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AMBULANCE SERVICE FACILITY LOCATION PROBLEM

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Abstract In this paper an ambulance service facility problem is considered in an urban area with a polygonal shape. The objective of this research is to locate the facility under bi-criteria. One criterion is to minimize the maximum weighted sum of distances measured by A-distance in the route which passes from the facility to the hospital by way of the scene of accident. The other is to maximize the preference function of the facility site. Usually an optimal site that optimizes both criteria does not exist and so we seek some non-dominated sites for the facility after defining notion of non-dominated site.

Keywords: Facility Planning, emergency facility, A-distance, preference function of facility site, weighted sum of distance, non-dominated site

1. Introduction

Models so far considered as facility location problems assume either Euclid distance or Rectilinear distance. But it is not enough to cover all actual cases, especially urban area case and so we adopt A-distance introduced by Widmayer et al. [7] which is a generalization of Rectilinear distance, that is, the distance determined by the given multiple directions (Rectilinear distance is determined by vertical and horizontal directions). Further we introduce preference function of the facility site ([4]). This implies that we must take construction cost, safety etc into consideration for determination of the site of the facility, that is, not only customer side but also the local government side responsible for the construction of the facility should be considered in an actual problem. Especially for site of an ambulance facility, safety and security are very important. Our model is an extension of the emergency facility model considered in [3]. Of course there exist many related works about emergency facility (please see [1] and [6] for a excellent summary) since Elzinga and Hearn [2] have considered mini-max model under rectilinear distance and given a geometric solution procedure. Section 2 formulates our model and derives useful properties. Based on the results in Section 2, Section 3 proposes a solution procedure for obtaining some non-dominated facility sites after definition of non-dominated site. Section 4 summarizes this paper and discusses further research problems.

2. Problem Formulation

We consider an ambulance service station location problem given as follows:

(1) If an accident (demand) occurs, the ambulance servers rush to the scene of accident (demand point) and bring the injured persons to the nearest hospital as soon as possible. We consider a polygonal area X where an ambulance service station should be located, demand occurs and there exist m hospitals $H_1, H_2, \cdots, H_m$.

(2) For each point $p = (x, y)$ in X membership function is attached denoting the preference with respect to construction of the station at the point.

(3) Our objective is to locate the station so as to minimize the maximum weighted A-distance of the route from the station to the hospital via the demand point (scene of accident) and maximize the preference of the station site. As is shown below, rectilinear distance is a special case of A- distance So we think using A-distance is more realistic than that of rectilinear distance.

(4) Let $S(Q)$ denote the nearest hospital to the point Q. Then we formulate an ambulance service station problem under the above setting (1),(2),(3) as the following problem $\mathbf{P_M}$.

$$\begin{aligned} \mathrm{P_M}: &\text{Minimize} \quad \mathrm{Max}_{Q\in X} WR(p,Q)(= w_1 d_A(p.Q) + w_2 d_A(Q.S(Q))) \\ &\text{Maximize} \ \ \mu_p(p) \\ &\text{subject to} \ \ p \in X \end{aligned} \tag{2.1}$$

where w_1, w_2 are positive weights corresponding to the importance (emergency) of A-distance $d_A(p, Q)$ from the demand point Q to the station site p and that of $d_A(Q, S(Q))$ from the demand point Q to the nearest hospital *S(Q)*, and we assume that $w_1 \geq w_2$ since for the purpose of the station is to rush to the accident site. We consider the satisfaction degree about A-distance instead of A-distance directly with respect to Q for fixed p, i.e., the following membership functions on A-distance.

$$\mu_1(d_A(p,Q)) = \begin{cases} 1 \ (d_A(p,Q) \leq f_1) \\ 1 - \frac{d_A(p,Q)-f_1}{e_1-f_1} \ (f_1 \leq d_A(p,Q) \leq e_1) \\ 0 \ (d_A(p,Q) \geq e_1) \end{cases} \tag{2.2}$$

$$\mu_2(d_A(Q,S(Q))) = \begin{cases} 1 \ (d_A(Q,S(Q)) \leq f_2) \\ 1 - \frac{d_A(Q,S(Q))-f_2}{e_2-f_2} \ (f_2 \leq d_A(Q,S(Q)) \leq e_2) \\ 0 \ (d_A(Q,S(Q)) \geq e_2) \end{cases} \tag{2.3}$$

where e_1, f_1 are critical distances of satisfaction with respect to the distance $d_A(p, Q)$ and e_2, f_2 those with respect to the distance $d_A(Q, S(Q))$. That is if $d_A(p, Q)$ is over e_1 , then the situation becomes severe (not satisfied at all) and ideal distance is less than f_1 (satisfied completely). So distance $d_A(p, Q)$ is usually considered between f_1 and e_1. For $d_A(Q, S(Q))$, the situation is similar but it does not depend on the station site. Anyway we should consider the weighted sum of A-distance given as $w_1 = \frac{1}{e_1-f_1}$ and $w_2 = \frac{1}{e_2-f_2}$. (we assume that $(e_1 - f_1) \leq (e_2 - f_2)$, i.e., $w_1 \geq w_2$ since usually rushing accident site and making minimal processing to injured person is more important) if the problem is meaningful, that is, not trivial (maximal minimum satisfaction degree is 1), nor infeasible (maximal minimum satisfaction degree is 0). This weighted sum has a meaning, especially maximum value of this sum among Q. Since usually there is no site that optimizes both criteria, that is, the maximum weighted A-distance and preference function, we seek some non-dominated sites.

First we introduce A-distance and derive some useful properties.

A-distance

There exists a set of directions $A = \{\alpha_1, \alpha_2, \cdots, \alpha_a\}$ where each $\alpha_i, i = 1, 2, \ldots, a$ is an angle from x axis in an orthogonal coordinate and let $0° \leq \alpha_1 < \alpha_2 < \cdot < \alpha_a < 180°$. Hereafter if no confusion occurs, directions $\alpha_i, i = 1, 2, \ldots, a$ and angles $\alpha_i, i = 1, 2, \ldots, a$ are used as the same meaning.

Directions $\alpha_j.\alpha_{j+1}$ are called neighboring where α_a, α_1 are also called neighboring, that is α_{a+1} is interpreted as α_1. Further a line, a half line and a line segment are called A-directional or A-oriented respectively (here we call them as A line, A half line and A line segment respectively also) if their directions are ones of $\alpha_i, i = 1, 2, \ldots, a$.

Then A distance d_A between two points $p^1, p^2 \in R^2$ is defined as follows.

$$d_A(p^1, p^2) = \begin{cases} d_2(p^1, p^2) \text{ if direction } \overline{p^1p^2} \text{ is } A\text{−oriented} \\ \min_{p^3 \in R^2} d_A(p^1, p^3) + d_A(p^3, p^2) \text{ otherwise} \end{cases} \tag{2.4}$$

where $d_2(p^1, p^2)$ is the Euclidean distance between p^1 and p^2. Figure 1 illustrates how to calculate A-distance, that is, making the smallest parallelogram with line segment p^1p^2 as one diagonal line (α_i, α_j are neighboring A line segments).

