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A MULTIPLE CLASS SEAT ALLOCATION MODEL WITH REPLENISHMENT

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Abstract In this paper we consider an airline seat allocation model where multiple fare classes are available and replenishment can be allowed for the lower fare classes. Such a replenishment likely occurs whenever the higher fare classes do not have demands large enough compared with the number of the pre-allocated seats. Then, the lower fare classes should be reopened while the prices may be discounted during the periods left before the departure. If the demands for high fare classes are large enough, the airlines should not discount the prices and the replenishment may not be allowed from the point of view of revenue management. We also allow the customers to cancel their reservations. Our model formulation is much closer to the airlines practice. We show under some conditions that the expected revenue with replenishment is greater than the one without replenishment and that there exists a simple optimal booking policy. Some numerical examples are provided to confirm analytical properties of the revenue function as well as of the optimal policy.

Keywords: Inventory, airline revenue management, capacity allocation, dynamic programming

1. Introduction

Many industries dealing with perishable products have the problem of how to sell these products to maximize the total revenue. Such perishable products possess a property that the products are worthless if they can not be sold beyond the certain time horizon. Airplane seats, hotel rooms, and rental-car are examples of such products. In this paper, we deal with airplane seats. Airline tickets are sold at several different prices for the same type of seats by imposing various restrictions on the tickets. To increase the road factor, the airlines sell out the seats by deep discount when there are some vacant seats at the last minute of the departure. However, if the customers can not change their itinerary of destination very easily, then the airlines can not expect a large amount of demands even if the airlines provide customers deep discount fares.

The purpose of this paper is to analyze the seat inventory model of allowing the discount during the sales period, not as sales at the end of period. We call this "replenishment" in this paper. If the number of demands is less than the pre-allocated seats in a fare class, then the airlines discount some seats for the next period. On the other hand, if the number of demands is larger than the pre-allocated seats, then they will not replenish in the following period. Moreover, there are three types of customers. First type of customers will buy their tickets after making the reservation. Second types are of customers who do not eventually buy the tickets even though they made the reservations. Third type of customers wish to buy their tickets without reservations. In this paper we distinguish these types of customers by taking account of this fact.

There are many papers that studied the model of dealing with multiple fare classes.

Curry [4], Wollmer [13], Brumelle and McGill [1] analyze the single-leg model in which the booking limit for each fare classes increase monotonically from low to high as the time approaches to the flight departure. Robinson [9] generalizes Brumelle and McGill's optimality conditions to the case when fares are nonmonotonic. Our model is different from the existing model in a sense that we allow the airlines to reopen the reservation for the lower fare classes, depending on whether the amount of demands is larger than the pre-allocated seats or not. On the other hand, there exist dynamic capacity control models which allow passengers to arrive in any order. These articles are Subramanian et al. [11], Lee and Hersh [7], Liang [8], and Brumelle and Walczak [3]. Moreover, there are several papers that analyze the stochastic models with discrete price changes (see Feng and Gallego [5], Feng and Xiao [6]).

In section 2, we introduce notations and assumptions, and formulate a multiple class seat allocation model. In section 3, we derive an optimal protection level for each fare class, and discuss analytical properties by comparing the model with non-replenishment model. In section 4, we extend the replenishment model into the model that the number of confirmations depends on the number of reservation. Finally, we provide some numerical examples to confirm analytical properties of the optimal protection level as well as the expected revenue.

2. The Airline Seat Inventory Control with Replenishment

In this section we present the airline seat inventory model with replenishment. We assume that there are N fare classes and the reservation requests come in a way of a mixture. The confirmation of each class will take place at the time of receipt of full payment for the tickets within the target period for each class. Assume that we set target period to one fare class and that arrivals of lower-fare classes come first (see Figure 1). The price of tickets and target period are under the one-to-one relationship. Let i be an index of the period with the time index running backward ($i = 1$ is the last period, and $i = N - 1$ is the first period). Also, we ascribe the number of confirmation to the demands for the target period, and assume that the demands for different booking classes are independent of each other.

When the demand for certain class does not fill the allocated seats, the airlines replenishes the leftover seats with two fare class which are cheaper than the present seats. Note that the demands for replenishment period are independent of one for target period.

