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A NEW COOPERATIVE SCENARIO FOR SUPPLY CHAINS USING COMMON REPLENISHMENT EPOCHS

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Abstract In this paper, we propose a new cooperative scenario for supply chains using common replenishment epochs based on the mathematical model presented in Viswanathan and Piplani's [25] paper. In order to solve an optimal solution for the cooperative scenario, we first conduct a theoretical analysis on the optimal cost function of the mathematical model. Then, we propose an efficient search algorithm by utilizing our theoretical results on the cooperative scenario. Also, we introduce a saving-sharing mechanism that could motivate the buyers to cooperate with the vendor. Importantly, the proposed saving-sharing mechanism assists the supply chain system to create a multi-win situation among the vendor and the buyers. Using the 48 sets of data in Viswanathan and Piplani's [25] paper, we demonstrate that the proposed cooperative scenario could attain significant improvement on the costs for both the vendor and the buyers in comparison to the Stackelberg-game scenario.

Keywords: Optimization, business policy, cost models, inventory control, operations management, search procedure.

1. Introduction

In the literature, it has been advocated that cooperation is an important way for creating win-win (or multi-win) relationships among the firms in supply chains. Researchers have been devoting their efforts to develop new scenarios and strategies to optimally coordinate inventory across the entire supply chain. In this study, we propose a new cooperative scenario for two-echelon supply chains using common replenishment epochs.

Over the last two decades, many researchers have devoted their attention to show that effective coordination could considerably reduce costs (or increase profit) for the firms in supply chains (see, e.g., Banerjee [2], Goyal [14, 15], Parlar and Wang [20], Lu [18], Weng [29], Viswanathan [24], Corbett and de Groote [9], Chen, Federgruen and Zheng [8], Wang [26] and Yao and Chiou [32]).

Some studies promote that when using centralized policies for inventory control, supply chains could become more efficient than decentralized cases. Please refer to the following papers for further reference: Lee and Whang [17], Chen, Federgruen and Zheng [8], Abdul-Jalbar et al. [1], Kim and Ha [16] and Chen and Chen [7]. We note that in these "centralized-control" cases, since the vendor (or the supplier) and the buyers (or the retailers) belong to the same firm, the firm has authority over the buyers to replenish the product with the designated lot quantities (or time intervals). Therefore, using centralized policies, the objective of the firm is to minimize the average total costs incurred in the whole supply chain system.

Other studies have formulated two-echelon supply chains such as the Stackelberg games in which the vendor(s) and the buyer(s) reside at the upper stream and the lower stream, respectively. Usually, the vendor plays the role of the leader and makes the first game decision (*e.g.*, the amount of price discount, and replenishment cycle) in these Stackelberg-game scenarios. And then, the followers (*i.e.*, the buyers) act on the leader's decision and make their own decisions (*e.g.*, their inventory policies, namely, whether to accept the price discount and to use the proposed replenishment cycle). Using a Stackelberg-game scenario, Eliashberg and Steinberg [11] characterize the properties of the optimal pricing and production policies in a production-distribution channel. However, they do not address the issue of channel coordination. Parlar and Wang [20] studied the pricing decision of a vendor and the subsequent ordering decisions of homogeneous buyers as a Stackelberg game. Interestingly, they derive the conditions in which the vendor offers quantity discounts to the buyers. Later, Wang and Wu [28] extend their work to the case with heterogeneous buyers. In Viswanathan and Piplani's [25] paper, they consider another Stackelberg-game scenario in which a vendor offers a discount to buyers as an incentive to attract the buyers to place their orders only at times specified by the vendor. Wang [26] considers another case similar to Viswanathan and Piplani's [25] scenario and compares the strategies using integer-ratio and power-of-two time coordination mechanisms. He concludes that integer-ratio time coordination provides a better coordination mechanism and power-of-two time coordination may not be stable to provide a stable coordination equilibrium strategy. Recently, Mishra [19] generalizes Viswanathan and Piplani's model to allow for a selective discount policy that could exclude some buyers in the supply chain to minimize the supplier total cost. Also, Bylka [4] compares competitive and cooperative policies for the vendor–buyer system and defines some conditional games. He shows that the competition scenario does not essentially degrade the system efficiency. However, our numerical experiments show that the optimal solution for the Stackelberg-game scenario does not necessarily lead to optimality in the supply chain. In fact, both the vendor and the buyers have opportunities to gain even more cost savings if they could cooperate with each other. Therefore, we are motivated to introduce a cooperative scenario based on a Stackelberg-game scenario to improve the performance of supply chains in this study.

Recently, researchers have paid more attention to the implementation of cooperative scenarios in supply chains. One category of the studies focuses on developing revenue-sharing contracts among the vendors and the buyers to improve their profit (see, *e.g.*, Giannoccaro and Pontrandolfo [12], Wang, Jiang and Shen [27], Cachon and Lariviere [5] and Chauhan and Proth [6]). Interestingly, Van der Veen and Venugopal [22] illustrate that revenue-sharing contracts create win-win among all parties in the video industry. On the other hand, many researchers employ price discounts as coordination mechanisms to conduct coordination in supply chains (or distribution channels); see Parlar and Wang [20], Weng [29], Weng and Wong [30] and Yue et al. [33], etc. Also, Viswanathan and Piplani (V&P) [25] present two mathematical models for a supply chain where a single vendor supplies a single product to many buyers. Under a Stackelberg-game scenario, V&P's models coordinate supply chain inventories through the use of common replenishment epochs (CRE) and price discounts. Interestingly, they demonstrate it based on their numerical experiments that if the vendor's order processing cost exceeds a threshold value on the vendor's order processing cost, both the vendor and the system gain more savings as the vendor's and the buyers' order processing costs increase.

In this study, on the top of V&P's model, we propose a new cooperative scenario in which the vendor cooperates with the buyers to determine their replenishment strategies so as to minimize the average total costs in the whole system. Importantly, we introduce a saving-sharing mechanism to reinforce the cooperation between the vendor and the buyers to create a multi-win situation for supply chains using common replenishment epochs.

The organization of this paper is as follows. We first review the problem definition and the mathematical model proposed by Viswanathan and Piplani [25] in Section 2. Then, in Section 3, we propose a cooperative scenario based on V&P's model. We then derive theoretical properties to explore the characteristics of the optimal cost function curve. Utilizing our theoretical results, we propose an efficient search algorithm to obtain the optimal solution for the cooperative scenario. Importantly, we introduce a saving-sharing mechanism for reinforcing the cooperation between the vendor and the buyers. Section 4 uses a numerical example to illustrate the implementation of our search algorithm and compares the proposed cooperative scenario and Stackelberg-game scenario using the 48 data sets in Viswanathan and Piplani's [25] paper. Finally, we provide some concluding remarks in Section 5.

2. Review of a Stackelberg-game Scenario

In this section, we review a mathematical model for a Stackelberg-game scenario presented in Viswanathan and Piplani's [25] paper.

2.1. Problem definition

In such a Stackelberg-game scenario, a single vendor supplies a single product to m buyers. The assumptions of the Economic Order Quantity (EOQ) model apply to each buyer. Namely, the annual demand rate, D_i and the cost parameters (*e.g.*, the ordering cost, K_i and the holding cost, h_i) are deterministic for each buyer i. Lead times are known or can be ignored. It is assumed that the vendor purchases the product from its upstream supplier following a *lot-for-lot* policy. Therefore, the vendor does not have to carry inventory (and no inventory holding cost incurred for the vendor consequently).

