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# RANKING BY RELATIONAL POWER BASED ON DIGRAPHS

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*Abstract* In this paper we examine the ranking of objects whose relative merits are given by a directed graph. We consider several measures and show their rationality through axiomatization as well as showing the relationship with the Shapley value of games whose characteristic function is derived from the directed graph. We also give some numerical examples and report the experience of the application to the best paper selection problem.

**Keywords**: Game theory, relational power, ranking, directed graph, axiomatization, Shapley value

## 1. Introduction

In this paper we consider the $n$ objects named $1, 2, \ldots, n$ that we need to rank hierarchically, for example, pianists, wines, papers nominated for the best paper award and so on. We assume that between every pair of objects a binary relation is available such as "$i$ wins $j$", "$j$ wins $i$" and "no match between $i$ and $j$". The binary relation between an object and itself is assumed to be "no match".

Van den Brink and Gilles [2] proposed some measures for ranking the objects based on the above setting, and showed the rationality of the measures through axiomatization and the discussion about the relationship with the Shapley value. The aim of this paper is to propose new measures and show their rationality.

Take $\{1, 2, \ldots, n\}$ as the node set, and draw a directed arc from $i$ to $j$ when $i$ wins $j$, then we have a *directed graph*, or *digraph* in short. The digraph thus constructed does not have self-loops or parallel arcs, i.e., $(i, i)$ is not an arc and at most one of $(i, j)$ and $(j, i)$ is an arc when $i \neq j$. We denote the node set by $N$, i.e., $N = \{1, 2, \ldots, n\}$, and the arc set by $D$, which is a subset of $N \times N$. Throughout this paper we will fix the node set and represent the digraph by its arc set $D$ alone. Let

$$S_D(i) := \{\, j \in N \mid (i, j) \in D \,\}$$

for $i \in N$, and call the nodes in $S_D(i)$ the *successors* of $i$ in $D$, and call the nodes in

$$P_D(i) := \{\, j \in N \mid (j, i) \in D \,\}$$

the *predecessors* of $i$ in $D$. We denote the cardinality of a set $A$ by $\kappa(A)$, and denote $\kappa(S_D(i))$ and $\kappa(P_D(i))$ by lower case letters $s_D(i)$ and $p_D(i)$, respectively for notational simplicity. Let $\mathbb{R}^N$ denote the set of $n$-dimensional real vectors whose components are given indices of $N$. We denote by $\mathcal{D}$ the collection of all digraphs on $N$ without self-loops or parallel arcs. We refer to the function $f : \mathcal{D} \to \mathbb{R}^N$ as a *measure* and to the $i$th component of $f(D)$ as $i$'s *relational power* with respect to digraph $D$.

After reviewing the existing measures and their axiomatization we will propose several new measures, show their rationality through axiomatizing them as well as showing the relationship with the Shapley value. We will demonstrate the character of measures by some numerical examples and also report the experience of their application to the best paper selection.

## 2. Existing Measures and Axioms

### 2.1. $\alpha$ plus measure and $\beta$ plus measure

Van den Brink and Gilles [2] introduced the $\alpha$ *plus measure*, which will be denoted by $\alpha^+ : \mathcal{D} \to \mathbb{R}^N$ in this paper, as

$$\alpha_i^+(D) := s_D(i) \quad (\forall i \in N, \forall D \in \mathcal{D})$$

and proposed a game theoretic axiomatization in terms of the following four axioms.

**Axiom 2.1** (Normalization)**.** The sum of all relational powers is equal to the number of arcs, i.e.,

$$\sum_{i \in N} f_i(D) = \kappa(D) \quad (\forall D \in \mathcal{D}).$$

**Axiom 2.2** (Dummy node property)**.** The relational power of the node which has no successors is zero, i.e.,

$$S_D(i) = \emptyset \text{ implies } f_i(D) = 0 \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

**Axiom 2.3** (Monotonicity)**.**

$$S_D(i) \supseteq S_D(j) \text{ implies } f_i(D) \geq f_j(D) \quad (\forall i, j \in N, \forall D \in \mathcal{D}).$$

For the fourth axiom they introduced an independent partition of $D$. A partition of $D$ is a collection $\mathcal{S} = \{ D_1, \ldots, D_m \}$ that satisfies

- $\bigcup_{k=1}^{m} D_k = D$,
- $D_k \cap D_l = \emptyset \quad (\forall k, l\,;\, 1 \leq k < l \leq m)$.

**Definition 2.4.** A partition $\mathcal{S} = \{ D_1, \ldots, D_m \}$ of $D$ is said to be *independent* when

$$\kappa(\{ k \mid P_{D_k}(i) \neq \emptyset \}) \leq 1 \quad (\forall i \in N).$$

Figure 2.1 shows an example of the independent partition $\{D_1, D_2\}$, depicted by solid arrows and broken arrows, respectively.

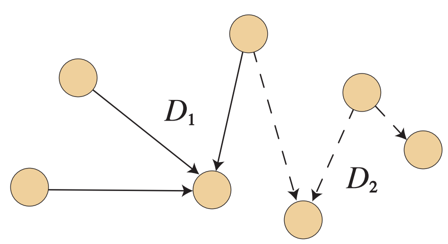


Figure 2.1: Independent partition

**Axiom 2.5** (Additivity over independent partitions)**.** The sum of the relational powers which are measured on an independent partition $\mathcal{S}$ of $D$ is equal to the relational power on $D$, i.e.,

$$f_i(D) = \sum_{D_k \in \mathcal{S}} f_i(D_k) \quad (\forall i \in N, \forall \text{independent partition } \mathcal{S} = \{D_1, \cdots, D_m\} \text{ of } D, \forall D \in \mathcal{D}).$$

The axiomatization theorem of the $\alpha$ plus measure is as follows.

**Theorem 2.6** (Theorem 3.3 [2])**.** *A function $f : \mathcal{D} \to \mathbb{R}^N$ is equal to the $\alpha$ plus measure on $N$ if and only if it satisfies Axiom 2.1, 2.2, 2.3 and 2.5.*

Van den Brink and Gilles also introduced the $\beta$ *plus measure* given by

$$\beta_i^+(D) := \sum_{j \in S_D(i)} \frac{1}{p_D(j)} \quad (\forall i \in N, \forall D \in \mathcal{D})$$

and showed that it is axiomatized as will be shown in Theorem 2.8 by replacing Axiom 2.1 with the following normalization axiom.