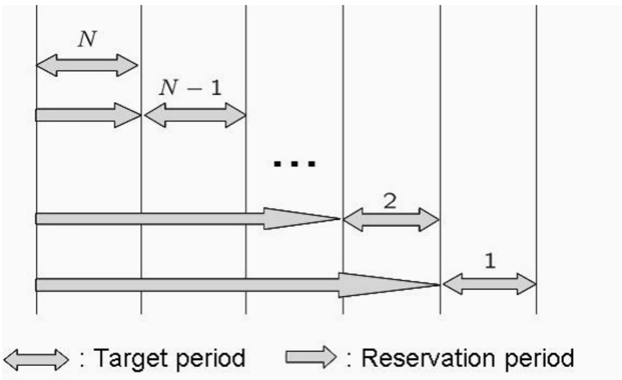

Figure 1: Target period and reservation period

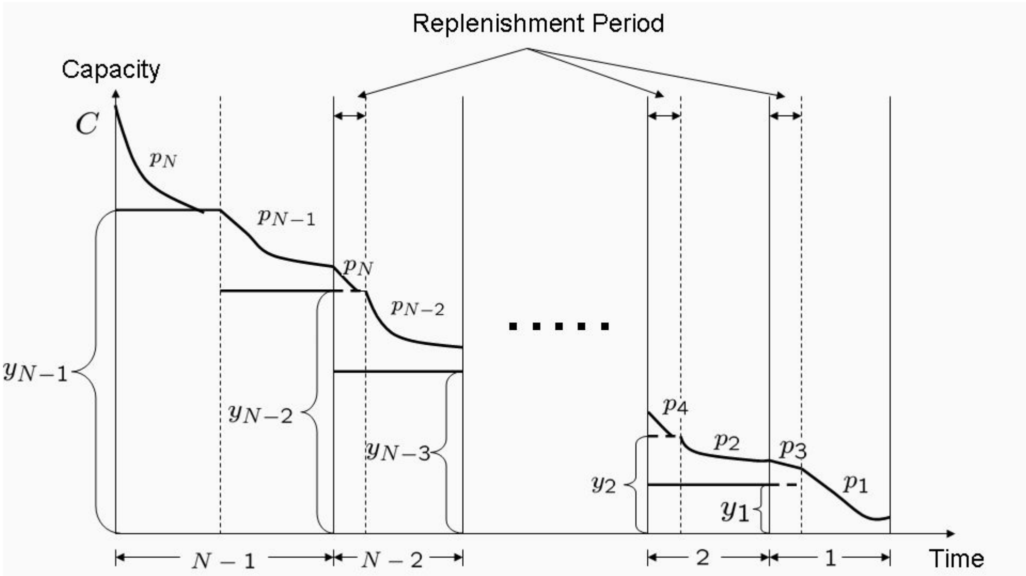

Figure 2: An inventory segmentation

We use the following notations;

- C : the capacity of the plane,
- N : the number of classes,
- p_i : the fare for class i ($p_N < \cdots < p_1$),
- x : the number of the remaining seats, (state variable),
- y_i : the nested protection level for class i after accepting the customers for class $i+2$, $y_{i-1} \le y_i \le x$, $i = 1, \cdots, N-1$, $y_0 \equiv 0$, (decision variables),
- D_i : the random demand of class i in period i,
- $D_i^{'}$: the random demand of class $i+2$ in period i.

In period i, a sequence of operations is as follows:

(i) At the beginning of period, if the remaining capacity x is larger than the protection level y_i for class i, then the class $i+2$ should be reopened and the airlines accepts the replenishment demands for class $i+2$ up to $x - y_i$. Note that the passengers who are accepted in replenishment period have to pay the fare at the time of booking. On the other hand, if the remaining capacity x is less than or equal to the protection level y_i, then the class $i+2$ should not be opened in this period.

(ii) The demand D_i occurs in target period, and the airlines accepts bookings up to the booking limit that is equal to the capacity remaining minus the protection level for class $i-1$.

(iii) We follow the same procedure for period $i-1$ as for period i. And the remaining seats at period $i-1$ is given by

$$x - \min\{D_i^{'}, x - y_i\} - \min\{D_i, (x - y_{i-1} - (x - y_i) \wedge D_i^{'})^+\} \tag{2.1}$$

where $a \wedge b := \min(a, b)$ and $a^+ := \max(a, 0)$.