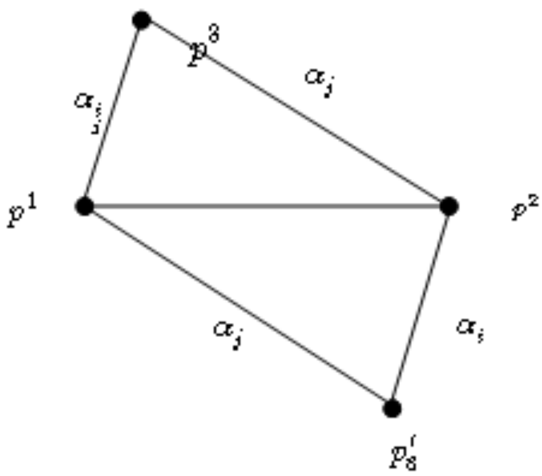

Figure 1: A-distance between p^1 and p^2

According to the result in [5], when $\alpha_j <$ an angle of the line connecting demand point i with the station site $(x, y) < \alpha_{j+1}$, parallelogram is constructed by using neighboring angles $\alpha_j, \alpha_{[j+1}$ and so

$$d_i = M_1|m_2(p_i - x) - (q_i - y)| + M_2|m_1(p_i - x) - (q_i - y)| \tag{2.5}$$

where (p_i, q_i) is coordinates of demand point i and

$m_1 = \max(\tan\alpha_j, \tan\alpha_{j+1}), m_2 = \min(\tan\alpha_j, \tan\alpha_{j+1}), M_1 = \frac{\sqrt{1+m_1^2}}{m_1-m_2}, M_2 = \frac{\sqrt{1+m_2^2}}{m_1-m_2}$
If either α_j or α_{j+1} is $90°$, then we interpret

$M_1 = \lim_{m_1 \to \infty} \frac{\sqrt{1+m_1^2}}{m_1-m_2} = 1,$

$$M_2|m_1(p_i - x) - (q_i - y)| = \lim_{m_1 \to \infty} \frac{\sqrt{1 + m_2^2}}{m_1 - m_2}|m_1(p_i - x) - (q_i - y)| = |p_i - x| \tag{2.6}$$

Since in the rectilinear distance case, we consider as $a = 2, \alpha_1 = 0°, \alpha_2 = 90°, m_1 \to \infty, m_2 = 0$ and so $d_i = |-(q_i - y)| + |p_i - x| = |p_i - x| + |q_i - y|$. This means rectilinear distance is a special case of A-distance.

A-circle with center p and radius r is $2a$ polygon where vertices are intersection points between All A half-lines from p and ordinary circle with center p and radius r. Please refer to Figure 2 as an example of A-circle where a_1, a_2, a_3 are A-line segments and so hexagonal shape contacted with an ordinary circle. Note that edges of A-circle are denoted by directions $\beta_1, \beta_2, \beta_3$ and distance to any points p' on the line segment $p'p''(= \beta_3)$ from p is r.

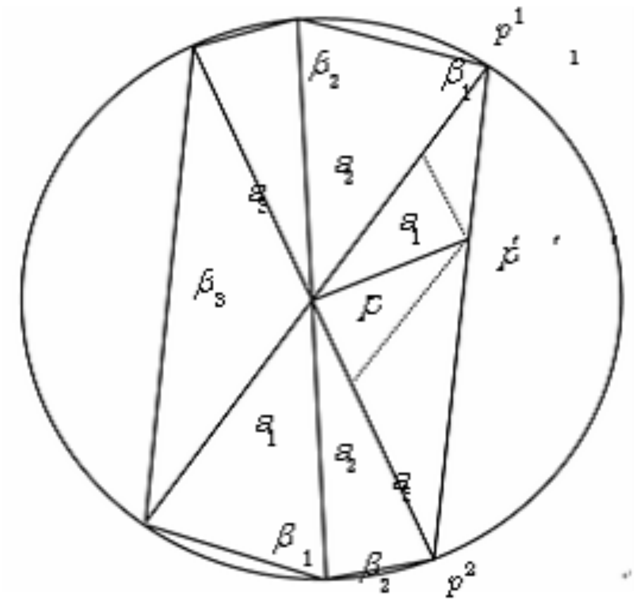

Figure 2: An example of A-circle

Of course when the line connecting demand point i with the station site $p = (x, y)$ is A-oriented, $d_i = \sqrt{(p_i - x)^2 + (q_i - y)^2}$ (Euclidean distance between demand point i and the station).

Now we define non-dominated site and review the Voronoi diagram.

The definition of Non-dominated site

If $R(p^2) \geq R(p^1), \mu_P(p^1) \geq \mu_P(p^2)$ and at least one inequality holds without equality for the sites $p^1 = (x_1, y_1), p^2 = (x_2, y_2) \in X$, then we call p^1 dominates p^2 where $R(p) = \max\{WR(p, Q)|Q \in X\}$, i.e., maximal weighted sum of A-distance in the route from the station site p to all demand points. If there exists no site that dominates p, then p is called non-dominated site.

Voronoi diagram

For a set of v points $\mathbf{VT} = \{VT_1, VT_2, \cdots, Vt_n\}$, Voronoi polygon $VT_A(T_i)$ with respect to VT_i and A-distance on X is defined as follows:

$$VT_A(VT_i) = \bigcap_{j \neq i} \{p | d_A(p, VT_i) \leq d_A(p, VT_j), p \in x\} \tag{2.7}$$

The set of all Voronoi polygons for the points in **it VT** is a partition of X and called Voronoi diagram. We construct Voronoi diagram $VD_A(\mathbf{H})$ with respect to the set of hospital points $\mathbf{H} = \{H_1, H_2, \cdots, H_m\}$ and A-distance on the area X. in order to solve the problem. Figure 3 illustrates Voronoi diagram with respect to Hospitals. It is done in *O(m log m)* computational time [7].

Then we have the same properties as the results in [5] though in that case not necessarily $w_1 \neq w_2$.

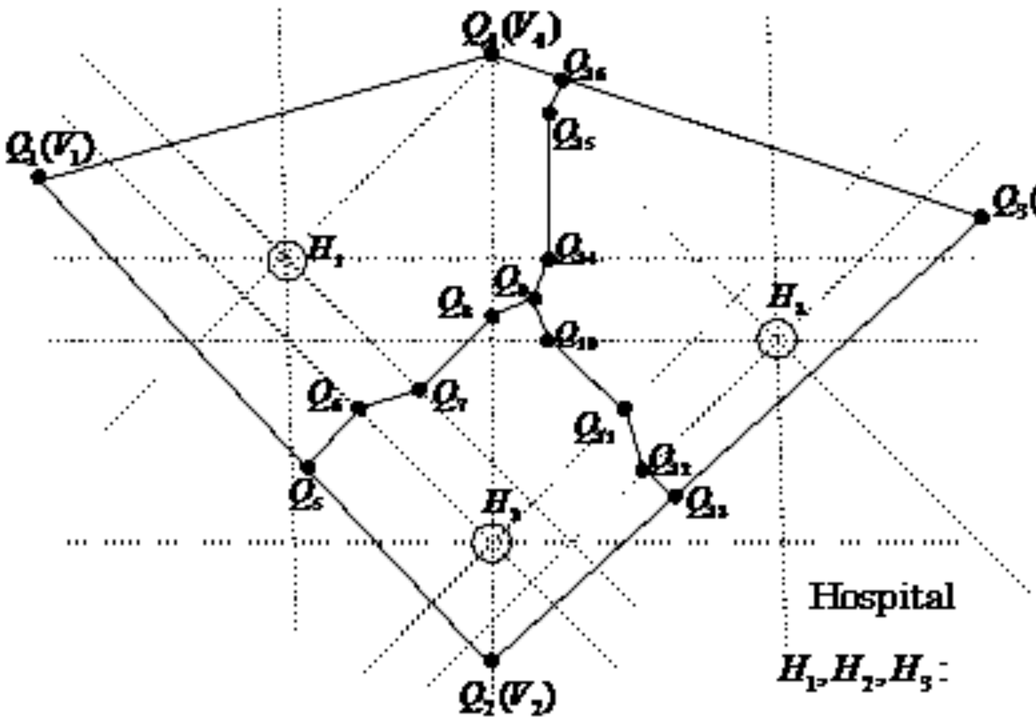

Figure 3: Voronoi diagram with respect to hospitals H_1, H_2, H_3

Theorem 1 For the line segment *DE* with endpoints *D,E* and points *B,C* not on *DE,* suppose *BD* and *BE* are *A*-oriented adjacent orientations α_j, α_{j+1}. Then the weighted sum of A-distance among paths between *B* and *C* via point *T* on the line segment *DE,* $w_1 d_A(B,T) + w_2 d_A(T,C)$ is maximized when *T=D or E.*