**Axiom 2.7** (Normalization)**.** The sum of all relational powers is equal to the number of the nodes which has some predecessors, i.e.,

$$\sum_{i \in N} f_i(D) = \kappa(\{\, j \in N \mid P_D(j) \neq \emptyset \,\}) \quad (\forall D \in \mathcal{D}).$$

**Theorem 2.8** (Theorem 2.7 [2])**.** *A function $f : \mathcal{D} \to \mathbb{R}^N$ is equal to the $\beta$ plus measure on $N$ if and only if it satisfies Axiom 2.2, 2.3, 2.5 and 2.7.*

They also showed how these measures are related to the Shapley value [7]. Let us consider the cooperative game with the node set $N$ as its player set. Each non-empty subset $C \subseteq N$ is called a *coalition*. The *characteristic function* $v : 2^N \to \mathbb{R}$ is a function that gives the coalition value $v(C)$, the maximum utility that coalition $C$ can obtain. The Shapley value of player $i$ is given by

$$\varphi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(V(i,\pi) \cup \{\, i \,\}) - v(V(i,\pi))], \tag{2.1}$$

where $\pi$ is a permutation of $N$ and $\Pi$ is the collection of all permutations of $N$, $V(i,\pi)$ is the set of players that precede player $i$ in permutation $\pi$, i.e., $V(i,\pi) = \{\, j \in N \mid \pi(j) < \pi(i) \,\}$. A permutation $\pi$ means the order in which the players enter the coalition, and $v(V(i,\pi) \cup \{\, i \,\}) - v(V(i,\pi))$ is the increment of the coalition value when player $i$ enters the coalition $V(i,\pi)$, namely player $i$'s contribution to coalition $V(i,\pi) \cup \{\, i \,\}$. See, for example, Owen [4] for further properties of the Shapley value.

Given $D \in \mathcal{D}$ and $C \subseteq N$, we define the set of successors of $C$ in $D$ as

$$S_D(C) := \bigcup_{i \in C} S_D(i)$$

and denote its cardinality by $s_D(C)$. Note that $s_D(C)$ is not always identical to $\sum_{i \in C} s_D(i)$. We define the set of predecessors $P_D(C)$ in the same manner and denote its cardinality by $p_D(C)$. The following theorem concerning the relationship between $\beta$ plus measure and the Shapley value is due to van den Brink and Gilles [2].

**Theorem 2.9** (Theorem 4.2 [2]). *For every* $C \subseteq N$ *let*

$$v_\beta(C) = s_D(C),$$

*then the Shapley value* $\varphi(v_\beta)$ *on characteristic function game* $(N, v_\beta)$ *is equal to the* $\beta$ *plus measure of digraph* $D$.

In the similar manner we see the following theorem, whose proof is omitted.

**Theorem 2.10.** *For every* $C \subseteq N$ *let*

$$v_\alpha(C) = \kappa(\{\, (i,j) \mid i \in C, (i,j) \in D \,\}),$$

*then the Shapley value* $\varphi(v_\alpha)$ *on characteristic function game* $(N, v_\alpha)$ *is equal to the* $\alpha$ *plus measure of digraph* $D$.

## 3. New Measures, Axiomatization and Relationship with Shapley Value

In the preceding discussion on the $\alpha$ and the $\beta$ plus measures of node $i$, we focused on only the nodes that lose node $i$. We will propose new measures by fully utilizing the information about who wins and who loses node $i$.

### 3.1. $\gamma$ plus measure

We define the $\gamma$ *plus measure* $\gamma^+ : \mathcal{D} \to \mathbb{R}^N$ by

$$\gamma_i^+(D) := \sum_{j \in S_D(i)} \frac{s_D(j)+1}{p_D(j)} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

The $\gamma$ plus measure considers the nodes that node $i$ wins, i.e., the sum is taken over all nodes of $S_D(i)$, and also how many times those nodes win and lose. We will give several axioms for the axiomatization of the $\gamma$ plus measure. The first axiom is concerning the normalization.

**Axiom 3.1** (Normalization).

$$\sum_{i \in N} f_i(D) = \sum_{\substack{j \in N \\ P_D(j) \neq \emptyset}} s_D(j) + \kappa(\{\, j \in N \mid P_D(j) \neq \emptyset \,\}) \quad (\forall D \in \mathcal{D}).$$

**Axiom 3.2** (Extended dummy node property).

$$S_D\big(S_D(i)\big) = \emptyset \text{ implies } f_i(D) = \sum_{j \in S_D(i)} \frac{1}{p_D(j)} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

**Axiom 3.3** (Monotonicity).

$$S_D(i) \supseteq S_D(j) \text{ implies } f_i(D) \geq f_j(D) \quad (\forall i, j \in N, \forall D \in \mathcal{D}).$$

In order to give the fourth axiom we define a subdigraph $D'_k$ for each $k \in N$.

**Definition 3.4.** For $k \in N$, the subdigraph $D'_k$ is defined as

$$D'_k := \{\, (i,k) \mid i \in P_D(k) \,\} \cup \{\, (k,j) \mid j \in S_D(k) \,\}.$$

**Axiom 3.5** (Additivity over subdigraphs). Node $i$'s relational power on the digraph $D$ is equal to the difference of two terms: the first term is the sum of node $i$'s relational power on subdigraphs $D'_k$, and the second term is the number of $i$'s successors, i.e.,

$$f_i(D) = \sum_{k \in N} f_i(D'_k) - s_D(i) \quad (\forall i \in N, \forall D \in \mathcal{D}). \tag{3.1}$$

Then we show that the four axioms introduced above axiomatize the $\gamma$ plus measure.

**Theorem 3.6.** *A function $f : \mathcal{D} \to \mathbb{R}^N$ is equal to the $\gamma$ plus measure on $N$ if and only if it satisfies Axiom 3.1, 3.2, 3.3 and 3.5.*

*Proof.* Since it is easily seen that the $\gamma$ plus measure satisfies Axiom 3.1, 3.2 and 3.3, we will show that it satisfies Axiom 3.5. Let $i \in N$ be fixed. For $k \in N$, there are four possible cases: $k \in S_D(i)$, $k \in P_D(i)$, $k = i$ and the rest. Then the right hand side of (3.1) with $f$ replaced by $\gamma^+$ is rewritten as

$$\sum_{k \in N} \gamma_i^+(D'_k) - s_D(i) = \sum_{k \in S_D(i)} \gamma_i^+(D'_k) + \sum_{k \in P_D(i)} \gamma_i^+(D'_k) + \gamma_i^+(D'_i) + \sum_{k \in N \setminus (S_D(i) \cup P_D(i) \cup \{ i \})} \gamma_i^+(D'_k) - s_D(i). \quad (3.2)$$

Next we calculate the value of $\gamma_i^+(D'_k)$ in each case.