Without using the CRE strategy, each buyer *i's* optimal replenishment cycle corresponds to their EOQ, *i.e.*, $T_i^u = \sqrt{K_i/H_i}$ where $H_i = D_i h_i/2$. The average total cost per year for buyer i is given by

$$g_i^u = 2\sqrt{K_i H_i}. \tag{1}$$

Under the CRE strategy, the vendor specifies a replenishment epoch T_0, and asks the buyers to replenish at epochs that are integer multiples of T_0 where T_0 could be one day or one week, *etc.* Denote T_i^c as the replenishment interval of buyer i. Then, $T_i^c = n_i T_0$ where n_i is a positive integer. Also, it is assumed that the vendor is the leader of the game, and the followers' (in this case, the buyers') cost parameters and demand rate are known to the vendor. In such a Stackelberg-game scenario, the vendor specifies a set of common replenishment epochs for the buyers, then the buyer decides its associated optimal replenishment intervals subject to the vendor's CRE. Then, the vendor has to determine a price discount for all the buyers so as to compensate the buyers for using the vendor's preferred replenishment epoch.

We may compare the costs incurred for both cases as follows. Suppose that the buyers in a set C place their orders *simultaneously* when using the CRE strategy, then the vendor will incur an order processing cost of $A_s + \sum_{i \in C} A_i$ where A_s is a common setup cost for processing the entire set of orders and A_i is the additional setup cost for processing the order from buyer i. On the other hand, without using the CRE strategy, those buyers in the set C place their orders *independently*, then the order processing cost for the vendor will be $\sum_{i \in C}(A_i + A_s)$. Therefore, by consolidating buyers' orders, the vendor may gain significant saving on order processing costs from adopting such a CRE mechanism. However, when following the CRE strategy, the buyers suffer from the increase in inventory holding costs.

In order to entice the buyers to accept the CRE strategy, the vendor not only offers a price discount Z (which is identical to all the buyers) to compensate the buyers for any increase in holding costs, but also provides a minimum savings of $100S$ percent additionally.

2.2. The mathematical model for the CRE strategy

Following the problem definition stated in Section 2.1, Viswanathan and Piplani [25] formulated the mathematical model for the CRE strategy as the problem (P) as follows.

Problem (P)

$$\text{Minimize} \quad g_0^c = AC(T_0, Z, n_1, \ldots, n_m) = A_S/T_0 + \sum_{i=1}^{m}(D_i Z + (A_i/n_i T_0)) \tag{2}$$

$$\text{Subject to} \quad D_i Z \geq (K_i/n_i T_0) + H_i n_i T_0 - (1-S)2\sqrt{K_i H_i}, i = 1, \ldots, m, \tag{3}$$

$$T_0 \in X, \tag{4}$$

$$X \in \{1/365,\ 1/52,\ 1/26,\ 1/12,\ 2/12,\ 1/4\}, \tag{5}$$

$$n_i \geq 1 \textit{ and integer},\ i = 1, \ldots, m. \tag{6}$$

Equation (2) defines the objective function where g_0^c denotes the annual total costs under the CRE strategy. The first term in the objective function is the annual major order processing cost. When the smallest frequency multiplier is larger than one, the so-called *empty replenishment occasions* or *empty epoch* problems occur. Dagpunar [10] proposed a correction factor based on the principle of inclusion and exclusion. However, in our study we assume that at least one buyer adopts the CRE, as the basic period addressed in the joint replenishment problem. (See Goyal [13] and Van Eijs [23]) In a real-world case, furthermore, it might be not so reasonable to set a CRE, but no buyer adopt it. The reader can refer to Widleman, Frenk and Dekker [31] for in-depth discussion. The second term in the objective function includes the sum of the annual revenue reduction due to the price discount and the average annual minor order processing cost for all buyers. Constraints (3) ensure buyer i will accept the CRE only if the price discount offered is large enough to compensate for the increase in inventory costs and in addition provide him with a minimum saving of $100S$ percent over the total annual cost based on using his EOQ. Constraints (4) and (5) specify the allowable set of CREs, which could be one day, one week, or one month, *etc.* Constraint (6) indicates that the replenishment interval for each buyer i should be a positive integer multiple of the CRE (*i.e.*, T_0). The decision variables for this model are Z, T_0, and n_i, $i = 1, \ldots, m$.

In addition to CRE policy which restricts on the times at which orders may be placed, the cooperation between the firms in the supply chain system can be implemented by specifying a set of rules such as pricing rule, the commitment to delivery in whole or in part, return policies, among others. (Refer to Chen, Federgruen and Zheng [8]) Obviously, the members may accept such cooperative strategies only if they allow each member to realize a profit increase (or cost reduction) superior to or at least equal to *status quo*.

Note that those annul fixed charges or costs (*e.g.*, franchise fee, *etc*) are not included in this model since they will not be involved in the process of problem solving. Also, in Viswanathan and Piplani's [25] paper, they presented another model that allows the vendor to offer non-identical price discounts to the buyers. Here we focus only on problem (P) that considers only identical discounts since non-identical discounts may violate some trade laws (see Stern, El-Ansary and Coughlan [21]).

2.3. Viswanathan and Piplani's heuristic for solving the problem (P)

Viswanathan and Piplani [25] propose a heuristic for solving the problem (P) as follows.
(i) For each value of $T_0 = x_j (\in X)$, each buyer i first obtains an optimal multiplier n_i^* such that

$$n_i^*(n_i^* - 1) \le K_i/(H_i T_0^2) \le n_i^*(n_i^* + 1). \tag{7}$$

Then, given the obtained n_i^*, each buyer i determines a minimum price discount of Z_i such that

$$Z_i \ge \frac{1}{D_i}[(K_i/n_i^* T_0) + H_i n_i^* T_0 - (1-S)2\sqrt{K_i H_i}]. \tag{8}$$

Set

$$Z^* = \max\{Z_i\},\ i = 1, ..., m. \tag{9}$$

Determine the objective function value given by (2), by substituting for $(n_1^*, ..., n_m^*)$ and Z^*.
(ii) Among all the $T_0 \in X$, choose the value that minimizes the objective function value given by (2).

Viswanathan and Piplani's (V&P's) heuristic works in the following fashion: The decision maker first finds the vector $(n_1^*, ..., n_m^*)$ for a given T_0, then decides price discount Z^* based on the obtained $(n_1^*, ..., n_m^*)$, and finally, determines the objective function value by substituting for $(n_1^*, ..., n_m^*)$ and Z^* in (2) for each T_0. One may observe that following such logic, the determination of Z^* is surely constrained by the vector $(n_1^*, ..., n_m^*)$ obtained at the beginning. Therefore, the decision making is done *sequentially and independently* in the Stackelberg-game scenario.

3. A Cooperative Scenario

Intuitively, if the vendor establishes a cooperative relationship with the buyers, the vendor may coordinate the replenishments from the buyers to achieve even more cost savings. Therefore, we are motivated to suggest a cooperative scenario to improve the collaboration between the vendor and the buyers.

In the following discussions, we fist introduce the problem definition of the proposed cooperative scenario in Section 3.1. Then, we propose a heuristic for solving an optimal solution for the cooperative scenario in Section 3.2. Finally, Section 3.3 presents a saving-sharing mechanism to create a multi-win situation among the vendor and the buyers in the cooperative scenario.