The value of $\min\{D_i^{'}, x - y_i\}$ is the sales when the airlines replenishes the demands for class $i+2$. The value of $\min\{D_i, (x - y_{i-1} - (x - y_i) \wedge D_i^{'})^+\}$ is the number of passengers for class i, which can be rewritten as follows;

$$\min\{D_i, (x - y_{i-1} - (x - y_i) \wedge D_i^{'})^+\} = \min\{D_i, y_i - y_{i-1} + (x - y_i - D_i^{'})^+\}.$$

For $i = N-1$, the airlines has no occasion to replenish the lower-fare class because there is no fare class $N+1$. Hence, we assume that the demands for class N come from the replenishment period in $i = N-1$. Thus, we handle the following substitution;

$$\begin{aligned} D'_{N-1} &= D_N, \\ p_{N+1} &= p_N. \end{aligned}$$

Figure 2 illustrates a sequence of operations for $i = 1, \cdots, N-1$. Let $g_i(x, D_i, D'_i, y_i)$ be the remaining seats at the end of period i, that is,

$$g_i(x, D_i, D'_i, y_i) = x - \min\{D'_i, x - y_i\} - \min\{D_i, y_i - y_{i-1} + (x - y_i - D'_i)^+\}.$$

Moreover, let $v_i(x, y_i)$ be the expected revenue starting in x seats remaining at the beginning of period i with a protection level y_i over the truncated horizon $i, \cdots, N-1$. Then, the total expected revenue for $i = i, \cdots, N-2$ is given by

$$\begin{aligned} v_i(x, y_i) = E[p_{i+2} \min\{D'_i, x - y_i\} + p_i \min\{D_i, y_i - y^*_{i-1} + (x - y_i - D'_i)^+\} \\ + V_{i-1}(g_i(x, D_i, D'_i, y_i))],\ i = 1, \cdots, N-2 \quad (2.2) \end{aligned}$$

and the expected revenue in period $N-1$ is given by

$$\begin{aligned} v_{N-1}(C, y_{N-1}) = E[p_N \min\{D_N, C - y_{N-1}\} \\ + p_{N-1} \min\{D_{N-1}, y_{N-1} - y^*_{N-2} + (C - y_{N-1} - D_N)^+\} \\ + V_{N-2}(g_{N-1}(C, D_{N-1}, D_N, y_{N-1}))], \quad (2.3) \end{aligned}$$

where y^*_i is the optimal protection level for class i and $V_{i-1}(\cdot)$ is the optimal expected revenue for class $i-1$.

The optimal expected revenue is given by

$$V_i(x) = \max_{0 \le y_i \le x} v_i(x, y_i), \quad (2.4)$$

with the boundary conditions

$$V_0(x) = 0,\ V_i(0) = 0 \quad \forall x \text{ and } i = 1, \cdots, N-1.$$

3. Optimal Booking Policy and Some Properties

In this section, we derive an optimal booking policy and investigate some properties of the model. To simplify the analysis, we consider the case that the capacity is continuous. Although the number of seats is discrete, the continuous model can be an approximation to the discrete one, provided that the number of seats is large enough.

Next, we investigate some analytical properties of $v_i(\cdot, \cdot)$ and $V_i(\cdot)$ to identify the optimal booking policy.

Lemma 3.1. (a) $v_i(x, y_i)$ is quasi-concave in y_i for each x.
(b) $V_i(x)$ is non-decreasing and concave in x for each i.

Proof. We prove (a) and (b) simultaneously, by induction. Letting $i = 1$, the expected revenue is given by

$$v_1(x, y_1) = E[p_3 \min\{D'_1, x - y_1\} + p_1 \min\{D_1, y_1 + (x - y_1 - D'_1)^+\}]. \quad (3.1)$$