- case 1: $k \in S_D(i)$
  It holds that $S_{D'_k}(i) = \{k\}$. Then by the construction of $D'_k$ we have

$$\gamma_i^+(D'_k) = \sum_{j \in S_{D'_k}(i)} \frac{s_{D'_k}(j) + 1}{p_{D'_k}(j)} = \frac{s_{D'_k}(k) + 1}{p_{D'_k}(k)} = \frac{s_D(k) + 1}{p_D(k)}. \quad (3.3)$$

- case 2: $k \in P_D(i)$
  Since $S_{D'_k}(i) = \emptyset$, we have

$$\gamma_i^+(D'_k) = 0. \quad (3.4)$$

- case 3: $k = i$
  It holds that $S_{D'_i}(j) = \emptyset$ and $P_{D'_i}(j) = \{i\}$ for every $j \in S_{D'_i}(i)$. Then

$$\gamma_i^+(D'_i) = \sum_{j \in S_{D'_i}(i)} \frac{s_{D'_i}(j) + 1}{p_{D'_i}(j)} = \sum_{j \in S_{D'_i}(i)} \frac{1}{1} = s_D(i). \quad (3.5)$$

- case 4: $k \in N \setminus (S_D(i) \cup P_D(i) \cup \{ i \})$
  Since $S_{D'_k}(i) = \emptyset$, we readily see

$$\gamma_i^+(D'_k) = 0. \quad (3.6)$$

Combining (3.3), (3.4), (3.5) and (3.6), we have

$$\begin{aligned} \sum_{k \in N} \gamma_i^+(D'_k) - s_D(i) &= \sum_{k \in S_D(i)} \frac{s_D(k) + 1}{p_D(k)} + 0 + s_D(i) + 0 - s_D(i) \\ &= \sum_{k \in S_D(i)} \frac{s_D(k) + 1}{p_D(k)} = \gamma_i^+(D). \end{aligned} \quad (3.7)$$

Supposing that a function $f : \mathcal{D} \to \mathbb{R}^N$ satisfies the four axioms, we show that $f$ is equal to the $\gamma$ plus measure. Given $D \in \mathcal{D}$ we consider the value of the function $f$ on subdigraph

$D'_k$. It follows from the above discussion and Axiom 3.3 that there is a constant, say $c \in \mathbb{R}$ such that for every $i \in P_D(k)$

$$f_i(D'_k) = c. \tag{3.8}$$

From Axiom 3.2 we also see that for every $i \notin P_D(k) \cup \{k\}$

$$f_i(D'_k) = 0 \tag{3.9}$$

and

$$f_k(D'_k) = \sum_{j \in S_{D'_k}(k)} \frac{1}{p_{D'_k}(j)} = \sum_{j \in S_{D'_k}(k)} \frac{1}{1} = s_D(k). \tag{3.10}$$

Applying Axiom 3.1 to the subdigraph $D'_k$, then we obtain that

$$\begin{aligned} \sum_{i \in N} f_i(D'_k) &= \sum_{\substack{j \in N \\ P_{D'_k}(j) \neq \emptyset}} s_{D'_k}(j) + \kappa(\{\, j \in N \mid P_{D'_k}(j) \neq \emptyset \,\}) \\ &= s_D(k) + (s_D(k) + 1) \\ &= 2s_D(k) + 1. \end{aligned} \tag{3.11}$$

By (3.8), (3.9) and (3.10) we have

$$\begin{aligned} \sum_{i \in N} f_i(D'_k) &= \sum_{i \in P_D(k)} f_i(D'_k) + \sum_{i \in S_D(k)} f_i(D'_k) + f_k(D'_k) + \sum_{i \in N \setminus (P_D(k) \cup S_D(k) \cup \{k\})} f_i(D'_k) \\ &= p_D(k) \times c + 0 + s_D(k) + 0 \\ &= p_D(k) \times c + s_D(k), \end{aligned} \tag{3.12}$$

which together with (3.11) yields that $c = \dfrac{s_D(k)+1}{p_D(k)}$. Then by Axiom 3.5, we obtain that for every $i$

$$\begin{aligned} f_i(D) &= \sum_{k \in N} f_i(D'_k) - s_D(i) \\ &= \left( \sum_{k \in S_D(i)} f_i(D'_k) + \sum_{k \in P_D(i)} f_i(D'_k) + f_i(D'_i) + \sum_{k \in N \setminus (S_D(i) \cup P_D(i) \cup \{i\})} f_i(D'_k) \right) - s_D(i) \\ &= \sum_{k \in S_D(i)} \frac{s_D(k)+1}{p_D(k)} + 0 + s_D(i) + 0 - s_D(i) \\ &= \sum_{k \in S_D(i)} \frac{s_D(k)+1}{p_D(k)} = \gamma_i^+(D). \end{aligned}$$

Therefore we conclude that $f = \gamma^+$. □

Indeed the second term of Axiom 3.1 is observed in Axiom 2.7, and $\beta$ plus measure emerges in Axiom 3.2. Also the term $s_D(i)$ in Axiom 3.5 is the $\alpha$ plus measure. Thus the $\gamma$ plus measure is based on both the $\alpha$ plus measure and the $\beta$ plus measure, however the major drawback is that the axioms do not admit a natural interpretation.

### 3.2. $\delta$ plus measure

Borm *et al.* [1] proposed the following measure, which we will call the $\delta$ *plus measure* in this paper, but they did not axiomatize it:

$$\delta_i^+(D) := \sum_{j \in S_D(i) \cup \{i\}} \frac{1}{p_D(j)+1} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

This measure can also be axiomatized as follows.

**Axiom 3.7** (Normalization)**.** The sum of all relational powers is equal to $n$, the number of nodes of $N$, i.e.,

$$\sum_{i \in N} f_i(D) = n \quad (\forall D \in \mathcal{D}).$$

**Axiom 3.8** (Dummy node property)**.** The relational power of the node without successors is one divided by the number of its predecessors incremented by one, i.e.,

$$S_D(i) = \emptyset \text{ implies } f_i(D) = \frac{1}{p_D(i)+1} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

**Axiom 3.9** (Monotonicity)**.**

$$S_D(i) \supseteq S_D(j) \text{ implies } f_i(D) \geq f_j(D) \quad (\forall i, j \in N, \forall D \in \mathcal{D}).$$

In order to give the fourth axiom we introduce another partition $\mathcal{T} = \{ D''_k \mid k \in N \}$ of $D$, where

$$D''_k = \{ (i,k) \mid i \in P_D(k) \}.$$

**Axiom 3.10** (Additivity over partition $\mathcal{T}$)**.**

$$f_i(D) - 1 = \sum_{k \in N} (f_i(D''_k) - 1) \quad (\forall i \in N, \forall D \in \mathcal{D}). \tag{3.13}$$

Then we see that the $\delta$ plus measure is characterized by the four axioms introduced above.

**Theorem 3.11.** *A function $f : \mathcal{D} \to \mathbb{R}^N$ is equal to the $\delta$ plus measure on $N$ if and only if it satisfies Axiom 3.7, 3.8, 3.9 and 3.10.*

*Proof.* It can easily be seen that the $\delta$ plus measure satisfies Axiom 3.7, Axiom 3.8 and Axiom 3.9. Now we will show that it satisfies Axiom 3.10. Let $i \in N$ be fixed. For each $k \in N$, consider the three possible cases: $k \in S_D(i)$, $k = i$ and the rest. Then the right hand side of (3.13) with $f$ replaced by $\delta^+$ is written as

$$\sum_{k \in N} (\delta_i^+(D''_k) - 1) \;=\; \sum_{k \in S_D(i)} \delta_i^+(D''_k) + \delta_i^+(D''_i) + \sum_{k \in N \setminus (S_D(i) \cup \{ i \})} \delta_i^+(D''_k) - n.$$

- case 1: $k \in S_D(i)$
  Since $S_{D''_k}(i) = \{k\}$ and $P_{D''_k}(i) = \emptyset$, we have

$$\delta_i^+(D''_k) = \sum_{j \in S_{D''_k}(i) \cup \{ i \}} \frac{1}{p_{D''_k}(j)+1} = \frac{1}{p_D(k)+1} + 1 \quad (\forall k \in S_D(i)). \tag{3.14}$$