(Proof)

The lines of all orientations of *A* through *C* partition the line segment *DE* into subintervals $[F_k, F_{k+1}], k = 0, 1, \ldots, q-1$ where $F_0 = D, F_q = E$ and $F_k, k \neq 0, q$ are cross points between *DE* and all *A*-oriented lines through *C*. Consider the certain subinterval $[F_k, F_{k+1}]$. By a suitable transformation, we assume *DE* is x axis, $F_k = (0,0), F_{k+1} = (e,0), B = (b_1, b_2)$ and $C = (c_1, c_2)$ without any loss of generality. Then for point $T = (x,0), (0 \leq x \leq e)$,

$$R_A^k(x) = w_1 d_A(B,T) + w_2 d_A(T,C) = w_1 M_1 |m_2(x - b_1) + b_2|$$

$$+ w_1 M_2 |m_1(x - b_1) + b_2| + w_2 M_3 |m_4(x - c_1) + c_2| + w_2 M_4 |m_3(x - c_1) + c_2| \qquad (2.8)$$

where

$m_1 = \max(\tan\alpha_j, \tan\alpha_{j+1}), m_2 = \mathrm{minx}(\tan\alpha_j, \tan\alpha_{j+1}), M_1 = \frac{\sqrt{1+m_1^2}}{m_1 - m_2}, M_2 = \frac{\sqrt{1+m_2^2}}{m_1 - m_2}, m_3 = \max(\tan\alpha_i, \tan\alpha_{j+1}), m_4 = \mathrm{minx}(\tan\alpha_i, \tan\alpha_{i+1}), M_3 = \frac{\sqrt{1+m_3^2}}{m_3 - m_4}, M_4 = \frac{\sqrt{1+m_4^2}}{m_3 - m_4}$

and α_j, α_{j+1} are the orientations corresponding to the subinterval $[F_k, F_{k+1}]$ (Figure 4 illustrates the situation). $R_A^k(x)$ is a convex function and so maximum value of $R_A^k(x)$ is

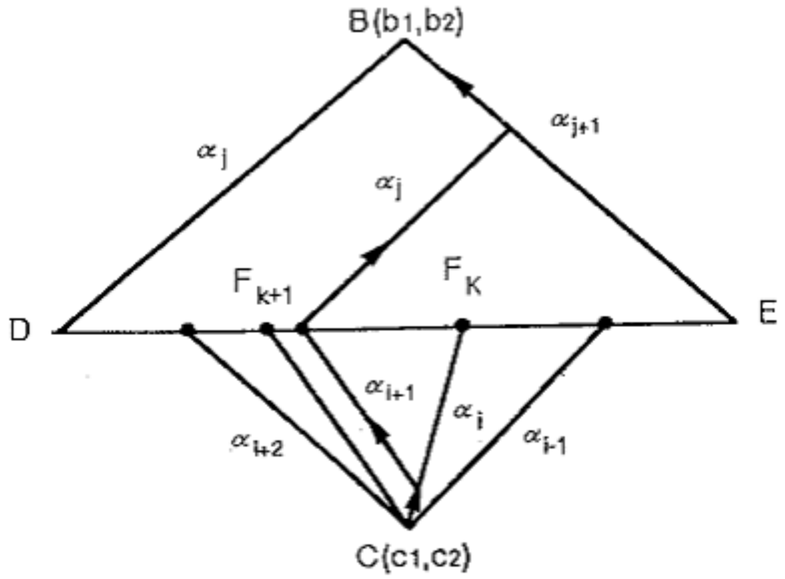

Figure 4: Relation between segment DE and neighboring direction

attained at x=0 or x=e, i.e. $T = F_k$ or F_{k+1}. Thus the candidate points of maximum A-distance path is $F_0, \cdots, F_q$. Since each $F_k, k = 1, 2, \ldots, q-1, d_A(F_k, C) = d_2(F_k, C)$, $d_A(F_0, C) \geq d_2(F_0, C), d_A(F_q, C) \geq d_2(F_q, C)$ and α_j, α_{j+1} are adjacent orientations, then $d_A(B < T) + d_2(T, C)$, for $T \in DE$, is considered as a path length between B and C via $T \in DE$.

Now let $D = (0,0), E = (e', 0), B = (b_1, b_2)$ and $C = (c_1, c_2)$ without any loss of generality. Then for $T = (x, 0), (0 \leq x \leq e')$,

$$w_1 d_A(B,T) + w_2 d_2(T,C) = w_1 M_1 |m_2(x - b_1) + b_2|$$

$$+ w_1 M_2 |m_1(x - b_1) + b_2| + w_2 \sqrt{(x - c_1)^2 + c_2^2} \tag{2.9}$$

Each term of right hand side in the above expression is convex function of$x$. So maximum of $w_1 d_A(B,T) + w_2 d_A(T,C)$ is attained at $x = 0$ or e', i.e. D or E. Further the path length through D or E is not less than $w_1 d_A(B,D) + w_2 d_2(D,C)$ or $w_1 d_A(B,E) + w_2 d_2(E,C)$, because either CD or EC is necessarily A-oriented. Therefore maximum is attained at D or E.

Q. E. D.

Further we relax the constraints that BD and BE have α_j and α_{j+1} oriented respectively from Theorem 1.

Theorem 2 For the line segment DE with endpoints *D,E* and points *B,C* not on *DE,* $w_1 d_A(B,T) + w_2 d_A(T,C), T \in DE$ is maximized when *T=D or E.*

Proof

We draw all A-oriented half lines from B and C, and let all intersections of these lines and DE be $T_1, T_2, \cdots, T_{t-1}$ by ordering from D. Further let $T_0 = D$ and $T_t = E$. Then the situation may be interpreted as Figure 5. By Theorem 1, when consider the subinterval $T \in [T_{i-1}, T_{i+1}], w_1 d_A(B,T) + w_2 d_A(T,C)$ is maximized at T_{i-1} or T_{i+1}. So T_i is dropped from candidates of maximizer. In turn, when considering $T \in [T_{i-2}, T_i], T_{i-1}$ is dropped by Theorem 1. Continuing this way, only remaining candidates are *D,E* and points as T_{i+7}

which are intersections between DE and certain A -lines from both B and C. Let all points on DE with same property as T_{i+7} be $T'_1 \cdots, T'_\ell$. Then

$$w_1 d_A(B, T'_i) + w_2 d_A(T'_i, C) = w_1 d_2(B, T'_i) + w_2 d_2(T'_i, C), i = 1, \ldots, \ell \tag{2.10}$$

since both BT'_i and CT'_i are A-oriented. Since Euclidean distance is a convex function, then $w_1 d_2(B, T) + w_2 d_2(T, C), T \in DE$ is maximized at $T = D$ or E. Thus

$$w_1 d_A(B, D) + w_2 d_A(D, C) \geq w_1 d_2(B, D) + w_2 d_2(D, C) \tag{2.11}$$

$$w_1 d_A(B, E) + w_2 d_A(E, C) \geq w_1 d_2(B, E) + w_2 d_2(E, C) \tag{2.12}$$

implies $w_1 d_A(B, T) + w_2 d_A(T, C), T \in DE$ is maximized when $T = D$ or E.