Then we have

$$\frac{\partial v_1(x,y_1)}{\partial y_1} = [p_1\overline{F}_1(y_1) - p_3]\overline{G}_1(x-y_1), \tag{3.2}$$

where F_i and G_i are the probability distribution of random variable D_i and $D_i^{'}$, respectively. Since the term in brackets in equation (3.2) is non-increasing in y_1 and $\overline{G}_1(x-y_1)$ is non-negative, it follows that $v_1(x,y_1)$ is quasi-concave in y_1. Therefore, the optimal protection level for class 1 is given by

$$y_1^* = \overline{F}_1^{-1}\left(\frac{p_3}{p_1}\right). \tag{3.3}$$

We take first and twice-derivative in x to see the concavity of $V_1(x)$,

$$\begin{aligned} \frac{dV_1(x)}{dx} &= p_3\overline{G}_1(x-y_1^*) + p_1\int_0^{x-y_1^*}\overline{F}_1(x-k)dG_1(k) > 0, &(3.4)\\ \frac{d^2V_1(x)}{dx^2} &= -[p_3 - p_1\overline{F}_1(y_1^*)]g_1(x-y_1^*) - p_1\int_0^{x-y_1^*} f_1(x-k)dG_1(k) \\ &= -p_1\int_0^{x-y_1^*} f_1(x-k)dG_1(k) < 0. &(3.5)\end{aligned}$$

The last equality of equation (3.5) follows equation (3.3). Hence, $V_1(x)$ is non-decreasing and concave in x. Suppose that (a) and (b) hold for $i \geq 2$. Differentiating $v_i(x,y_i)$ with respect to y_i, then we have

$$\begin{aligned} \frac{\partial v_i(x,y_i)}{\partial y_i} &= \left[-p_{i+2} + p_i\overline{F}_i(y_i - y_{i-1}^*) + \int_0^{y_i-y_{i-1}^*}\frac{d}{dy_i}V_{i-1}(y_i-j)dF_i(j)\right]\overline{G}_i(x-y_i) \\ &= \left[p_i - p_{i+2} - \int_0^{y_i-y_{i-1}^*}(p_i - \frac{d}{dy_i}V_{i-1}(y_i-j))dF_i(j)\right]\overline{G}_i(x-y_i). &(3.6)\end{aligned}$$

When $i = N-1$, we substitute $\overline{G}_{N-1}(x-y_{N-1})$ for $\overline{F}_{N-1}(C-y_{N-1})$.

Let $\phi(y_i)$ be the term in brackets in (3.6) and we differentiate $\phi(y_i)$ with respect to y_i, then we obtain,

$$\frac{d\phi(y_i)}{dy_i} = -(p_i - p_{i+1})f_i(y_i - y_{i-1}^*) + \int_0^{y_i-y_{i-1}^*}\frac{d^2}{dy_i^2}V_{i-1}(y_i-j)dF_i(j). \tag{3.7}$$

By the inductive assumption, $\frac{d^2}{dy_i^2}V_{i-1}(y_i-j) < 0$. Since $\frac{d\phi(y_i)}{dy_i}$ is non-positive, $\phi(y_i)$ is non-increasing in y_i. Hence, $v_i(x,y_i)$ is quasi-concave in y_i.

On the other hand, we obtain the first and second derivatives of $V_i(x)$ as follows;

$$\begin{aligned} \frac{dV_i(x)}{dx} &= p_{i+2}\overline{G}_i(x-y_i^*) + p_i\int_0^{x-y_i^*}\overline{F}_i(x-y_{i-1}^*-k)dG_i(k) \\ &\quad + \int_0^{x-y_i^*}\int_0^{x-y_{i-1}^*-k}\frac{d}{dx}V_{i-1}(x-k-j)dF_i(j)dG_i(k) > 0, &(3.8)\\ \frac{d^2V_i(x)}{dx^2} &= -p_i\int_0^{x-y_i^*} f_i(x-y_{i-1}^*-k)dG_i(k) \\ &\quad + \int_0^{x-y_i^*}\int_0^{x-y_{i-1}^*-k}\frac{d^2}{dx^2}V_{i-1}(x-k-j)dF_i(j)dG_i(k) < 0. &(3.9)\end{aligned}$$

