\ n-1\}$, it is clear that there exists no tie in M_G. Since $S = \{0,\ 1,\ \ldots,\ n-1\}$ and Equation (2.3), the number of noncircular triads in M_G is

$$\frac{0 \times (0-1)}{2} + \frac{1 \times (1-1)}{2} + \sum_{k=2}^{n-1} \binom{k}{2}. \tag{6.1}$$

It follows that the number of the circular triads in M_G is

$$\binom{n}{3} - \left(\frac{0 \times (0-1)}{2} + \frac{1 \times (1-1)}{2} + \sum_{k=2}^{n-1} \binom{k}{2} \right) = \binom{n}{3} - \sum_{k=2}^{n-1} \binom{k}{2} = 0, \tag{6.2}$$

which means that there exists no circuit in M.

On the other hand since any circular n-ad contains at least one circular triad, it follows easily from the assumption that all items are ranked linearly without a tie. So we have clearly $S = \{0,\ 1,\ \ldots,\ n-1\}$. □

We give an example to understand Theorem 6.1.

Example 6.1 It follows from Theorem 6.1 that the following pairwise comparison table has no circular triad. In fact we have $n = 7$ and $S = \{0,\ 1,\ \ldots,\ 6\}$ in Theorem 6.1. There were numerous similar patterns when we researched the cases in the AHP.

Table 8: Trivial example of a pairwise comparison table without a circular triad

	O_1	O_2	O_3	O_4	O_5	O_6	O_7	m_i
O_1	1	2	3	4	5	7	9	6
O_2	1/2	1	2	3	3	4	5	5
O_3	1/3	1/2	1	2	2	3	3	4
O_4	1/4	1/3	1/2	1	2	2	3	3
O_5	1/5	1/3	1/2	1/2	1	2	2	2
O_6	1/7	1/4	1/3	1/2	1/2	1	2	1
O_7	1/9	1/5	1/3	1/3	1/2	1/2	1	0

C.I.=0.023

7. Conclusions

In this paper we proposed a test, using the pairwise comparison matrix in the binary AHP or one without a tie between different items in the AHP, to ascertain whether or not we can accept that the items which are factors or alternatives are ranked linearly, respectively. Originally, this consistency test is a consistency test used in a sensory test. Indeed since elements of a pairwise comparison matrix M in these AHPs are expressed on a numerical scale and a numerical scale are considered as an ordinal scale, items in M can be ranked linearly by it which is sufficiently consistent if those items are essentially ranked linearly.

On the other hand, the purpose of making a pairwise comparison matrix in these AHPs is to calculate each weight of items. As a result items are ranked linearly by the weights. So it is useful that before calculating weights of items we test whether or not these items are ranked linearly using the pairwise comparison matrix in the sense of ranking by a pick-the-winner ordinal scale. We think that this test enhances the utility of these AHPs that deal with ranking problems. For instance, we recommend using this test after we checked that the consistency index C.I. is less than 0.1.

In this paper we used the significant level $\alpha = 0.05$ in the consistency tests as in a sensory test, but a decision maker can suitably decide any significant level by using Tables 1 and 2. The general value in these AHPs might be requested. And in this paper we couldn't apply the test to items in a pairwise comparison matrix with a tie in the AHP. It is very hard to interpret a tie between two different items in the sense of inconsistency by circular triads. These are research topics which remain to be investigated.

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