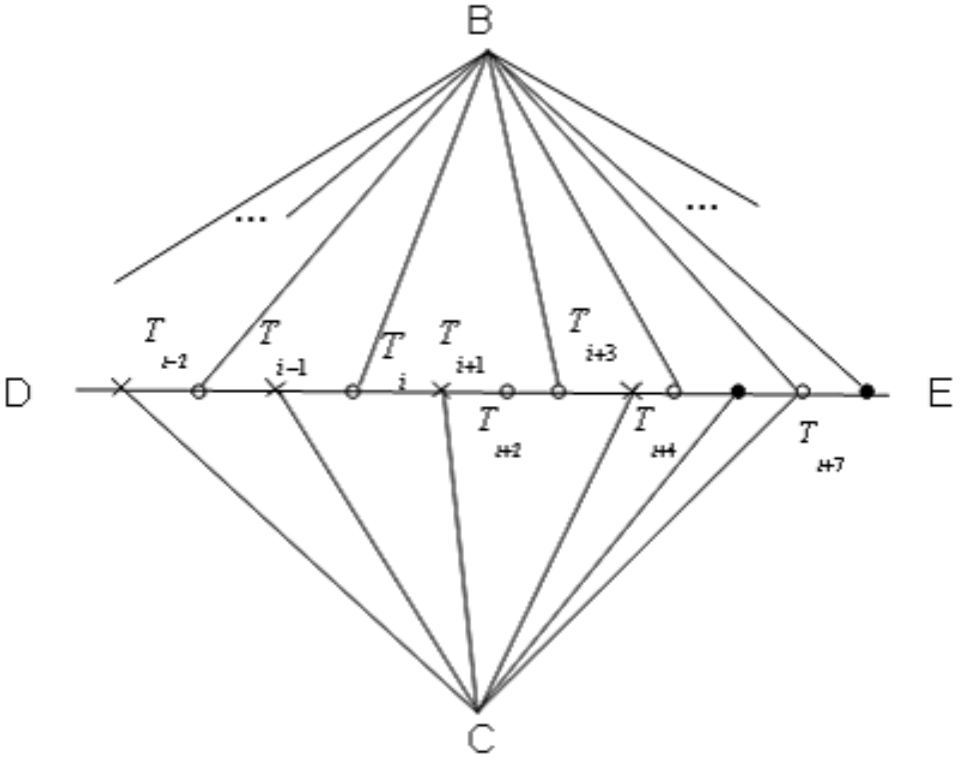

Figure 5: Intersections between each A half line from B, C and line segment DE

Q. E. D.

Figure 6 illustrates a small example (m=6) of Voronoi diagram with respect to $H = \{H_1, \cdots, H_m\}$. Consider any interior point E of X on a Voronoi edge and draw the half line originating from the facility P and through E. Let the intersection of this half line and the other Voronoi edge of same Voronoi polygon as E be F.

Further let the intersection of this half line and boundary of X be G. It is sufficient to consider the situation of Figure 6, in order to show $w_1 d_A(P, G) + w_2 d_A(G, S(G)) \geq w_1 d_A(P, E) + w_2 d_A(E, S(E))$ It holds that

$$w_1 d_A(P, E) + w_2 d_A(E, S(E)) l w_1 d_A(P, E) + w_1 d_A(E, F) + w_2 d_A(F, S(F)) \tag{2.13}$$

$= w_1 d_A(P, F) + w_2 d_A(F, S(F))$

by the triangular property of A-distance. Since F is on Voronoi edge of Voronoi polygons with respect to both H_2 and H_4, then $w_2 d_A(F, S(F)) = w_2 d_A(F, H_2) = w_2 d_A(F, H_4)$. Further $w_1 d_A(F, G) + w_2 d_A(G, S(G)) \geq w_2 d_A(F, H_4)$ holds by the triangular inequality of A-distance. Thus

$$\begin{aligned} w_1 d_A(P, G) + w_2 d_A(G, S(G)) &= w_1 d_A(P, F) + w_1 d_A(F, G) + w_2 d_A(G, S(G)) \\ &\geq w_1 d_A(P, F) + w_2 d_A(F, H_4) = w_1 d_A(P, F) + w_2 d_A(F, S(F)) \\ &\geq w_1 d_A(P, E) + w_2 d_A(E, S(E)) \qquad (2.14) \end{aligned}$$

From above discussion and Theorem 2, we have the following Theorem 3.

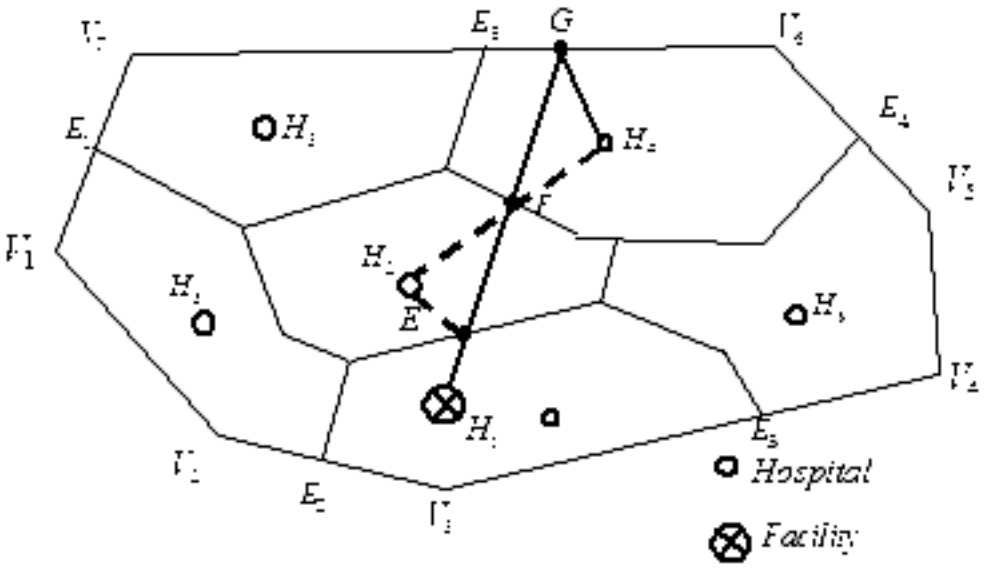

Figure 6: Voronoi diagram with respect to $H = \{H_1, H_2, \cdots, H_6\}$ illustrating the situation of Theorem 2

Theorem 3 For fixed p, candidates of maximizer of $WR(p, Q)$ are

(a) Vertices of boundary of X.

(b) The intersection points of Voronoi edges and boundary of X.

Proof It is directly shown from above discussion and Theorem 2.