3.1. The problem definition

In our cooperative scenario, we drop the leader-follower relationship in the Stackelberg-game scenario. But, we keep the vendor's incentive policies to entice the buyers to cooperate with the vendor. Therefore, in the proposed cooperative scenario, the vendor still offers an identical price discount Z to all the buyers to compensate any of their increases in inventory holding costs and additionally provides a minimum savings of $100S$ percent (as the Stackelberg-game scenario does).

The proposed cooperative scenario is to entice the buyers to cooperate with the vendor especially when they have long term business relationship. We assume that the vendor knows the buyers' historical ordering pattern and the buyers' cost and demand parameters are known to the vendor. Hence the vendor's CRE and the buyers' cost and demand information are available *before* the vendor announces the amount of price discounts. The major characteristics of the cooperative scenario show in the decision-making process. In the cooperative scenario, the vendor not only decides a common replenishment epoch and a price discount, but also determines the replenishment intervals for the buyers. The vendor optimizes the replenishment policies for all the members in the supply chain *simultaneously*, and all the buyers are willing to cooperate with the vendor. (The proposed saving-sharing mechanism in Section 3.3 provides the buyers with on enticement to cooperate with the vendor.) We note that the decision makings are done *sequentially and independently* in the Stackelberg-game scenario which is significantly different from the cooperative scenario.

3.2. The proposed search algorithm for the cooperative scenario

We note that our cooperative scenario also utilizes Viswanathan and Piplani's [25] mathematical model presented in Section 2.2. (Only the decision-making process is different from the Stackelberg-game scenario.) In this subsection, we first conduct a theoretical analysis to explore the optimality properties of the model. Then, utilizing our theoretical results, we propose an efficient search algorithm to obtain an optimal solution for the cooperative scenario.

3.2.1. Theoretical analysis

In fact, problem (P) is a complex nonlinear integer programming problem which is very difficult to solve optimally. Before presenting the details, we discuss the rationale of our search algorithm. First, instead of directly attacking problem (P), we explore its optimality structure. We will show that the optimal-cost curve (with respect to the value of Z) is *piece-wise linear*. Furthermore, we conduct a theoretical analysis on the piece-wise linear curve (*e.g.*, to derive the closed form formula for the location of the break points, *etc*). These theoretical properties allow us to devise an efficient algorithm to obtain an optimal solution for problem (P). In the following sub-sections, we will present the details of these topics, respectively.

First, we explore the optimality structure of problem (P) for a given value of . Here, we treat the optimal value of the objective function as a single-variable function with respect to price discount Z. That is, for each value of $Z=z'$ on the Z-axis, we solve the vector of optimal multipliers $(n_1^*(z'), ..., n_m^*(z'))$, and keep track of the associated optimal objective value of problem (P) as a function of Z.

We denote $\underline{AC}(Z\,|T_0)$ as the optimal objective function value for problem (P) given T_0 where $\underline{AC}(Z\,|T_0)$ is a function of Z. Then, $\underline{AC}(Z\,|T_0){=}AC(Z, n_1^*(z'), ..., n_m^*(z')\,|T_0)$ is actually obtained by solving
$\min_{(n_1,...,n_m)} AC(T_0, Z = z', n_1(z'), ..., n_m(z'))$ for all $Z=z'$. By using the data in Section 4, one may plot the $\underline{AC}(Z\,|T_0)$ curve using a small step-size of ΔZ as shown in Figure 1. Figure 1 illustrates an interesting property of $\underline{AC}(Z\,|T_0)$, namely, it is a *piece-wise linear* function in Z. Let us formalize this observation by the following theoretical results.

Lemma 1 *For a given value of* T_0, *the optimal objective function value for the following problem* (Pi), denoted by $\min_{n_i} g_i^c(Z\,|T_0)$, *is a piece-wise linear function with respect to Z.*

Problem (Pi)

$$\min_{n_i} \quad g_i^c(Z\,|T_0) = D_i Z + (A_i/n_i T_0) \tag{10}$$

$$\textit{Subject to} \quad D_i Z \geq (K_i/n_i T_0) + H_i n_i T_0 - (1-S)2\sqrt{K_i H_i} \tag{11}$$

$$\textit{where } n_i \geq 1 : \textit{integer}. \tag{12}$$

[Proof] For each (integer) value of n_i, there exists a feasible set of Z for (Pi), and such a

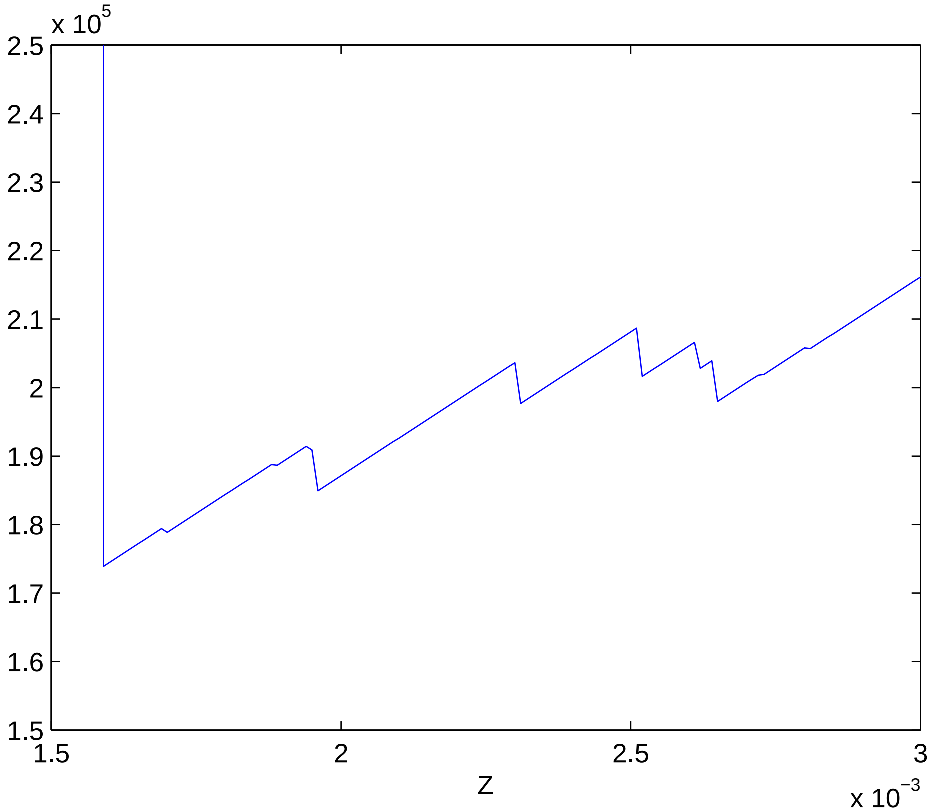

Figure 1: The $\underline{AC}(Z\,|T_0)$ curve of the problem (P)

feasible set is an interval on the Z-axis defined by constraint (11). On the other hand, for each Z=z', its optimal multiplier n_i^* for (Pi), *i.e.*,$n_i^*(z'\,|T_0)$, is obviously the largest value of positive integer that satisfies constraint (11). So, the set of Z=z' such that $n_i^*(z'\,|T_0)$ is optimal for (Pi) forms an interval on the Z-axis. Also, when n_i remains the same, the first term in the objective function, *i.e.*,D_iZ, will increase linearly as Z increases while the second term, *i.e.*,(A_i/n_iT_0), is a constant given any value of T_0. Therefore, we conclude that the optimal objective function value of (Pi) is a piece-wise linear function with respect to Z.