- case 2: $k = i$
  Since $S_{D''_i}(i) = \emptyset$, we see

$$\delta_i^+(D''_i) = \sum_{j \in S_{D''_i}(i) \cup \{ i \}} \frac{1}{p_{D''_i}(j) + 1} = \frac{1}{p_D(i) + 1} \quad (k = i). \tag{3.15}$$

- case 3: $k \in N \setminus (S_D(i) \cup \{ i \})$
  We have $S_{D''_k}(i) = \emptyset$ and $P_{D''_k}(i) = \emptyset$, and then

$$\delta_i^+(D''_k) = \sum_{j \in S_{D''_k}(i) \cup \{ i \}} \frac{1}{p_{D''_k}(j) + 1} = 1 \quad (\forall k \in N \setminus (S_D(i) \cup \{ i \})). \tag{3.16}$$

According to (3.14), (3.15) and (3.16), we obtain that (3.14) is

$$\begin{aligned}
\sum_{k \in N} (\delta_i^+(D''_k) - 1) &= \sum_{k \in S_D(i)} \delta_i^+(D''_k) + \delta_i^+(D''_i) + \sum_{k \in N \setminus (S_D(i) \cup \{ i \})} \delta_i^+(D''_k) - n \\
&= \sum_{k \in S_D(i)} \left( \frac{1}{p_D(k) + 1} + 1 \right) + \frac{1}{p_D(i) + 1} + 1 \times (n - (s_D(i) + 1)) - n \\
&= \sum_{k \in S_D(i)} \frac{1}{p_D(k) + 1} + s_D(i) + \frac{1}{p_D(i) + 1} - s_D(i) - 1 \\
&= \sum_{k \in S_D(i) \cup \{i\}} \frac{1}{p_D(k) + 1} - 1 \\
&= \delta_i^+(D) - 1.
\end{aligned}$$

This proves that $\delta$ plus measure satisfies Axiom 3.10.

Next supposing that a function $f : \mathcal{D} \to \mathbb{R}^N$ satisfies the four axioms, we will show that $f = \delta^+$. For $k \in N$ we consider the value of the function $f$ on $D''_k$. Since for every $i \in P_D(k)$ i.e., $k \in S_D(i)$ it follows from the above discussion and Axiom 3.9 that there is a constant $c \in \mathbb{R}$ such that

$$f_i(D''_k) = c. \tag{3.17}$$

For $i = k$ it follows from the above discussion and Axiom 3.8 that

$$f_k(D''_k) = \frac{1}{p_{D''_k}(k) + 1} = \frac{1}{p_D(k) + 1}. \tag{3.18}$$

For $i \in N \setminus (P_D(k) \cup \{k\})$ we have from the above discussion and Axiom 3.8 that

$$f_i(D''_k) = \frac{1}{0 + 1} = 1. \tag{3.19}$$

We apply Axiom 3.7 to $D''_k$, and we obtain that

$$\sum_{i \in N} f_i(D''_k) = n, \tag{3.20}$$

while according to (3.17), (3.18) and (3.19),

$$\begin{aligned}
\sum_{i \in N} f_i(D''_k) &= \sum_{i \in P_D(k)} f_i(D''_k) + f_i(D''_i) + \sum_{i \in N \setminus (P_D(k) \cup \{k\})} f_i(D''_k) \\
&= p_D(k) \times c + \frac{1}{p_D(k) + 1} + n - (p_D(k) + 1).
\end{aligned}$$

Then from (3.20) we see $c = 1 + \dfrac{1}{p_D(j)+1}$. Since digraphs $D''_k$ form a partition $\mathcal{T}$, it follows from Axiom 3.10 that for every $i \in N$

$$\begin{aligned}
f_i(D) &= \sum_{k\in N}(f_i(D''_k)-1)+1 \\
&= \sum_{k\in N} f_i(D''_k) - n + 1 \\
&= \sum_{k\in S_D(i)} f_i(D''_k) + f_i(D''_i) \sum_{k\in N\setminus(S_D(i)\cup\{i\})} f_i(D''_k) - n + 1 \\
&= \sum_{k\in S_D(i)} \left(1+\frac{1}{p_D(k)+1}\right) + \frac{1}{p_D(i)+1} + n - (s_D(i)+1) - n + 1 \\
&= s_D(i) + \sum_{k\in S_D(i)} \frac{1}{p_D(k)+1} + \frac{1}{p_D(i)+1} + n - s_D(i) - 1 - n + 1 \\
&= \sum_{k\in S_D(i)\cup\{i\}} \frac{1}{p_D(k)+1} = \delta_i^+(D).
\end{aligned}$$

Therefore we conclude that $f = \delta^+$. □

### 3.3. Two-stage game

We will show in this subsection a relationship between the $\gamma$ plus measure and the Shapley value. The key idea is, say *two-stage game*. Namely, based on the given digraph $D$ we construct the first stage game and obtain the Shapley value. The second stage game is constructed based on not only the digraph $D$ but also thus obtained Shapley value. We will show that if appropriately constructed, the second stage game will provide us with the $\gamma$ plus measure as the Shapley value. The idea is illustrated below, where $\mathbf{e}$ is the vector of ones.

$$\left.\left.\begin{array}{l}\mathbf{e}\in\mathbb{R}^N\\ \text{1st configuration}\end{array}\right\} \rightarrow \begin{array}{c}(N,v_1) \;\rightarrow\; \varphi(v_1)\\ \\ \text{2nd configuration}\end{array}\right\} \rightarrow (N,v_2) \rightarrow \varphi(v_2)=\gamma^+$$

For $C \subseteq N$ let us denote the set of nodes of $C$ and their successors by

$$\overline{S}_D(C) := C \cup S_D(C).$$

**Lemma 3.12.** *Let the characteristic function $v_D : 2^N \to \mathbb{R}$ be defined as*

$$v_D(C) = \sum_{i\in C} \kappa\big(\overline{S}_D(\{i\})\big)$$

*for $C \subseteq N$. Then node i's Shapley value $\varphi_i(v_D)$ is $s_D(i)+1$.*

*Proof.* Let [ ] be the indicator function which gives one if the statement in the brackets is true and zero otherwise, see Graham *et al.* [3]. Then

$$\begin{aligned} v_D(C) &= \sum_{i\in C} \kappa\big(\overline{S}_D(\{i\})\big) \\ &= \sum_{i\in C} \sum_{j\in \overline{S}_D(\{i\})} 1 \\ &= \sum_{i\in C}\sum_{j\in N} [\, j \in \overline{S}_D(\{i\})\, ] \\ &= \sum_{i\in C}\sum_{j\in N} [\, j \in S_D(\{i\}) \cup \{i\}\, ] \\ &= \sum_{i\in N}\sum_{j\in N} [\, i \in C\, ][\, j \in S_D(\{i\}) \cup \{i\}\, ] \\ &= \sum_{i\in N}\sum_{j\in N} [\, i \in C, (i,j) \in D\, ] + \sum_{i\in N} [\, i \in C\, ] \\ &= \sum_{i\in N}\sum_{j\in N} [\, i \in C, (i,j) \in D\, ] + \kappa(C). \qquad (3.21) \end{aligned}$$