From Theorem 3 we can reduce the number of demand points which should be considered in the solution procedure for $\mathbf{P_M}$ to be finite. In this case, we have the following solution algorithm which is the very same as that in [5]. But we consider the solution procedure in [5] since this solution procedure becomes a base for the solution procedure of our model and an optimal solution corresponds to one non-dominated solution of $\mathbf{P_M}$ (that is, so-called minimizer of one objective $\mathrm{Max}_{Q \in X} WR(p, Q) (= w_1 d_A(p.Q) + w_2 d_A(Q.S(Q)))$ of $\mathbf{P_M}$). Let consider all points in (a) and (b) of Theorem 3. Let vertices of boundary of X be $V_1, V_2, \cdots, V_n$. Further let the intersections of Voronoi edges and boundary of X be $E_1, E_2, \cdots, E_e$. By a suitable numbering of $V_1, V_2, \cdots, V_n$ and $E_1, E_2, \cdots, E_e$, let those points be $Q_1, Q_2, \cdots, Q_N$ where N is the number of different points of them (for example,

decreasing order of $k_i = \frac{w_2}{w_1} d(Q_i, S(Q_i)), i = 1, 2, \ldots, N$) Then by Theorem 3, $\mathbf{P_M}$ is reduced to the following messenger boy problem $\mathbf{P_E}$ since
$\text{Max}_{Q\in X} WR(p, Q) (= w_1 d_A(p.Q) + w_2 d_A(Q.S(Q))) = \text{Max}\{w_1 d_A(p.Q_i) + w_2 d_A(Q_i.S(Q_i)) | i = 1, 2, \ldots, N\} = w_1 \text{Max}\{d_A(p, Q_i) + k_i | i = 1, 2, \ldots, N\}$.

$$\mathbf{P_E} : \text{Minimize} \quad \max\{d_A(p, Q_\ell) + k_i | i = 1, 2, \ldots, N\} \qquad \text{subject} \quad \text{to} \quad p \in X \tag{2.15}$$

$\mathbf{P_E}$ is further transformed into the following problem $\mathbf{P_L}$

$$\mathbf{P_L} : \text{Minimize } z \qquad \text{subject} \quad \text{to} \quad d_A(p, Q_\ell) + k_i \le z, i = 1, 2, \ldots, N, p \in X \tag{2.16}$$

3. Solution Procedure for P_M

P_L is the very same as that in [5] and clearly an optimal solution p^0 of P_L is a a non-dominated site of P_M since the center of A-circle with minimal radius covering all A-circle with radius k_i at a center Q_i is optimal for P_L. So we now consider the solution procedure for P_L. Let C_i denote A-circle with radius k_i at a center Q_i. Then P_L is the determination problem of minimum radius A-circle covering all A-circles $C_1, C_2, \cdots, C_N$ where A-circle with radius r at center c is defined as follows:

$$\{p | d_A(c, p) \le r\} \tag{3.1}$$

Usually it becomes a polygon consisting of A-line segments with length r. We define

$$C_A(C_i, C_j) = \{p \in R^2 | d_A(p, Q_i) + k_i = d_A(p, Q_j) + k_j\} \text{for} i \ne j, i, j = 1, \ldots, N. \tag{3.2}$$

where $C_A(C_i, C_j)$ is a bisector between Q_i and Q_j taking account of the weighted distance to hospitals. Then p^0 is obtained as follows.

[Solution procedure for p^0]

Step 1: Draw A-circle $C_1, C_2, \cdots, C_N$ and let C_θ denote the biggest A-circle which has the largest radius among $C_1, C_2, \cdots, C_N$. If C_θ covers all other $C_i, i \ne \theta$, then C_θ is the optimal A-circle. Stop as Q_θ is an optimal solution. Otherwise, find C_s, C_t such that

$$\max\{d_A(Q_i, Q_j) + k_i + k_j | i \ne j, i, j = 1, \ldots, N\} = d_A(Q_s, Q_t) + k_s + k_t \tag{3.3}$$

and go to Step 2.

Step 2 Let P_0 be the intersection of $C_A(C_s, C_t)$ and the line segment connecting Q_s with Q_t. Draw the A-circle C_0 centered at P_0 with minimum radius covering C_s, C_t. If C_0 covers all C_i, then C_0 is an optimal A-circle. Stop as P_0 is an optimal location of the ambulance service station. Otherwise, choose one A-circle C_u which is not covered by C_0 and go to Step 3.

Step 3 Let P_1 be an intersection of $C_A(C_s, C_t)$, $C_A(C_t, C_u)$ and $C_A(C_u, C_s)$. Draw A-circle C_c covering C_s, C_t, C_u with minimum radius centered at P_1, that is, externally tangent to these three A-circles. If C_c covers all $_i$, then C_c is an optimal A-circle. Stop as P_1 is an optimal solution. Otherwise, choose one A-circle C_v which is not covered by C_c. Go to Step 4. (Please refer to Figure 7)

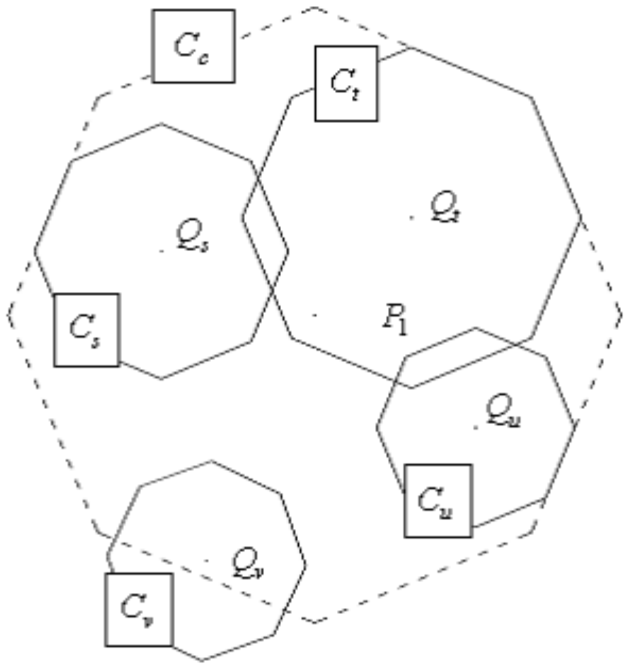

Figure 7: An illustration of Step 3 in Solution Procedure for p_0

Step 4 Draw a half line from P_1 which through Q_v and let an intersection of the line and boundary be Z_v which is the farthest from P_1. By the same manner obtain Z_s, Z_t, Z_u. Let $D = Z_v$ and farthest point from D among Z_s, Z_t, Z_u be P_A. Divide an area X into two sub-areas by line through both P_A and P_1. Let a point which does not belong to the same sub-area with D be C. Let $Q_s = P_A, Q_t = C, Q_u = D$ and return to Step 3.

The above procedure finds p_0 in at most $O(\max(n,m)^3 \cdot T)$ computational time where T is the computational time constructing A-circle covering given three A-circles. Validity of the algorithm is clear since basically, a center of A-circle covering suitable three A-circles is an optimal solution p_0 (this is shown already in [5]).

Next we seek other non-dominated sites. First let $X(\beta) = \{p \in X | \mu_P(p) \geq \beta\}$ for $1 \geq \beta > \mu_P(p_0)$, that is, the set of sites that their preference is not less than β. Then we consider the following parametric problem $P_L(\beta)$, that is, the problem where possible sites are restricted to $X(\beta)$ instead of X.

$$P_L(\beta) : \text{Minimize } z \quad \text{subject to} \quad d_A(p, Q_i) + k_i \leq z, i = 1, 2, \ldots, N, p \in X(\beta) \qquad (3.4)$$

As is easily seen, p_0 is in the area X though it is not explicitly restricted. But $P_L(\beta)$ is a little bit difficult to solve since we explicitly restrict the feasible region of $P_L(\beta)$ Let $P(\beta)$ be an optimal site of $P_L(\beta)$ and $R(\beta) = R(p(\beta))$. Now we propose a solution method for $P_L(\beta)$.