Theorem 1 $\underline{AC}(Z\,|T_0)$ *is a piece-wise linear function of* Z *for a given value of* T_0.

[Proof] For a given value of *T0*, one may rewrite problem (P) by (2) as

$$\begin{aligned} &Minimize \qquad g_0^c(Z\,|T_0) = A_S/T_0 + \sum_{i=1}^{m} g_i^c(Z\,|T_0) \qquad (13)\\ &Subject\ to \qquad (3)-(6). \end{aligned}$$

Obviously, problem (P) can be divided into m independent (Pi), and $\underline{AC}(Z\,|T_0)$ is a sum of m piece-wise linear functions. Clearly, $\underline{AC}(Z\,|T_0)$ is also a piece-wise linear function.

Naively, one can obtain an optimal solution for the problem (P) by a small-step search algorithm which enumerates a “reasonable” range of Z using a very small step-size $\Delta Z \to 0$. But this is neither efficient nor accurate since the step-size of the search algorithm determines its performance. Obviously, such a search algorithm could become very time-consuming.

In order to propose an efficient solution approach, we must utilize our theoretical results on the optimality structure, especially, the breakpoints on the piece-wise linear curve of

$g_0^c(Z|T_0)$. Before locating the breakpoints, we first derive the upper and the lower bounds to define the search range.

3.2.2. The bounds for the search range

It is a critical issue to set a "reasonable" range of Z in devising a search algorithm for solving problem (P). Importantly, the search range of Z must include the optimal solution of problem (P).

Lower bound

First, we focus on deriving a lower bound for the search range. One may observe that there exists a lower bound for the value of Z in Figure 1. Denote by Z^{lb} the lower bound value of Z. The following proposition indicates the location of Z^{lb} for a given value of T_0.

Proposition 1 *For a given value of T_0, a lower bound for Z, is given by*

$$Z^{lb} = Max\{Z_i^{'}\} \tag{14}$$

$$where\ Z_i^{'} = \frac{((K_i/n_i^s T_0) + H_i n_i^s T_0) - (1-S)2\sqrt{K_i H_i}}{D_i} \tag{15}$$

$$and\ n_i^s = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4K_i}{H_i T_0^2}} \right\rceil, \tag{16}$$

for $i = 1, \ldots, m$.

[Proof] By (3), for each buyer i, the minimum feasible value of Z_i, denoted by $Z_i^{'}$, is given by $Z_i^{'} = \underset{n_i}{Min}\left[((K_i/n_i T_0) + H_i n_i T_0) - (1-S)2\sqrt{K_i H_i}\right]/D_i$, $i = 1, \ldots, m$. As shown in Wildeman, Frenk and Dekker [31],$Min((K_i/n_i T_0) + H_i n_i T_0)$ is $(K_i/n_i^s T_0) + H_i n_i^s T_0$, where $n_i^s = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4K_i}{H_i T_0^2}} \right\rceil$, $i = 1, \ldots, m$. Therefore, the minimum Z value, which satisfies the constraints for all the buyers, is $Z^{lb} = Max\{Z_i^{'}\}$, which is exact (14).

We note that the proposed search algorithm searches from the lower bound toward larger values of Z until it meets the upper bound. Therefore, Proposition 1 also provides the starting point for the proposed search algorithm. From another point of view, one may interpret Z^{lb} as the minimum price discount that the vendor needs to offer to sufficiently compensate the buyers' cost increase in inventory so as to induce their continued business.

Upper bound

The following proposition provides us with an easy way to obtain the vector of optimal multipliers $(n_1^*(Z), \ldots, n_m^*(Z))$ for the given values of T_0 and Z. Also, it assists to locate an upper bound for the search range.

Proposition 2 *For the given values of T_0 and Z, the optimal value of $n_i^*(Z)$, $(i = 1, \ldots, m)$, is given by*

$$n_i^*(Z) = \left\lfloor \frac{P_i + \sqrt{P_i^2 - 4Q_i R_i}}{2R_i} \right\rfloor \tag{17}$$

where $P_i = D_i Z + (1-S)2\sqrt{K_i H_i}$, $Q_i = (K_i/T_0)$, and $R_i = H_i T_0$.

[Proof] Recall that $AC(T_0, Z, n_1, ..., n_m) = A_S/T_0 + \sum_{i=1}^{m}(D_i Z + (A_i/n_i T_0))$, *i.e.*, eq. (2). By (10)- (13), problem (P) can be divided into m independent problems (Pi) when T_0 and Z are given. The objective function value of each (Pi) decreases as n_i increases. Also from constraint (11), one may obtain the feasible range for n_i as $\left(\left\lfloor \frac{P_i - \sqrt{P_i^2 - 4Q_i R_i}}{2R_i} \right\rfloor, \left\lfloor \frac{P_i + \sqrt{P_i^2 - 4Q_i R_i}}{2R_i} \right\rfloor\right)$.

Thus the minimum objective function value is achieved at $n_i^*(Z)=\left\lfloor \frac{P_i+\sqrt{P_i^2-4Q_iR_i}}{2R_i} \right\rfloor$.

To locate an upper bound, denoted by Z^{ub}, for the search range, we employ a relaxation of problem (P), denoted as problem (R). The first step in formulating the problem (R) is to plug the optimal $n_i^*(Z)$ from (17) into the objective function of the problem (P), which leads to

$$\underset{Z}{Minimize}\ g_0^c(Z\,|T_0) = A_S/T_0 + \sum_{i=1}^{m}\left(D_iZ + A_i \Bigg/ \left[\left\lfloor \frac{P_i + \sqrt{P_i^2 - 4Q_iR_i}}{2R_i} \right\rfloor T_0\right]\right). \quad (18)$$

Next, we replace the optimal $n_i^*(Z)$ in (18) with its continuous relaxation to derive the objective value of the problem (R) as follows.
Problem (R)

$$Minimize\ g_0^R(Z\,|T_0) = A_S/T_0 + \sum_{i=1}^{m}\left(D_iZ + A_i \Bigg/ \left[\left(\frac{P_i + \sqrt{P_i^2 - 4Q_iR_i}}{2R_i} \right) T_0\right]\right). \quad (19)$$

where $P_i = D_iZ + (1-S)2\sqrt{K_iH_i}$, $Q_i = (K_i/T_0)$, and $R_i = H_iT_0$.

Clearly, it holds that $g_0^c(Z\,|T_0) \geq g_0^R(Z\,|T_0)$, *i.e.*, the objective value of the problem (P) is bounded from below by that of the problem (R) for all Z=z' on the Z-axis. The following lemma will be used to prove another proposition later.

Lemma 2 *For any given T_0, the function $g_0^R(Z\,|T_0)$ is a convex function with respect to Z.*

[Proof] The proof is easily accomplished by showing $d^2g_0^R/dZ^2 > 0$ for all $Z > 0$.