Hence, $V_i(x)$ is non-decreasing and concave in x. □

Theorem 3.1. The optimal protection level y_i^*, $i = 1, \cdots, N-1$, is given by

$$y_i^* = \begin{cases} 0 & \text{if } \frac{\partial v_i(x,0)}{\partial y} < 0, \\ \max\left\{ y : H_i\left(p_i - \frac{d}{dy}V_{i-1}(y-j)\right) < p_i - p_{i+2} \right\}, & \text{if } \frac{\partial v_i(x,C)}{\partial y} < 0 < \frac{\partial v_i(x,0)}{\partial y}, \\ C & \text{if } 0 < \frac{\partial v_i(x,C)}{\partial y}, \end{cases} \quad (3.10)$$

where

$$H_i(u_j) = \int_0^{y_i - y_{i-1}^*} u_j dF_i(j). \quad (3.11)$$

Proof. By Lemma 1 (a), $v_1(x, y_1)$ is quasi-concave and an optimal booking policy given by (3.3) is optimal for period 1. By Lemma 1 (b), $V_1(x)$ is concave. Thus, the same argument as well can be applied backward through the periods in the sequence $i = 1, \cdots, N-1$. □

Corollary 3.1. If the replenishment demand for $i = 1, \cdots, N-1$ is zero, then equation (2.2) is simplified as follows;

$$\overline{v}_i(x, y_{i-1}) = E[p_i \min\{D_i, x - y_{i-1}\} + \overline{V}_{i-1}(x - \min\{D_i, x - y_{i-1}\})]. \quad (3.12)$$

The optimal protection level is given by

$$\overline{y}_{i-1} \equiv \max\left\{ y : p_i < \frac{d}{dy}\overline{V}_{i-1}(y) \right\}. \quad (3.13)$$

The proof has been carried out by Curry [4]. We call this model a non-replenishment model.

Proposition 3.1. If $\frac{dV_i(x)}{dx} \geq \frac{d\overline{V}_i(x)}{dx}$ for all x, then the optimal protection level for the replenishment model is greater than or equal to the one for non-replenishment model, that is, $y_i^* \geq \overline{y}_i$, $i = 1, \cdots, N-1$.

Proof. In order to compare the optimal protection level for equation (3.10) with the one for (3.13), we compute the first derivative of $\overline{v}_{i+1}(x, y_i)$ with respect to y_i as follows;

$$\frac{\partial \overline{v}_{i+1}(x, y_i)}{\partial y_i} = \left[-p_{i+1} + \frac{d}{dy_i}\overline{V}_i(y_i) \right] \overline{F}_i(x - y_i). \quad (3.14)$$

Letting $\overline{\phi}_i(y_i)$ be the term in bracket in equation (3.14), for $i = 1$ we have

$$\phi_1(y_1) - \overline{\phi}_1(y_1) = p_2 - p_3 > 0. \quad (3.15)$$

Since the ϕ_1 and $\overline{\phi}_1$ are non-increasing function for y_1, the relationship between the optimal protection levels for replenishment and non-replenishment is given by $y_1^* \geq \overline{y}_1$. To apply induction, we assume that proposition 3.1 holds for fare class $1, \cdots, i-1$. Then, we have

$$\phi_i(y_i) - \overline{\phi}_i(y_i) = p_i - p_{i+2} - \int_0^{y_i - y_{i-1}^*} \left(p_i - \frac{d}{dy_i}V_{i-1}(y_i - j) \right) dF_i(j) + p_{i+1} - \frac{d}{dy_i}\overline{V}_i(y_i), \quad (3.16)$$

where

$$\frac{d}{dy_i}\overline{V}_i(y_i) = p_i - \int_0^{y_i-\overline{y}_{i-1}} \left(p_i - \frac{d}{dy_i}\overline{V}_{i-1}(y_i - j)\right) dF_i(j).$$

Therefore, equation (3.16) can be rewritten as follows;

$$\phi_i(y_i) - \overline{\phi}_i(y_i) = p_{i+1} - p_{i+2} + \int_0^{y_i-y^*_{i-1}} \left(\frac{d}{dy_i}V_{i-1}(y_i - j) - \frac{d}{dy_i}\overline{V}_{i-1}(y_i - j)\right) dF_i(j)$$
$$+ \int_{y_i-y^*_{i-1}}^{y_i-\overline{y}_{i-1}} \left(p_i - \frac{d}{dy_i}\overline{V}_{i-1}(y_i - j)\right) dF_i(j) \geq 0. \quad (3.17)$$

From the assumption, the second term in integral of (3.17) is positive. And third term in integral is also positive within $[y_i - y^*_{i-1},\, y_i - \overline{y}_{i-1}]$. Hence, we obtain $y^*_i \geq \overline{y_i}$. □

The following Corollary represents that the expected revenue of the replenishment model is greater than one of the non-replenishment model when the demands for normal period is smaller than the allocated seats.