Solution Procedure for $P_L(\beta)$

Step 1: Draw A-circle $C_1, C_2, \cdots, C_N$ and let C_θ denote the biggest A-circle which has the largest radius among $C_1, C_2, \cdots, C_N$. If C_θ covers all other $C_i, i \neq \theta$ and $Q_\theta \in X(\beta)$, then C_θ is the optimal A-circle. Stop as Q_θ is an optimal solution $P_L(\beta)$. Otherwise, find C_s, C_t such that

$\max\{d_A(Q_i, Q_j) + k_i + k_j | i \neq j, i, j = 1, \ldots, N\} = d_A(Q_s, Q_t) + k_s + k_t$ and go to Step 2.

Step 2 Let P_0 be the intersection of $C_A(C_s, C_t)$ and the line segment connecting Q_s with Q_t. If $P_0 \in X(\beta)$, draw the A-circle C_0 centered at P_0 with minimum radius covering C_s, C_t. If C_0 covers all C_i, then C_0 is an optimal A-circle. Stop as P_0 is an optimal location of $P_L(\beta)$. Otherwise, choose one A-circle C_u which is not covered by C_0 and go to Step 3. If $P_0 \notin X(\beta)$, go to Step 4.

Step 3 Let P_1 be an intersection of $C_A(C_s, C_t)$, $C_A(C_t, C_u)$ and $C_A(C_u, C_s)$. If $P_1 \in X(\beta)$ draw A-circle C_c covering C_s, C_t, C_u with minimum radius centered at P_1, that is, externally tangent to these three A-circles. If C_c covers all C_i , then C_c is an optimal A-circle. Stop as P_1 is an optimal solution. Otherwise, choose one A-circle C_v which is not covered by C_c. Go to Step 5. If $P_1 \notin X(\beta)$, go to Step 4.

Step 4 Consider each case that Q_i is maximum among all $Q_1, Q_2, \cdots, Q_N$ from $X(\beta)$ with respect to A-distance and find the point p_i in minimizing $d_A(Q_i, p)$ corresponding to A-circle covering all $C_1, C_2, \cdots, C_N$. Let p_c minimizing $d_A(Q_j, p_j) + k_j, j = 1, 2, \ldots, N$ be an optimal solution of $P_L(\beta)$ and stop.

Step 5 Draw a half line from P_1 which passes Q_v and let an intersection of the line and boundary of $P_L(\beta)$ be Z_v which is the farthest from P_1. By the same manner, obtain Z_s, Z_t, Z_u. Let $D = Z_v$ and farthest point from D among Z_s, Z_t, Z_u be P_A. Divide an area $X(\beta)$ into two sub-areas by the line through both P_A and P_1. Let a point which does not belong to the same sub-area with D be C. Let $Q_s = P_A, Q_t = C, Q_u = D$ and return to Step 3.

$P_L(\beta)$ can be solved efficiently if Step 4 is executed efficiently and this depend the shape of $P_L(\beta)$. That is to find p_i efficiently is critical. Anyway, based on the solution procedure for $P_L(\beta)$, we consider the following solution procedure for P_M though for a general case, how to determine $\varepsilon_0, \varepsilon_\beta$ is not clear.

(Solution Procedure for P_M)

Step 1 Set $\beta = \mu_P(P^0) + \varepsilon_0$ (ε_0 is suitable small positive number), $DS = \{P^0\}$ and go to Step 2.

Step 2 Solve $P_L(\beta)$ and obtain $P(\beta)$ and $R(\beta)$. If there exists no site $p \in DS$ dominating $P(\beta)$, then go to Step3. Otherwise go to Step 4.

Step 3 Update $DS \leftarrow DS \bigcup\{p(\beta)\}$. If $\beta = 1$, terminate (DS is a set of some non-dominated sites). Otherwise, update $\beta \leftarrow \min\{\mu_P(p(\beta)) + \varepsilon_\beta, 1\}$ (ε_β is a suitable small positive number) and return to Step 2.

Step 4 If $\beta = 1$, terminate (DS is a set of some non-dominated sites). Otherwise update $\beta \leftarrow \min\{\mu_P(p(\beta)) + \varepsilon_\beta, 1\}$ (ε_β is a suitable small positive number) and return to Step 2.

Whether this solution procedure terminates or not is not clear since its preference function is not specified and so suitable small numbers are not concrete. But when we give up to find all non-dominated solutions and change β by constant amount ε and each $P_L(\beta)$ can be solved in at most $O(\max(n,m)^3 \cdot T)$ computational time, the above solution algorithm terminates in at most $O(\max(n,m)^3 \cdot \frac{T}{\varepsilon})$ computational time. However for the following special but important case, we can obtain the non-dominated solutions by changing $\beta = t_1, t_2, \cdots, t_q$ in the above solution procedure.

(Case that the preference function is constant block-wisely)

We assume that

$$\mu_P(p) = \begin{cases} t_1 & p \in A_1 \\ t_2 & p \in A_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ t_q & p \in A_q \\ 0 & p \in X - A_1 - A_2 - \cdots - A_q \end{cases} \tag{3.5}$$

where $1 \geq t_1 > t_2 > \cdots > t_q > 0$ and $A_1, A_2, \cdots, A_q \subset X$ are disjoint polygons. In this case, for solution procedure for P_M, as β only $t_1, t_2, \cdots, t_q$ should be considered. That is, following problems $P^{t_1}, P^{t_2}, \cdots, P^{t_q}$ should be considered and each feasible region of P^{t_k} is set A_k. that is, a polygon.

$$P^{t_k} : \text{Minimize } z \quad \text{subject to } d_A(p, Q_i) + k_i \leq z, i = 1, 2, \ldots, N, p \in A_k \tag{3.6}$$

Then following theorem is very useful in this case.

Theorem 4 *If $p^0 \notin A_k$, then an optimal solution of P^{t_k} exists on the boundary of A_k.*
Proof

We derive a contradiction by assuming the optimal solution $p(t_k)$ is the interior point of A_k. Following cases (i) (ii) (iii) should be checked.
(i) The optimal value is attained by a certain only one Q_i. Then a point moved by small amount toward Q_i from $p(t_k)$ along the line segment connecting between Q_i and $p(t_k)$ is better point with respect to z as is easily shown.
(ii)The optimal value is attained by certain two Q_i and Q_j.
Then $p(t_k)$ is on the bisector $C_A(C_I, C_J) \equiv \{p \in R^2 | d_A(p, Q_i) + k_i = d_A(p, Q_j) + k_j\}$.
(ii a) $p(t_k)$ is the anchor point, that is, an intersection point between the line segment $\overline{Q_i Q_J}$ and the bisector $C_A(C_i, C_j)$.
Then in this subcase clearly $p(t_k)$ is also an optimal solution of P_L and it derives a contradiction with $p^0 \notin A_k$.
(ii b) $p(t_k)$ is not the anchor point. Then moving by small amount toward anchor point along the bisector from $p(t_k)$ makes z decrease.
This means $p(t_k)$ is not optimal.
(iii) The optimal value is attained by certain three Q_i, Q_s and Q_j.
Then again clearly $p(t_k)$ is also an optimal solution of P_L and it derives a contradiction with $p^0 \notin A_k$.