Recall that *without* adopting the CRE strategy, it is assumed that the vendor purchases the product from its upstream supplier following a *lot-for-lot* policy, and each buyer *i's* optimal replenishment cycle corresponds to their EOQ, *i.e.*, $T_i^U = \sqrt{2K_i/(h_id_i)}$. No inventory holding cost is incurred for the vendor since the vendor does not keep any inventory. The average annual total cost (*i.e.*, total ordering costs) for the vendor is given by

$$g_0^U = \sum_{i=1}^{m}(A_s + A_i)/T_i^U. \quad (20)$$

Obviously, g_0^U provides a *ceiling* for the optimal objective function value of the problem (P) since it does not make sense to implement the CRE coordination strategy if it results in a higher cost than g_0^U. Define

$$Z^{ub} = \max\{\arg[g_0^R(Z\,|T_0) = g_0^U]\}. \quad (21)$$

Proposition 3 concludes that Z^{ub} provides an upper bound on the search range for the optimal value of Z.

Proposition 3 *For any given T_0, one obtains no better objective value than g_0^U for all Z>Z^{ub} using the CRE strategy.*

[Proof] By Lemma 2, the objective function of the function $g_0^R(Z\,|T_0)$ is *convex*. By eq.(21), g_0^U provides a *ceiling* for the optimal objective function value of the problem (P) since it does not make sense to implement the CRE coordination strategy if it results in a higher cost than g_0^U. Note that g_0^U, obtained from eq.(20), is not a function of Z. If we set $Z^{ub} = \max\{\arg[g_0^R(Z\,|T_0) = g_0^U]\}$, then it holds that $g_0^R(Z\,|T_0) > g_0^U$ for all Z>Z^{ub}. Since $g_0^R(Z\,|T_0)$ is a continuous relaxation of $g_0^c(Z\,|T_0)$, it holds that $g_0^c(Z\,|T_0) \geq g_0^R(Z\,|T_0)$, for all $Z > 0$ and any given T_0.

In summary, because $g_0^c(Z) \geq g_0^R(Z)$, for all $Z > 0$ and $g_0^R(Z) > g_0^U$ for all $Z > Z^{ub}$, we conclude that $g_0^c(Z) \geq g_0^R(Z) > g_0^U$, for all $Z > Z^{ub}$.

One may obtain the value of Z^{ub} by using some line search algorithm (*e.g.*, bisection search, or quadratic fit search, see Bazarra, Sherali and Shetty [3]).

Next, we will show how to efficiently locate all the breakpoints and the corresponding vector of optimal multipliers between two consecutive breakpoints.

3.2.3. The breakpoints and the vector of optimal multipliers

We define as Z=$\delta_i(k)$ a breakpoint for a Pi where the optimal n_i^* changes from $n_i^s + k$ to $n_i^s + (k+1)$ where n_i^s is expressed in (16). The following formula indicates the location of a breakpoint for (Pi).

$$\delta_i(k) = \frac{((K_i/\hat{n}_i T_0) + H_i \hat{n}_i T_0) - (1-S)2\sqrt{K_i H_i}}{D_i} \tag{22}$$

where $\hat{n}_i = n_i^s + k$ and k: integer, $k \geq 0$.

Hence the relation between the lower bound (Z^{lb}) and the breakpoints can be expressed as $Z^{lb} = \max_i\{\delta_i(0)\}$. The following corollary, which is an immediate result of Theorem 1, lays an important foundation for our proposed search algorithm.

Corollary 1 *For any given T_0, all the breakpoints for (Pi) will be inherited by the piece-wise linear function $\underline{AC}(Z\,|T_0)$.*

In other words, if a breakpoint $\delta_i(k)$ shows on one piece-wise linear curve for (Pi), then, $\delta_i(k)$ must also show on the piece-wise linear curve of the $\underline{AC}(Z\,|T_0)$ function as a breakpoint.

Also, Corollary 2 provides an easier (than V&P's heuristic proposed in [25]) way to obtain the vector of optimal multipliers $(n_1^*(Z), ..., n_m^*(Z))$ for any given Z. (For clarity, we assume that Z is greater than the lower bound for the search range.) The relation between the lower bound (Z^{lb}) and the breakpoints can be expressed as $Z^{lb} = \max_i\{\delta_i(0)\}$.

Corollary 2 *For any given Z, one can obtain n_i^* for each buyer i by*

$$n_i^* = \begin{cases} n_i^s, & if\ Z \in [\delta_i(0), \delta_i(1)]. \\ n_i^s + k, & if\ Z \in [\delta_i(k), \delta_i(k+1)]. \end{cases} \tag{23}$$

[**Proof**] It is an immediate result from Propositions 2 and Corollary 1.

Corollaries 1 and 2 indicate that the sorted (ascending) sequence of all the breakpoints,*i.e.*, $\{\delta_i(k) : i = 0, \cdots, m\}$ serves as the backbone of our search algorithm since each breakpoint specifies the location where to update one of the multipliers n_i^* when searching along the Z-axis (from Z^{lb} toward Z^{ub}).

3.2.4. The proposed search algorithm

We are now ready to state the proposed search algorithm. Recall that the algorithm searches from Z^{lb} toward larger values of Z until it meets the upper bound Z^{ub}. In the search process, we use a sequence of (sorted) breakpoints as the backbone, obtain the objective values for all the breakpoints, and pick the one with the minimum value as the optimal solution.

For any given T_0, denote as $AC^* = AC(T_0, Z^*, n^*)$, the optimal objective function value of the optimal solution for the problem (P) where Z^* and $n = (n_1^*(Z^*), ..., n_m^*(Z^*))$ are the corresponding optimal price discount and the vector of optimal multipliers, respectively. Let Z^c be the breakpoint that the search algorithm currently visits. We summarize the step-by-step procedure of the proposed search algorithm as follows:

i) For each $T_0 \in X$:

1. Obtain the lower bound Z^{lb} by (14) – (16).
2. Compute g_0^U by (20) and use a line search algorithm to secure the upper bound Z^{ub} by solving (21).
3. Compute the breakpoints within the search range for each buyer i, *i.e.*, $\{\delta_i(k)\} \in [Z^{lb}, Z^{ub}]$ by (22). Sort all the breakpoints (of all the buyers) in an ascending order and record the corresponding changed n_i^* for each breakpoint for this sorted sequence.
4. Use Corollary 2 to obtain the vector of optimal multipliers at Z^{lb}, *i.e.*, $\left(n_1^*(Z^{lb}), ..., n_m^*(Z^{lb})\right)$. Then, set $Z^*= Z^{lb}$, $n^* = \left(n_1^*(Z^{lb}), ..., n_m^*(Z^{lb})\right)$, $AC^* = AC(T_0, Z^{lb}, n_1^*(Z^{lb}), ..., n_m^*(Z^{lb}))$, and $Z^c = Z^{lb}$.
5. If all the breakpoints in the sorted sequence are visited, stop. Otherwise, move to the next breakpoint by setting $Z^c = \min\{\delta_i(k) : \delta_i(k) > Z^c\}$ and do the following items at new Z^c:
   (a) Update the vector of optimal multipliers $(n_1^*(Z^c), ..., n_m^*(Z^c))$ according to Corollary 2.
   (b) Compute $AC(T_0, Z^c, n_1^*(Z^c), ..., n_m^*(Z^c))$, *i.e.*, the optimal objective value at Z^c.
   (c) If $AC(T_0, Z^c, n_1^*(Z^c), ..., n_m^*(Z^c)) < AC^*$, then set $Z^* = Z^c$, $n^* = (n_1^*(Z^c), ..., n_m^*(Z^c))$, and $AC^* = AC(T_0, Z^c, n_1^*(Z^c), ..., n_m^*(Z^c))$.
   (d) Go to Step 5.

ii) Among all the $T_0 \in X$, choose the value that minimizes the objective function value given by (2).