Using $D''_j = \{\, (i,j) \mid i \in P_D(j)\, \}$ introduced in Section 3.2, we consider the sum $\sum_{j\in N} v_{D''_j}(C)$:

$$\begin{aligned} \sum_{j\in N} v_{D''_j}(C) &= \sum_{j\in N} \left( \sum_{i\in N}\sum_{k\in N} [\, i \in C, (i,k) \in D''_j\, ] + \kappa(C) \right) \\ &= \sum_{j\in N} \left( \sum_{i\in N} [\, i \in C, (i,j) \in D\, ] + \kappa(C) \right) \\ &= \sum_{i\in N}\sum_{j\in N} [\, i \in C, (i,j) \in D\, ] + n\kappa(C) \\ &= v_D(C) + (n-1)\kappa(C). \qquad (3.22) \end{aligned}$$

Note that $\kappa$ is the function that gives the cardinality of a set, and then we obtain from the the additivity of the Shapley value that

$$\begin{aligned} \varphi_i(v_D) &= \varphi_i \left( \sum_{j\in N} v_{D''_j} - (n-1)\kappa \right) \\ &= \sum_{j\in N} \varphi_i(v_{D''_j}) - (n-1)\varphi_i(\kappa). \qquad (3.23) \end{aligned}$$

Let $C$ be a coalition and node $i$ be outside of $C$. Then

$$v_{D''_j}(C \cup \{i\}) - v_{D''_j}(C) = \begin{cases} 2 & \text{if } i \in P_D(j) \\ 1 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases} \qquad (3.24)$$

It should be noted that node $i$'s contribution to a coalition is independent of the coalition. Take the average of node $i$'s contributions over all permutations of $N$, and we obtain from (2.1) and (3.24) that

$$\varphi_i(v_{D''_j}) = \begin{cases} 2 & \text{if } i \in P_D(j) \\ 1 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases} \qquad (3.25)$$

Note that $\varphi_i(\kappa) = 1$ since $\kappa(C \cup \{i\}) - \kappa(C) = 1$ whenever $i \notin C$. According to (3.23) and (3.25), we obtain that

$$\begin{aligned}
\varphi_i(v_D) &= \sum_{j \in N} \varphi_i(v_{D''_j}) - (n-1)\varphi_i(\kappa) \\
&= \sum_{j \in S_D(i)} \varphi_i(v_{D''_j}) + \varphi_i(v_{D''_i}) + \sum_{j \in N \setminus (S_D(i) \cup \{i\})} \varphi_i(v_{D''_j}) - (n-1)\varphi_i(\kappa) \\
&= \sum_{j \in S_D(i)} 2 + 1 + \sum_{j \in N \setminus (S_D(i) \cup \{i\})} 1 - (n-1)1 \\
&= 2 s_D(i) + 1 + n - s_D(i) - 1 - n + 1 \\
&= s_D(i) + 1.
\end{aligned}$$

□

**Lemma 3.13.** *Given $\theta \in \mathbb{R}^N$, let the characteristic function $v_{(D,\theta)}$ be*

$$v_{(D,\theta)}(C) := \sum_{j \in S_D(C)} \theta_j,$$

*and consider the game $(N, v_{(D,\theta)})$. Then node $i$'s the Shapley value is given as*

$$\varphi_i(v_{(D,\theta)}) = \sum_{j \in S_D(i)} \frac{\theta_j}{p_D(j)}.$$

*Proof.* First we rewrite $v_{(D,\theta)}(C)$ by using the bracket notation.

$$v_{(D,\theta)}(C) = \sum_{j \in S_D(C)} \theta_j = \sum_{j \in N} \theta_j [\, \exists i \in C, (i,j) \in D \,].$$

Similarly for $D''_j = \{\, (i,j) \mid i \in P_D(j) \,\}$ we have

$$\begin{aligned}
\sum_{j \in N} v_{(D''_j,\theta)}(C) &= \sum_{j \in N} \sum_{k \in S_{D''_j}(C)} \theta_k \\
&= \sum_{j \in N} \sum_{k \in N} \theta_k [\, \exists i \in C, (i,k) \in D''_j \,] \\
&= \sum_{j \in N} \theta_j [\, \exists i \in C, (i,j) \in D \,]. \qquad (3.26)
\end{aligned}$$

Therefore we see that

$$v_{(D,\theta)}(C) = \sum_{j \in N} v_{(D''_j,\theta)}(C) \qquad (3.27)$$

holds for every $C \subseteq N$.

For the digraph $D''_j$, coalition $C$ and node $i \in N \setminus C$ we have

$$v_{(D''_j,\theta)}(C \cup \{i\}) - v_{(D''_j,\theta)}(C) = \begin{cases} \theta_j & \text{if } i \in P_D(j) \text{ and } C \cap P_D(j) = \emptyset \\ 0 & \text{otherwise.} \end{cases} \qquad (3.28)$$

Let the elements of $P_D(j)$ be $\{i_1, i_2, \ldots, i_{p_D(j)}\}$. Focusing on the element of $P_D(j)$, we divide all the permutations into $p_D(j)$ groups: for $l = 1, 2, \ldots, p_D(j)$ group $G_l$ consists of all the permutations where $i_l$ precedes other nodes of $P_D(j)$. Then each group consists of $n!/p_D(j)$ permutations. Therefore from (2.1) together with (3.28), the Shapley value of node $i$ in game $(N, v_{(D''_j,\theta)})$ is given by

$$\varphi_i(v_{(D''_j,\theta)}) = \begin{cases} \dfrac{\theta_j}{p_D(j)} & \text{if } i \in P_D(j) \\ 0 & \text{otherwise.} \end{cases} \tag{3.29}$$

Applying the additivity property of the Shapley value to (3.27) and (3.29) yields that

$$\begin{aligned} \varphi_i(v_{(D,\theta)}) &= \varphi_i\left(\sum_{j\in N} v_{(D''_j,\theta)}\right) = \sum_{j\in N} \varphi_i(v_{(D''_j,\theta)}) \\ &= \sum_{j\in S_D(i)} \varphi_i(v_{(D''_j,\theta)}) + \sum_{j\in N\setminus S_D(i)} \varphi_i(v_{(D''_j,\theta)}) = \sum_{j\in S_D(i)} \frac{\theta_j}{p_D(j)} \end{aligned}$$

for all $i \in N$. □

**Theorem 3.14.** *Let*

$$v_D^1(C) := \sum_{i\in C} \kappa(\overline{S}_D(\{i\}))$$

*for every $C \subseteq N$, and let $\varphi(v_D^1)$ be the Shapley value of the game $(N, v_D^1)$. Let $v_D^2(C)$ be defined as*

$$v_D^2(C) := \sum_{j\in S_D(C)} \varphi_j(v_D^1).$$

*Then the Shapley value of the game $(N, v_D^2)$ is equal to the $\gamma$ plus measure.*

*Proof.* Straightforward from Lemma 3.12 and Lemma 3.13. □

### 3.4. Minus measures and plus minus measures

Exchanging simply the roles of $S_D(i)$ and $P_D(i)$, we obtain the *minus measures.* Note that the more a node loses, the higher relational power the minus measure gives to the node. A combination of the plus and minus measures will give the *plus minus measures.* We list the definition of each measure and the characteristic function giving it in Table 3.1. Since the $\gamma$ plus and minus measures are obtained through the two-stage game, we do not provide the characteristic function for the $\gamma$ plus minus measure.