Q. E. D

First we construct the farthest point Voronoi diagram with respect to $C_1, C_2, \cdots, C_N$ as follows: For each pair of Q_i and Q_j , we draw the bisector

$C_A(C_I, C_J) \equiv \{p \in R^2 | d_A(p, Q_i) + k_i = d_A(p, Q_j) + k_j\}$ and we make the farthest region X_i about Q_i , that is, $X_i \equiv \{p \in R^2 | d_A(p, Q_i) + k_i \geq d_A(p, Q_j) + k_j, j \neq i\}$ for each i based on these bisectors. Then the farthest point Voronoi diagram is constructed similarly as the usual Voronoi diagram with respect to $C_1, C_2, \cdots, C_N$ and draw all A-half lines from each Q_i, obtain the intersection points $FP_1^{i,k}, FP_2^{i,k} \cdots, FP_{n_{i,k}}^{i,k}$ between these half lines and the boundary of $A_k \cap X_i$ where $n_{i,k}$ is the number of different intersection points. Further let vertices of the boundary with respect to $A_k \cap X_i$ be $FP_{n_{i,k}+1}^{i,k}, FP_{n_{i,k}+2}^{i,k} \cdots, FP_{n_{i,k}+b_{i,k}}^{i,k}$ by suitable numbering of vertices where $b_{i,k}$ is the number of their vertices. Next we solve the following single objective problem:

P_L : Minimize z subjectto $d_A(p, Q_i) + k_i \leq z, i = 1, 2, \ldots, N, p \in X$ and obtain an optimal solution p^0. Then we have the following theorem.

Theorem 5 *If $p^0 \notin A_k$, then an optimal solution of P^{t_K} exists among*

$$FP_1^{i,k}, FP_2^{i,k} \cdots, FP_{n_{i,k}}^{i,k}$$

Proof

From the result of Theorem 4, an optimal solution exists on boundary of $A_k \cap X_i$ Note that the boundary of $A_k \cap X_i$ consists of line segments and so $FP_1^{i,k}, FP_2^{i,k} \cdots, FP_{n_{i,k}}^{i,k}$ divide boundaries of A_k into line segments where maximizer $\ell \in \{1, 2, \ldots, N\}$ such that $d_A(p, Q_\ell) + k_\ell \geq d_A(p, Q_i) + k_i, \ell \neq i, i = 1, 2, \ldots, N$ from a point on each line segment is same. Further $d_A(p, Q_\ell)$ is determined some neighboring pair of A-directions, say, α_j, α_{j+1} for each line segment. That is, for $Q_\ell = (q_1^\ell, q_2^\ell)$ and $p = (x, y)$, $d_A(p, Q_\ell) = M_1|m_2(q_1^\ell - x) - (q_2^\ell - y)| + M_2|m_1(q_1^\ell - x) - (q_2^\ell - y)|$ where $m_1 = \max(\tan\alpha_j, \tan\alpha_{j+1}), m_2 = \min(\tan\alpha_j, \tan\alpha_{j+1}), M_1 = \frac{\sqrt{1+m_1^2}}{m_1 - m_2}, M_2 = \frac{\sqrt{1+m_2^2}}{m_1 - m_2}$ as is shown in Section 1.Then minimum of $d_A(p, Q_\ell) + k_i$ with respect to point on this line segment is attained at either end point of this line segment since $d_A(p, Q_\ell)$ includes two absolute values with linear functions of *x,y* inside and its minimum is attained at the points with coordinates making either absolute value zero (corresponding point is one of $FP_1^{i,k}, FP_2^{i,k} \cdots, FP_{n_{i,k}}^{i,k}, i = 1, 2, \ldots, N$) or the terminal points of corresponding line segment ($FP_{n_{i,k}+1}^{i,k}, FP_{n_{i,k}+2}^{i,k} \cdots, FP_{n_{i,k}+b_{i,k}}^{i,k}, i = 1, 2, \ldots, N$).

Q.E.D.

Since we solve all P^{t_K} and check non-domination of optimal solutions for P^{t_K} in order to solve this special case, we only show the solution procedure for P^{t_K}.

Solution Procedure for P^{t_k}

Step 1: Check whether $p^0 \in A_k$ or not. If $p^0 \notin A_k$, then go to Step 2. Otherwise, terminate with p^0 as an optimal solution of P^{t_k}.

Step 2: Based on Theorem 5, we obtain
$FP_1^{i,k}, FP_2^{i,k} \cdots, FP_{n_{i,k}}^{i,k}, FP_{n_{i,k}+1}^{i,k}, FP_{n_{i,k}+2}^{i,k} \cdots, FP_{n_{i,k}+b_{i,k}}^{i,k} i = 1, 2, \ldots, N$ and
seek the minimizer $FP(k)$ among
$FP_1^{i,k}, FP_2^{i,k} \cdots, FP_{n_{i,k}}^{i,k}, FP_{n_{i,k}+1}^{i,k}, FP_{n_{i,k}+2}^{i,k} \cdots, FP_{n_{i,k}+b_{i,k}}^{i,k} i = 1, 2, \ldots, N$
with respect to $\min[\min\{d_A(FP_j^{i,k}, Q_i)|j = 1, 2, \ldots, n_{i,k} + b_{i,k}\} + k_i|i = 1, 2, \ldots, N]$.
Terminate with $FP(k)$ as an optimal solution of P^{t_k}.

In this special case, non-dominated solutions are found in at most $O(\{a \cdot \max(n, m)^3 + EB\} \cdot q)$ computational time based on the result of Theorem 5 where EB is the maximum vertices number of polygon $A_k, k = 1, 2, \ldots q$ since of course, only nearest edge of boundary on $A_k \cap X_i$ should be considered as an intersection point between boundary and A-half lines from Q_i, that is, at most one edge of boundary for each A-half line from Q_i.

An example

We consider the following toy example in this case to illustrate the solution procedure. We set$w_1 = w_2, A = \{0°, 45°, 90°, 135°\}$ (see Figure 8). Data of this example is shown in Figure 9. X is a hexagon determined by vertices sited at (0,0), (60,0), (100,30), (60,60) and (0,60). There are two hospitals H_1, H_2 sited at (40,40) and (60,20) respectively and

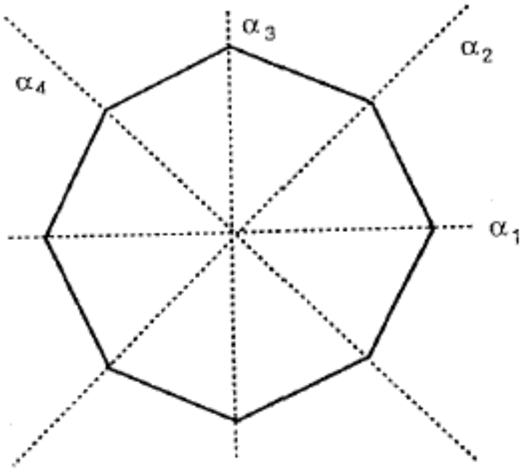

Figure 8: $A = \{0°, 45°, 90°, 135°\}$

two blocks $A_1 A_2$ with preferences 0.8 and 0.3 respectively. The line segment connecting with boundary points sited at (20, 0) and (71.43, 51.43) is a Voronoi edge. Applying our method, results are illustrated in Figure 10. Small bold A-circles are centered one at Q_i. Dotted A-circle centered at (37.72, 37.72) and covering all small A-circles corresponds to a solution minimizing weighted sum of distances. Big bold A-circle centered at (20, 20) denotes optimal covering circle where construction site is restricted toA_1. Therefore non-dominated sites are (37.72, 37.72) and (20,20) respectively. Their weighted sum of distance and preference are (109.92, 0.3) and (134.14, 0.8).