3.3. A saving-sharing mechanism between the vendor and the buyers

One may observe that the problem definition stated in Section 3.1 and the proposed algorithm presented in Section 3.2 works impeccably for so-called "centralized-control" cases. In these cases, the vendor and the buyers belong to the same firm. Since the firm should pay for the average total costs incurred in the whole supply chain, the firm surely intends to minimize the objective function value in eq. (2). Therefore, it makes perfect sense to authorize the vendor to coordinate the replenishments from the buyers by *simultaneously* determining the optimal values of , Z and the vector of optimal multipliers $(n_1^*, ..., n_m^*)$.

The readers may be more interested in the answers of the following questions. What if the vendor and the buyers do not belong to the same firm? How could the vendor ask the buyers to follow the designated replenishment intervals (obtained by the proposed search algorithm in Section 3.2) in such a case?

From our numerical experiments, we have observed that the vendor actually gains most of the cost savings in the supply chain for both the cooperative and Stackelberg-game scenarios. Recall that in Stackelberg-game scenario, the vendor not only offers a price discount Z to compensate the buyers for any increase in holding costs, but also provides a minimum savings of $100S$ percent additionally so as to entice the buyers to accept the CRE strategy (*i.e.*, to ask the buyers to replenish at epochs that are integer multiples of some CRE T_0). Provided that the vendor enjoys most of the cost savings from the cooperative scenario, the vendor should offer more incentives to motivate the buyers to take the designated replenishment intervals. In this section, we propose a "saving-sharing mechanism" to attain this purpose. Later, our numerical example in Section 4.1 will show that the proposed saving-sharing mechanism assists to create a multi-win situation among the vendor and the buyers in the cooperative scenario.

The vendor and the buyers would like to pursue for lower average total costs even if they do not belong to the same firm. Once ensuring that the vendor's saving-sharing mechanism could lead to further reduction in the average total costs, the buyer will take the replenishment intervals designated by the vendor. Since the vendor enjoys most of the cost savings from adopting the cooperative scenario, the vendor should share part of its cost savings to the buyers who are willing to change from the Stackelberg-game scenario to the cooperative scenario, provided that such a move brings more cost savings to the vendor. In practice, the vendor may realize the saving-sharing mechanism to return part of the cost savings to the buyer(s) by waiving some annul fixed charge (or cost), *e.g.*, franchise fee, *etc.*

Such a saving-sharing opportunity may be offered to a buyer only when the vendor's cost saving results from altering the Stackelberg-game scenario to the cooperative scenario and will be more than the buyer's cost increase. Proposition 4 provides an easy, but a *necessary*, condition for the change from the Stackelberg-game scenario to the cooperative scenario.

Proposition 4 Given a CRE T_0, the vendor will offer the buyer i a saving-sharing opportunity (to change from Stackelberg-game scenario to the cooperative scenario) only if the following condition holds:

$$\sqrt{\frac{A_i + K_i}{2H_i}} \geq T_0. \tag{24}$$

[**Proof**] Please refer to Appendix A for the proof of Proposition 4.

Note that when the condition in (24) holds, if buyer i changes from the Stackelberg-game scenario to the cooperative scenario, the vendor's cost saving will be more than the buyer i's cost increase. In such a case, the vendor could have sufficient cost savings to realize the saving-sharing mechanism to encourage the buyer i to follow the replenishment interval suggested by the vendor.

4. Numerical Study

In this section, we first present a numerical example to demonstrate the implementation of both, the Stackelberg-game scenario and the proposed cooperative scenario. Then, we show that the cooperative scenario outperforms the Stackelberg-game scenario using 48 sets of data presented in Viswanathan and Piplani's [25].

4.1. A demonstrative example

Here, we demonstrate an example in which a vendor supplies a product to 10 buyers, *i.e.,* m=10. The demand rate and the ordering cost of each buyer i, *i.e.*, D_i and K_i are given in Table 1. The setup costs for the vendor and each buyer i, are A_s=\$200 and A_i=\$500, respectively. The rate of the holding cost h is 0.1. The minimum buyer cost saving required (100S) is set to 10%.

In the following discussions, we first solve the solutions for the Stackelberg-game scenario by V&P's heuristic; then we obtain an optimal solution for the cooperative scenario using the proposed search algorithm; finally, we will verify the necessary condition for changing from the Stackelberg-game scenario to the cooperative scenario.

4.1.1. Stackelberg-game scenario

For each value of T_0, we obtain its corresponding solution for the Stackelberg-game scenario by V&P's heuristic. Part (a) of Table 3 presents our solutions solved by V&P's heuristic. Clearly, the best-obtained objective value obtained by V&P's heuristic is $g_0^c(Z)$=\$188,905 with T_0=1/26, Z^*=0.001587 and $n^* = (1,\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 1)$ (where T_0=1/26 is approximately a planning period of two weeks).

4.1.2. The cooperative scenario

Here, we show only the case for T_0=1/26 since it obtains the optimal solution for this example. The other values of T_0 repeat the same procedure.

Table 1: The ordering cost and the demand rate data of the buyers in the example

Buyer index i	Ordering cost K_i	Annual demand D_i
1	\$100	1,000,000
2	1,000	2,000,000
3	100	3,000,000
4	5,000	4,000,000
5	100	5,000,000
6	2,000	6,000,000
7	100	7,000,000
8	5,000	8,000,000
9	100	9,000,000
10	1,000	10,000,000

We first obtain a lower bound Z^{lb} by 0.001587. And, we compute g_0^U as \$208,047 and use a line search procedure to get Z^{ub} by Z^{ub}=0.00295. Next, we find a total of 12 breakpoints in the search range $[Z^{lb}, Z^{ub}]$. After sorting the breakpoints in ascending order, we record the corresponding n_i^* for each. By Corollary 1, the vector of optimal multipliers $\big(n_1^*(Z^{lb}), ..., n_m^*(Z^{lb})\big)$ corresponding to Z^{lb} is given by (2, 3, 1, 4, 1, 3, 1, 3, 1, 2). Also, we set $Z^* = Z^c = Z^{lb}$, $n^* = \big(n_1^*(Z^{lb}), ..., n_m^*(Z^{lb})\big)$, and $AC^* = AC(T_0, Z^{lb}, n_1^*(Z^{lb}), ..., n_m^*(Z^{lb}))$ $= \$173,738$.

Then, the search algorithm goes through the 12 breakpoints in the search range from Z^{lb} in an ascending order by Step 5. Table 2 summarizes the search process of the proposed search algorithm.