## 4. Experiment and Experience

### 4.1. Numerical experiment

Generating possible digraphs on the node set $N$, we computed the relational power given by the measures discussed so far, and compare them with that provided by the Analytic Network Process, ANP for short.

First we consider the digraphs every pair of which is connected by an arc, which we will refer to as a *complete* digraph. When the digraph $D$ expresses a total order on $N$, the transitivity is met, namely $(i, j), (j, k) \in D$ implies $(i, k) \in D$. The digraph lacking in this property has a directed cycle, and hence the number of directed cycles in $D$ can be viewed as the degree of inconsistency.

Table 3.1: List of measures

| name | symbol | definition | characteristic function |
|---|---|---|---|
| $\alpha$ plus [2] | $\alpha_i^+(D)$ | $s_D(i)$ | $\kappa(\{(i,j) \mid (i,j) \in D, i \in C\})$ |
| $\alpha$ minus | $\alpha_i^-(D)$ | $p_D(i)$ | $\kappa(\{(j,i) \mid (j,i) \in D, i \in C\})$ |
| $\alpha$ plus minus | $\alpha_i^\pm(D)$ | $\alpha_i^+(D) - \alpha_i^-(D)$ | $\kappa(\{(i,j) \mid (i,j) \in D, i \in C\})$ $-\kappa(\{(j,i) \mid (j,i) \in D, i \in C\})$ |
| $\beta$ plus [2] | $\beta_i^+(D)$ | $\sum_{j \in S_D(i)} \frac{1}{p_D(j)}$ | $\kappa(\{k \mid i \in C, (i,k) \in D\})$ [2] |
| $\beta$ minus | $\beta_i^-(D)$ | $\sum_{j \in P_D(j)} \frac{1}{s_D(j)}$ | $\kappa(\{k \mid j \in C, (k,j) \in D\})$ |
| $\beta$ plus minus | $\beta_i^\pm(D)$ | $\beta_i^+(D) - \beta_i^-(D)$ | $\kappa(\{k \mid i \in C, (i,k) \in D\})$ $-\kappa(\{k \mid j \in C, (k,j) \in D\})$ |
| $\gamma$ plus | $\gamma_i^+(D)$ | $\sum_{j \in S_D(i)} \frac{s_D(j)+1}{p_D(j)}$ | first : $\sum_{i \in C} \kappa(\overline{S}_D(\{i\}))$ second : $\sum_{j \in S_D(C)} \varphi_j(v_D^1)$ |
| $\gamma$ minus | $\gamma_i^-(D)$ | $\sum_{j \in P_D(i)} \frac{p_D(j)+1}{s_D(j)}$ | first : $\sum_{i \in C} \kappa(\overline{P}_D(\{i\}))$ second : $\sum_{j \in P_D(C)} \varphi_j(v_D^1)$ |
| $\gamma$ plus minus | $\gamma_i^\pm(D)$ | $\gamma_i^+(D) - \gamma_i^-(D)$ | |
| $\delta$ plus [1] | $\delta_i^+(D)$ | $\sum_{j \in S_D(i) \cup \{i\}} \frac{1}{p_D(j)+1}$ | $\kappa(\{j \in N \mid P_D(j) \cup \{j\} \subset C\})$ [1] |
| $\delta$ minus | $\delta_i^-(D)$ | $\sum_{j \in P_D(i) \cup \{i\}} \frac{1}{s_D(j)+1}$ | $\kappa(\{j \in N \mid S_D(j) \cup \{j\} \subset C\})$ |
| $\delta$ plus minus | $\delta_i^\pm(D)$ | $\delta_i^+(D) - \delta_i^-(D)$ | $\kappa(\{j \in N \mid P_D(j) \cup \{j\} \subset C\})$ $-\kappa(\{j \in N \mid S_D(j) \cup \{j\} \subset C\})$ |

Let the $n \times n$ matrix $T := [t_{ij}]_{i,j \in N}$ be constructed as

$$\begin{cases} t_{ij} := \theta \\ t_{ji} := 1/\theta \end{cases} \quad \text{if } (i,j) \in D$$

for some $\theta > 1$. The ANP uses the principal eigenvector of the matrix $T$ as the vector of relational powers. In the following computation we took $\theta = 2$.

We generated all possible complete digraphs on five nodes and computed relational powers. We will show several examples of them. The first example is a complete digraph whose adjacency matrix is given by

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table 4.1 shows the relational powers given by the measures and ANP as well as the rankings based on them. Upper case alphabets $A$ to $E$ are node names, and the sign of minus measures is reversed for ease of comparison. Clearly all measures as well as ANP provide the same ranking as expected.

Table 4.1: Relational powers for a complete digraph with no cycle

| | relational power | | | | | ranking | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| measure | $A$ | $B$ | $C$ | $D$ | $E$ | $A$ | $B$ | $C$ | $D$ | $E$ |
| ANP | 0.323 | 0.245 | 0.185 | 0.141 | 0.107 | 1 | 2 | 3 | 4 | 5 |
| $\alpha^+$ | 4.000 | 3.000 | 2.000 | 1.000 | 0.000 | 1 | 2 | 3 | 4 | 5 |
| $\alpha^-$ | 0.000 | -1.000 | -2.000 | -3.000 | -4.000 | 1 | 2 | 3 | 4 | 5 |
| $\alpha^\pm$ | 4.000 | 2.000 | 0.000 | -2.000 | -4.000 | 1 | 2 | 3 | 4 | 5 |
| $\beta^+$ | 2.083 | 1.083 | 0.583 | 0.250 | 0.000 | 1 | 2 | 3 | 4 | 5 |
| $\beta^-$ | 0.000 | -0.250 | -0.583 | -1.083 | -2.083 | 1 | 2 | 3 | 4 | 5 |
| $\beta^\pm$ | 2.083 | 0.833 | 0.000 | -0.833 | -2.083 | 1 | 2 | 3 | 4 | 5 |
| $\gamma^+$ | 6.417 | 2.417 | 0.917 | 0.250 | 0.000 | 1 | 2 | 3 | 4 | 5 |
| $\gamma^-$ | 0.000 | -0.250 | -0.917 | -2.417 | -6.417 | 1 | 2 | 3 | 4 | 5 |
| $\gamma^\pm$ | 6.417 | 2.167 | 0.000 | -2.167 | -6.417 | 1 | 2 | 3 | 4 | 5 |
| $\delta^+$ | 2.283 | 1.283 | 0.783 | 0.450 | 0.200 | 1 | 2 | 3 | 4 | 5 |
| $\delta^-$ | -0.200 | -0.450 | -0.783 | -1.283 | -2.283 | 1 | 2 | 3 | 4 | 5 |
| $\delta^\pm$ | 2.083 | 0.833 | 0.000 | -0.833 | -2.083 | 1 | 2 | 3 | 4 | 5 |

We observed the following fact concerning the digraphs that have a single cycle on three nodes. The digraph is divided into three strongly connected components, one of which is the cycle on three nodes and the other two consist of a single node, and there is a total order among these three strongly connected components. The observation is that all nodes on the cycle receive the same value of relational power, and the rankings provided by all measures as well as ANP coincide with the total order among the strongly connected components.