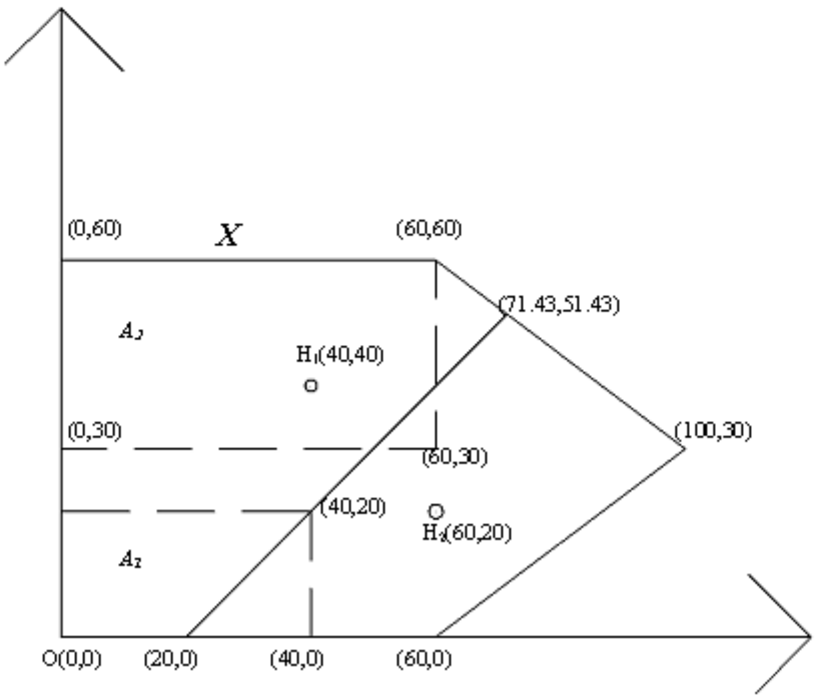

Figure 9: Data of the example

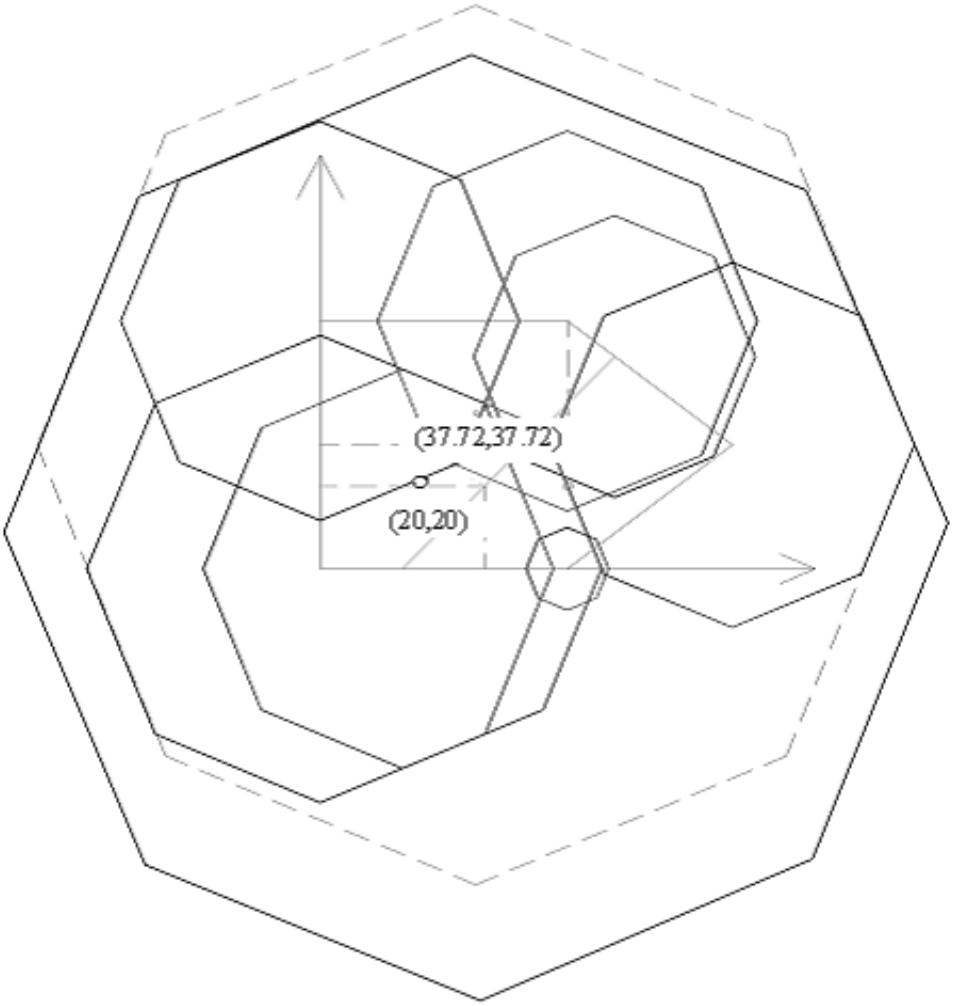

Figure 10: An illustration of the result applying our method to this toy example

4. Conclusion

We have proposed a model about ambulance facility location including the preference of the candidate sites, which is not explicitly considered so far in spite of importance from the actual point of view. Since we have used A-distance, solution procedure is complicated a little bit, we have endeavored to find non-dominated solutions in order to give information to decision makers. But as further research problems, followings are left.

(1) Our solution method for our model is straightforward and so refinement should be done in order to find non-dominated sites efficiently.

(2) Though our model includes both benefits of citizens and local government since it takes preference on the construction site into account, actual possibility of accident occurrence is not uniform with respect to sites in X.

(3) Preference function reflecting on actual conditions about construction of an ambulance station is not necessary simple as is considered in this paper. So more realistic assumptions about preference function should be considered.

References

[1] M.S. Danskin and L.K. Dean: Location of health care facilities. In M.L. Brandeau, F. Sainfort and W.P. Pierskalla (eds.): *Operations Research and Health Care: A Handbook of Methods and Applications* (Kluwer Academic Publishers, 2004), 43–76.

[2] J. Elzinga and D.W. Hearn: Geometrical solutions for some minmax location problems. *Transportation Science*, **6** (1972), 379–394.

[3] H. Ishii, Y.L. Lee and K.Y. Yeh: Fuzzy facility location problem with preference of candidate sites and A-distance. *Proceedings of 11th Asia Pacific Management Conference 1*, (2005), C-1-1C-4-6.

[4] H. Ishii, Y.L. Lee and K.Y. Yeh: Fuzzy facility location problem with preference of candidate sites. *Fuzzy Sets and Systems*, **158** (2007), 1922–1930.

[5] T. Matutomi and H. Ishii: Minimax location problem with A-distance. *Journal of the Operations Research Society of Japan*, **41** (1998), 181–195.

[6] A.J. Swersey: The development of police, fire and emergency median unit. In S.M. Pollock, M.H. Rothkopf and A. Barn (eds.): *Operations Research and the Public Sector: Handbooks in Operations Research and Management Science* (Elsevier, 1994).

[7] P. Widmayer, Y.F. Wu and C.K. Wong: On some distance problem with fixed orientations. *SIAM Journal on Computing*, **16** (1987), 728–746.

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