Table 2: The vector of optimal multipliers for the 12 breakpoints in the search process

Breakpoint	n_1^*	n_2^*	n_3^*	n_4^*	n_5^*	n_6^*	n_7^*	n_8^*	n_9^*	n_{10}^*	g_0^c
0.001587= Z^{lb}	2	3	1	4	1	3	1	3	1	2	\$173,738
0.001693	2	3	1	4	1	3	1	↓4	1	2	179,537
0.001885	2	3	1	↓5	1	3	1	4	1	2	188,399
0.001942	2	↓4	1	5	1	3	1	4	1	2	191,544
0.001956	2	4	↓2	5	1	3	1	4	1	2	184,697
0.002306	2	4	2	5	↓2	3	1	4	1	2	197,472
0.0025105	2	4	2	5	2	↓4	1	4	1	2	207,628
0.0025106	2	4	2	5	2	4	↓2	4	1	2	201,132
0.002611	2	4	2	5	2	4	2	4	1	↓3	202,320
0.002649	2	4	2	5	2	4	2	4	↓2	3	197,909
0.002725	2	4	2	5	2	4	2	↓5	2	3	202,085
0.002803	2	4	2	↓6	2	4	2	5	2	3	205,303

The proposed search algorithm obtains the same value of Z^*(Z^*=0.001587) as the Stackelberg-game scenario. But, the cooperative scenario secures a different vector of optimal multipliers $n^* = (n_1^*, ..., n_m^*)$, namely ($\underline{2}$, 3, 1, 4, 1, $\underline{3}$, 1, 3, 1, $\underline{2}$). (Please refer to Part (b) of Table 3 for the optimal solutions of all values of CRE obtained by the proposed search algorithm.)

Table 3 compares the solutions of the Stackelberg-game scenario and the cooperative scenario. The optimal objective function value for the cooperative scenario is obtained by \$173,738 which is **8.73%** less than the Stackelberg-game scenario.

Table 3: The optimal solutions for the Stackelberg-game and the cooperative scenarios

Part (a) Stackelberg-game scenario							
T_0	n_i^*					$Z*$	g_0^c
1/365	16	37	9	58	7	0.001581	\$314,665.35
	30	6	41	5	16		
1/52	2	5	1	8	1	0.001587	246,971.53
	4	1	6	1	2		
1/26	1	3	1	4	1	0.001587	188,904.86
	2	1	3	1	1		
1/12	1	1	1	2	1	0.002951	222,109.76
	1	1	1	1	1		
1/6	1	1	1	1	1	0.007058	419,409.76
	1	1	1	1	1		
1/4	1	1	1	1	1	0.011204	63,6954.21
	1	1	1	1	1		
Part (b) The cooperative scenario							
T_0	n_i^*					$Z*$	g_0^c
1/365	32	51	24	57	22	0.001581	\$220,224.95
	45	20	54	19	32		
1/52	4	7	3	8	3	0.001592	178,033.44
	6	2	7	2	4		
1/26	2	3	1	4	1	0.001587	173,738.20
	3	1	3	1	2		
1/12	1	2	1	2	1	0.002951	216,109.76
	1	1	2	1	1		
1/6	1	1	1	2	1	0.007055	417,909.76
	1	1	1	1	1		
1/4	1	1	1	1	1	0.011204	63,6954.21
	1	1	1	1	1		

Based on Viswanathan and Piplani's experimental results, they comment that the vendor may obtain very significant cost saving by changing from independent replenishments to the Stackelberg-game scenario when the vendor']s order processing costs is relatively larger. (An example in their numerical study achieved more than 50% cost savings by adopting the Stackelberg-game scenario.) It will be interesting to observe the performance of the cooperative scenario in such cases. For this purpose, we replace the order processing costs for the vendor and each buyer i with A_s=\$5,000 and A_i=\$5,000, respectively, in the following discussions.

Again, we solve the optimal solutions for both the Stackelberg-game and the cooperative scenarios by V&P's heuristic and the proposed search algorithm, respectively. Table 4 presents the optimal solutions for the case with A_s=\$5,000 and A_i=\$5,000. The total cost savings for the whole supply chain system is \$141,940. One may observe that the vendor and the whole supply chain system achieve more significant cost savings in the cooperative scenario than in the Stackelberg-game scenario. In the example, the vendor and the whole supply chain system enjoys a cost saving of 65.35% and 61.02% by changing from independent replenishments to the cooperative scenario. We present our discussions on the implementation of the proposed saving-sharing mechanism next.

Table 4: A comparison between the Stackelberg-game and the cooperative scenarios for the case with A_s=\$5,000 and A_i=\$5,000

	Independent	Stackelberg -game (S)	Saving (%)	Cooperative (C)	Saving (%)	Improvement (S - C)
Buyers' cost	\$313,866	\$241,057	23.19	\$250,783	20.09	-\$9,726
Vendor's cost	2,972,103	1,181,454	60.24	1,029,788	65.35	151,666
System cost	3,285,969	1,422,511	56.70	1,280,571	61.02	141,940

4.1.3. The saving-sharing mechanism

For the case with A_s=\$5,000 and A_i=\$5,000, we observe that the buyers' cost is increased by \$9,726 (as shown in Table 4) when changing from the Stackelberg-game scenario to the cooperative scenario. Although the buyers' cost is increased in this case, the cost of the buyers in cooperative scenario is still 20.09% superior to that of the independent way. In order to entice the buyers to join the cooperative scenario, the vendor should not only compensate the buyers' cost increase, but also share part of the net saving (\$141,940) with the buyers. For instance, if the vendor and the buyers evenly share the net saving (*i.e.*, \$70,970 for each party), the vendor and the buyers obtain cost savings of 2.38% and 22.61% respectively in comparison with the Stackelberg-game scenario. Basically, once the setup cost is over a threshold value, the effectiveness of the cooperative scenario is significant compared to the EOQ results, In fact, the proposed cooperative scenario can guarantee that the buyers' cost will not exceed their EOQ status quo. Therefore the vendor can entice the buyers to join the cooperative scenario instead of their EOQ comparison baseline. If the buyer wants to share the vendor's net saving in a justifiable way, then a contract or agreement between the vendor and the buyers is need. Regarding the splitting the net saving, it depends on the negotiation power in the supply chain system. In our paper just propose an easy example to split the saving evenly. In such a case, the buyers will be glad to join the cooperative scenario. Importantly, the cooperative scenario creates a multi-win situation where both the vendor and the buyers enjoy more cost saving than the Stackelberg-game scenario.

Next, we use this example with A_s=\$200 and A_i=\$500 to verify the necessary condition for the change from the Stackelberg-game scenario to the cooperative scenario in Proposition 4. (For the case with A_s=\$5,000 and A_i=\$5,000, the condition in (24) can be satisfied for each buyer since the values of A_i are relatively large. Since the condition in (24), will become insignificant for the readers in such a case, we would return to the case with A_s=\$200 and A_i=\$500.)

Recall that in this example, the optimal solution for the Stackelberg-game scenario is given by $n^* = (1, 3, 1, 4, 1, 2, 1, 3, 1, 1)$, Z^*=0.001587 , and T_0=1/26=0.385 with the optimal objective function value being $g_0^c(Z)$=\$188,905. On the other hand, the optimal solution for the cooperative scenario uses the same values of T_0 and Z^*, but a different vector of optimal multiplier $n^* = (\underline{2}, 3, 1, 4, 1, \underline{3}, 1, 3, 1, \underline{2})$ with $g_0^c(Z)$= \$173,738. Table 5 summarizes the results if the 10 buyers satisfy the condition in (24).