We also observed that different rankings emerge when there are more than one cycle. Table 4.2 shows the result for the digraph with three cycles given by

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next we generated all possible incomplete digraphs on four nodes. We excluded the $\alpha$ measure from comparison because we did not think it useful for incomplete digraphs since it only counts the number of nodes linked directly to each node. We focused on how many arcs are linked to the node, which we call the *comparison number*. We observed that when all nodes have the identical comparison number, the same ranking is provided by all measures. Conversely, when comparison number varies, significantly different rankings emerged. Table 4.3 shows the rankings for the digraph defined by the following matrix, where the comparison number is one for node $A$, two for $B$ and $C$, and three for $D$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Table 4.2: Relational powers for a complete digraph with three cycles

| | relational power | | | | | ranking | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| measure | $A$ | $B$ | $C$ | $D$ | $E$ | $A$ | $B$ | $C$ | $D$ | $E$ |
| ANP | 0.259 | 0.240 | 0.186 | 0.145 | 0.170 | 1 | 2 | 3 | 5 | 4 |
| $\alpha^+$ | 3.000 | 3.000 | 2.000 | 1.000 | 1.000 | 1 | 1 | 3 | 4 | 4 |
| $\alpha^-$ | -1.000 | -1.000 | -2.000 | -3.000 | -3.000 | 1 | 1 | 3 | 4 | 4 |
| $\alpha^\pm$ | 2.000 | 2.000 | 0.000 | -2.000 | -2.000 | 1 | 1 | 3 | 4 | 4 |
| $\beta^+$ | 1.833 | 1.167 | 0.667 | 0.333 | 1.000 | 1 | 2 | 4 | 5 | 3 |
| $\beta^-$ | -1.000 | -0.333 | -0.667 | -1.167 | -1.833 | 3 | 1 | 2 | 4 | 5 |
| $\beta^\pm$ | 0.833 | 0.833 | 0.000 | -0.833 | -0.833 | 1 | 1 | 3 | 4 | 4 |
| $\gamma^+$ | 6.167 | 2.833 | 1.333 | 0.667 | 4.000 | 1 | 3 | 4 | 5 | 2 |
| $\gamma^-$ | -4.000 | -0.667 | -1.333 | -2.833 | -6.167 | 4 | 1 | 2 | 3 | 5 |
| $\gamma^\pm$ | 2.167 | 2.167 | 0.000 | -2.167 | -2.167 | 1 | 1 | 3 | 4 | 4 |
| $\delta^+$ | 1.583 | 1.333 | 0.833 | 0.500 | 0.750 | 1 | 2 | 3 | 5 | 4 |
| $\delta^-$ | -0.750 | -0.500 | -0.833 | -1.333 | -1.583 | 2 | 1 | 3 | 4 | 5 |
| $\delta^\pm$ | 0.833 | 0.833 | 0.000 | -0.833 | -0.833 | 1 | 1 | 3 | 4 | 4 |

Table 4.3: Different comparison number

| | ranking | | | |
|---|---|---|---|---|
| measure | $A$ | $B$ | $C$ | $D$ |
| $\beta^+$ | 4 | 2 | 2 | 1 |
| $\beta^-$ | 1 | 3 | 1 | 3 |
| $\beta^\pm$ | 4 | 3 | 2 | 1 |
| $\gamma^+$ | 4 | 1 | 3 | 1 |
| $\gamma^-$ | 1 | 3 | 1 | 3 |
| $\gamma^\pm$ | 4 | 1 | 1 | 1 |
| $\delta^+$ | 4 | 2 | 2 | 1 |
| $\delta^-$ | 4 | 3 | 1 | 1 |
| $\delta^\pm$ | 4 | 3 | 2 | 1 |

### 4.2. Application to the best paper selection at University of Tsukuba

Owing to the contribution of emeritus professor Y. Kuratani, College of Policy and Planning Sciences, University of Tsukuba opened up an award for the best undergraduate thesis in 2007, when a total of 57 theses were submitted for the major of management science. An agreement was that the ranking should be decided according to the evaluation by a total of 22 faculty members who attended the thesis presentation. Each faculty member gave a grade on the scale of one to ten to each presentation that he or she attended. Due to the lack of prior consensus, it turned out impossible to compare the grades given by different faculty members, namely grade 7 of a member does not mean the same as grade 7 of another member. Furthermore, the presentations failed to have the audience of the same size, some had 6 present while some had 12 present. Therefore the total of the grades that each presentation received is totally useless, and the average is not reliable, either. We concluded that the ranking of presentations should be made based on the order that each faculty member placed on the presentations.

Let $N := \{1, 2, \ldots, 57\}$ be the set of theses, and let $F := \{1, 2, \ldots, 22\}$ be the set of faculty members. We define for each $f \in F$ and $i, j \in N$

$\triangleright$ $D_f := \{\, (i, j) \mid f \text{ gives higher grade to } i \text{ than } j \,\}$

▷ $E_f := \{ \{i,j\} \mid f$ gives the same grade to $i$ and $j \}$
▷ $p(i,j) := \kappa(\{ f \in F \mid (i,j) \in D_f \})$
▷ $t(i,j) := \kappa(\{ f \in F \mid \{i,j\} \in E_f \})$
▷ $suc(i) := \{ j \in N \mid (i,j) \in D_f$ for some $f \in F \}$
▷ $tie(i) := \{ j \in N \mid \{i,j\} \in E_f$ for some $f \in F \}$
▷ $pre(i) := \{ j \in N \mid (j,i) \in D_f$ for some $f \in F \}$

Based on these definitions we modified the measures, among which we show the modified plus measures below:

$$\hat{\beta}_i^+ := \sum_{j \in suc(i) \cup tie(i)} \frac{p(i,j) + \frac{1}{2}t(i,j)}{\sum_{k \in pre(j)} p(k,j) + \frac{1}{2}\sum_{k \in tie(j)} t(k,j)}$$

$$\hat{\gamma}_i^+ := \sum_{j \in suc(i) \cup tie(i)} \frac{p(i,j) + \frac{1}{2}t(i,j) + \sum_{k \in suc(j)} p(j,k) + \frac{1}{2}\sum_{k \in tie(j)} t(j,k)}{\sum_{k \in pre(j)} p(k,j) + \frac{1}{2}\sum_{k \in tie(j)} t(k,j)}$$

$$\hat{\delta}_i^+ := \sum_{j \in suc(i) \cup tie(i) \cup \{i\}} \frac{p(i,j) + \frac{1}{2}t(i,j)}{1 + \sum_{k \in pre(j)} p(k,j) + \frac{1}{2}\sum_{k \in tie(j)} t(k,j)}.$$

In Table 4.4 and 4.5 we list the 14 theses that were listed top-eight by some of measures. Note that C and L vie with each other for the lead by all the measures except $\hat{\gamma}$ plus.