From Table 5, one may observe that the buyers 5, 7 and 9 fail the condition in (24). Therefore, it is impossible for the vendor to offer these buyers a saving-sharing opportunity

Table 5: A summary of the results if the 10 buyers satisfy the condition in Proposition 4

Buyer	1	2	3	4	5	6	7	8	9	10
$\sqrt{\frac{A_i+K_i}{2H_i}}$	0.0775	0.0866	0.0447	0.1173	0.0346	0.0645	0.0293	0.0829	0.0258	0.0387
Satisfy the condition?	Yes	Yes	Yes	Yes	No	Yes	No	Yes	No	Yes

(to change from the Stackelberg-game scenario to the cooperative scenario). However, since the condition in (24) is not a *sufficient* condition, we are not able to assert if the other seven buyers will change to the cooperative scenario. In fact, in this example, changing only the buyers 1, 6 and 10 from the Stackelberg-game scenario to the cooperative scenario brings the vendor further cost savings.

4.2. Comparison with Stackelberg-game Scenario

In this subsection, we use the 48 sets of data in Viswanathan and Piplani's [25] paper to compare the performance (*i.e.*, the average total costs) of the whole supply chain system of both, the Stackelberg-game and the cooperative scenarios.

We note that among these 48 sets of data, both scenarios obtain the same solutions for the Data Sets 17 to 32. Changing from the Stackelberg-game scenario to the cooperative scenario achieves an average of 2.67% improvement in the average total costs of the whole supply chain system for these 16 data sets. And, impressively, 19 (out of the rest 32 data sets) examples gain an improvement of more than 5%. One may observe that the cooperative scenario outperforms that in the Stackelberg-game scenario for these 32 data sets in their solution quality. Also, the proposed search algorithm for the cooperative scenario (coded in Matlab 5.3) solves each instance with 10 buyers in only a fraction of a second (less than 0.08 seconds) on a PC with Pentium IV-1.6GHz processor. Therefore, the proposed search algorithm is very efficient for decision support.

5. Concluding Remarks

In this paper, we propose a new cooperative scenario for supply chains using common replenishment epochs based on the mathematical model presented in Viswanathan and Piplani's [25] paper. We summarize our contributions of this study as follows. First, we conduct a theoretical analysis on the optimal cost function of the mathematical model, and show that it is piece-wise linear with respect to Z (*i.e.*, the value of price discount). Second, utilizing our theoretical results, we propose an efficient search algorithm to solve an optimal solution for the cooperative scenario. Third, we introduce a saving-sharing mechanism that could motivate the buyers to cooperate with the vendor. Also, importantly, the saving-sharing mechanism assists the supply chain system to create a multi-win situation among the vendor and the buyers. Finally, using the 48 sets of data in Viswanathan and Piplani's [25] paper, we demonstrate that the proposed cooperative scenario could attain significant improvement on the costs for both the vendor and the buyers in comparison to the Stackelberg-game scenario. Also, our numerical results show that the proposed search algorithm is efficient since it solves the optimal solution for the cooperative scenario within a very short run time.

According to our literature review in Section 1, it is well-known that cooperation among the firms provides opportunities to create multi-win situations in supply chains. In this study, we advocate the adoption of a new cooperative scenario to replace the Stackelberg-game scenario supply chains using common replenishment epochs. The authors are currently working on extending the mathematical model to more general cases. For instances, we will consider multi-echelon supply chains rather than two-echelon ones studied in this paper. Also, recall that the vendor merely plays a role of a distributor in this study. By regarding the vendor as a manufacturer, we are revising the mathematical model to take into account the vendor's capacity constraints and the scheduling of the replenishment lots from the buyers.

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Appendix A. The Proof of Proposition 4

Given the values of CRE T_0 and Z, the vectors of optimal multipliers for the Stackelberg-game scenario and the cooperative scenario are $\hat{n} = (\hat{n}_1, ..., \hat{n}_m)$ and $n^* = (n_1^*, ..., n_m^*)$, respectively. Recall that the average total cost function for the vendor is given by $g_0^c =$

$AC(T_0, Z, n_1, ..., n_m) = A_S/T_0 + \sum_{i=1}^{m}(D_i Z + (A_i/n_i T_0))$, and the cost function for buyer i is given by $g_i^c = (K_i/n_i T_0) + H_i n_i T_0$.

When changing from the Stackelberg-game scenario to the cooperative scenario, the cost increase for buyer i is $(K_i/n_i^* T_0) + H_i n_i^* T_0 - [(K_i/\hat{n}_i T_0) - H_i \hat{n}_i T_0]$, and the vendor may have a cost saving of $A_S/T_0 + D_i Z + (A_i/\hat{n}_i T_0) - [A_S/T_0 + D_i Z + (A_i/n_i^* T_0)]$. Therefore, only when the vendor could get a cost saving which is larger than the buyer's cost increase, the vendor would like to offer a saving-sharing opportunity. In such a case, it leads to the following condition:

$$\frac{A_i + K_i}{T_0}\left(\frac{1}{\hat{n}_i} - \frac{1}{n_i^*}\right) > H_i T_0 (n_i^* - \hat{n}_i) \tag{25}$$

We define $W_i \equiv \frac{A_i + K_i}{H_i T_0^2}$ to simplify the notation in our presentation. The inequality in (25) gives $W_i(\frac{1}{\hat{n}_i} - \frac{1}{n_i^*}) > (n_i^* - \hat{n}_i)$, or equivalently,

$$n_i^* + \frac{W_i}{n_i^*} < \hat{n}_i + \frac{W_i}{\hat{n}_i}. \tag{26}$$

Again, let $U_i = \hat{n}_i + \frac{W_i}{\hat{n}_i}$. Then, from (26), it holds that

$$n_i^* \in \left[\frac{U_i - \sqrt{U_i^2 - 4W_i}}{2}, \frac{U_i + \sqrt{U_i^2 - 4W_i}}{2}\right]. \tag{27}$$

Obviously, one could have an opportunity to obtain n_i^* which is different from $\hat{n}_i$ only when the following condition on the upper bound of n_i^* holds:

$$\frac{U_i + \sqrt{U_i^2 - 4W_i}}{2} \geq \hat{n}_i + 1 \tag{28}$$

Then, from (28), we have

$$U_i^2 - 4W_i \geq (2(\hat{n}_i + 1) - U_i)^2 = U_i^2 - 4(\hat{n}_i + 1)U_i + 4(\hat{n}_i + 1)^2. \tag{29}$$

Or, equivalently,

$$(\hat{n}_i + 1)^2 - (\hat{n}_i + 1)U_i + W_i \leq 0. \tag{30}$$

By plugging $U_i = \hat{n}_i + \frac{W_i}{\hat{n}_i}$ into (30), we will reach the following condition:

$$\hat{n}_i + 1 - \frac{W_i}{\hat{n}_i} \leq 0 \tag{31}$$

We assert that $\hat{n}_i^2 + \hat{n}_i - W_i \leq 0$ since $\hat{n}_i \geq 1$. And, it gives

$$\hat{n}_i \in \left[0, \frac{-1 + \sqrt{1 + 4W_i}}{2}\right]. \tag{32}$$

Since $\hat{n}_i$ is a positive integer, the following condition holds for the upper bound on $\hat{n}_i$.

$$\frac{-1 + \sqrt{1 + 4W_i}}{2} \geq 1 \tag{33}$$

It gives $\sqrt{1 + 4W_i} \geq 3$ or $W_i \geq 2$.

By plugging $W_i = \frac{A_i + K_i}{H_i T_0^2}$ into (33), we have $\sqrt{\frac{A_i + K_i}{2H_i}} \geq T_0$ which is exact (24).

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