Table 4.4: Relational power given by measures

| | $\hat{\beta}^+$ | $\hat{\beta}^-$ | $\hat{\beta}^\pm$ | $\hat{\gamma}^+$ | $\hat{\gamma}^-$ | $\hat{\gamma}^\pm$ | $\hat{\delta}^+$ | $\hat{\delta}^-$ | $\hat{\delta}^\pm$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 1.77 | -0.31 | 1.46 | 93.83 | -44.13 | 49.69 | 1.77 | -0.31 | 1.46 |
| B | 1.19 | -1.44 | -0.26 | 98.92 | -91.91 | 7.01 | 1.18 | -1.44 | -0.26 |
| C | 3.02 | -0.11 | 2.91 | 89.57 | -11.19 | 78.38 | 3.03 | -0.11 | 2.92 |
| D | 2.26 | -0.56 | 1.70 | 102.20 | -44.82 | 57.39 | 2.25 | -0.56 | 1.69 |
| E | 1.78 | -0.56 | 1.23 | 102.42 | -37.83 | 64.59 | 1.77 | -0.56 | 1.22 |
| F | 1.37 | -0.19 | 1.17 | 98.47 | -24.41 | 74.07 | 1.39 | -0.20 | 1.19 |
| G | 1.24 | -0.36 | 0.88 | 63.98 | -31.88 | 32.10 | 1.24 | -0.36 | 0.88 |
| H | 1.40 | -0.95 | 0.45 | 103.20 | -81.43 | 21.77 | 1.39 | -0.94 | 0.45 |
| I | 1.07 | -0.34 | 0.73 | 89.34 | -35.96 | 53.39 | 1.07 | -0.34 | 0.73 |
| J | 0.85 | -0.29 | 0.56 | 84.43 | -27.34 | 57.09 | 0.86 | -0.30 | 0.56 |
| K | 1.70 | -0.51 | 1.19 | 96.27 | -44.47 | 51.80 | 1.69 | -0.51 | 1.18 |
| L | 2.65 | -0.10 | 2.55 | 89.74 | -7.33 | 82.41 | 2.67 | -0.10 | 2.57 |
| M | 1.95 | -0.31 | 1.63 | 101.10 | -29.74 | 71.36 | 1.93 | -0.32 | 1.62 |
| N | 1.77 | -0.62 | 1.15 | 86.12 | -51.37 | 34.75 | 1.76 | -0.62 | 1.14 |

## 5. Conclusion and Remarks

We proposed the $\gamma$ measure, considered other several measures, proved their rationality, and compared them by some numerical experiment. We also applied the measures to the best paper selection problem. One lesson we learned from the numerical experiment and the application is that the design of the thesis-jury combination counts very much for determining a reliable ranking. The combination should be designed so that every thesis has the same comparison number, or at minimum every thesis has nearly the same number of attendance of faculty members.

Table 4.5: Ranking by relational powers

| | $\hat{\beta}^+$ | $\hat{\beta}^-$ | $\hat{\beta}^\pm$ | $\hat{\gamma}^+$ | $\hat{\gamma}^-$ | $\hat{\gamma}^\pm$ | $\hat{\delta}^+$ | $\hat{\delta}^-$ | $\hat{\delta}^\pm$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 7 | 5 | 5 | 8 | 10 | 10 | 6 | 5 | 5 |
| B | 20 | 46 | 30 | 5 | 51 | 21 | 20 | 46 | 30 |
| C | 1 | 2 | 1 | 10 | 2 | 2 | 1 | 2 | 1 |
| D | 3 | 14 | 3 | 3 | 13 | 6 | 3 | 14 | 3 |
| E | 5 | 13 | 6 | 2 | 8 | 5 | 5 | 13 | 6 |
| F | 14 | 3 | 8 | 6 | 3 | 3 | 14 | 3 | 7 |
| G | 19 | 8 | 11 | 30 | 6 | 15 | 19 | 8 | 11 |
| H | 13 | 28 | 18 | 1 | 24 | 18 | 13 | 28 | 18 |
| I | 22 | 7 | 15 | 11 | 7 | 8 | 22 | 7 | 15 |
| J | 31 | 4 | 16 | 18 | 4 | 7 | 30 | 4 | 16 |
| K | 8 | 11 | 7 | 7 | 12 | 9 | 8 | 11 | 8 |
| L | 2 | 1 | 2 | 9 | 1 | 1 | 2 | 1 | 2 |
| M | 4 | 6 | 4 | 4 | 5 | 4 | 4 | 6 | 4 |
| N | 6 | 17 | 9 | 16 | 15 | 13 | 7 | 17 | 9 |

To compare the measures with ANP we introduce three matrices

$$A = [\alpha_{ij}]_{i,j\in N} := \begin{cases} 1 & \text{if } (i,j) \in D \\ 0 & \text{otherwise,} \end{cases}$$
$$B = [\beta_{ij}]_{i,j\in N} := \begin{cases} 1/p_D(j) & \text{if } (i,j) \in D \\ 0 & \text{otherwise,} \end{cases}$$
$$\Delta = [\delta_{ij}]_{i,j\in N} := \begin{cases} 1/(p_D(j)+1) & \text{if } (i,j) \in D \text{ or } i=j \\ 0 & \text{otherwise} \end{cases}$$

and let $\mathbf{e}$ be the vector of ones. Then it is readily seen that $A\mathbf{e}$, $B\mathbf{e}$ and $\Delta\mathbf{e}$ are equal to the $\alpha$ plus measure, the $\beta$ plus measure and the $\delta$ plus measure, respectively. Therefore the Shapley value given by the multi-stage game is simply written as, for example $A^k\mathbf{e}$ when the game that provides the $\alpha$ plus measure is played $k$ times. In the same way we see that the $\gamma$ plus measure is given as $(A + \mathbf{e}\mathbf{e}^\top)B\mathbf{e}$. Furthermore we can regard the matrix $\Delta$ as a transposed transition probability matrix since $\delta_{ij} \geq 0$ for all $i, j \in N$ and $\sum_{i\in N} \delta_{ij} = 1$. The principal eigenvector of $\Delta$, that is a nonzero vector $\mathbf{w} \in \mathbb{R}^N$ satisfying $\mathbf{w} = \Delta\mathbf{w}$, is known as the stationary distribution and obtained as $\Delta^\infty\mathbf{e} = \lim_{k\to\infty} \Delta^k\mathbf{e}$. Since the solution of ANP is the principal eigenvector of what is called the super matrix, it could be considered as the Shapley value of a game repeatedly played infinitely many times. Some drawbacks of ANP might have roots in this infinite repetition. See Sekitani [5, 6] for the detailed discussion about merits and demerits of ANP.

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