Corollary 3.2. If $D_i < y^*_i - y^*_{i-1} + (x - y^*_i - D'_i)^+$, then $V_i(x) > \overline{V}_i(\overline{x})$ for $i = 1, \cdots, N-1$.

Proof. We follow an induction argument. For $i = 1$, we have

$$\begin{aligned} &V_1(x) - \overline{V}_1(\overline{x}) \\ &\quad = E[v_1(x, y^*_1) - \overline{v}_1(\overline{x}, \overline{y}_1) \mid D_1 < y^*_1 + (x - y^*_1 - D'_1)^+] \\ &\quad = E[p_3 \min\{D'_1, x - y_1\} + p_1(D_1 - \min\{D_1, \overline{x}\}) \mid D_1 < y^*_1 + (x - y^*_1 - D'_1)^+] > 0. \end{aligned} \quad (3.18)$$

Suppose that Corollary 3.2 holds for fare class $1, \cdots, i-1$, and then show that it holds for i. Then, we obtain

$$\begin{aligned} &V_i(x) - \overline{V}_i(\overline{x}) \\ &\quad = E[v_i(x, y^*_i) - \overline{v}_i(\overline{x}, \overline{y}_{i-1}) \mid D_i < y^*_i - y^*_{i-1} + (x - y^*_i - D'_i)^+] \\ &\quad = E[p_{i+2} \min\{D'_i, x - y^*_i\} + p_i(D_i - \min\{D_i, \overline{x} - \overline{y}_{i-1}\}) + V_{i-1}(g_i(x, D_i, D'_i, y_i)) \\ &\quad - \overline{V}_{i-1}(x - \min\{D_i, x - \overline{y}_{i-1}\}) \mid D_i < y^*_i - y^*_{i-1} + (x - y^*_i - D'_i)^+] > 0. \end{aligned} \quad (3.19)$$

Hence, $V_i(x) > \overline{V}_i(\overline{x})$. □

The next proposition shows that the more demands for replenishment period increases, the more optimal expected revenue increases.

Proposition 3.2. If $\tilde{D}'_i$ dominates D'_i for $i = 1, \cdots, N-1$, then we obtain $V_i(x, \tilde{D}'_i) \geq V_i(x, D'_i)$.

Proof. From equation (3.8), $\min\{D'_i, x - y^*_i\}$, $\min\{D_i, y^*_i - y^*_{i-1} + (x - y^*_i - D'_i)^+\}$ and $V_{i-1}(g_i(x, D_i, D'_i, y^*_i))$ are non-decreasing in x. Hence, it can be shown from first-oder stochastic dominance that

$$V_i(x, \tilde{D}'_i) \geq V_i(x, D'_i). \quad (3.20)$$

□

4. The Model with Demands Depending on the Number of Reservation

In this section we analyze the model that there is a dependency between the number of reservation requests and the number of confirmation. In the proceeding section, we assumed that the airlines accepts the replenishment demands as long as the remaining seats are larger than the protection level of the next higher class. However, if the number of reservation requests is large enough, then it may not be profitable for the airlines to accept the replenishment demands. Therefore, we extend the previous model into the model that the sales volume for the replenishment period depend on the number of the reservation requests.

Let $\overline{D}_i$ be the total number of reservation requests for class i up to the starting time of target period i. Suppose that the reservation request is accepted up to the allocated seats for class i, $y_i - y_{i-1}$. Also, we assume that $\overline{D}_i$ and $D_i^{'}$ are independent to each other, but it is not assumed that $\overline{D}_i$ and D_i are independent. We consider the case that $\overline{D}_i$ is greater than the allocated seats for class i, $y_i - y_{i-1}$. Because, if $\overline{D}_i$ is less than $y_i - y_{i-1}$, then the airlines does not lose the revenue from accepting the replenishment demands. The conditional probability distribution function of D_i given that $\overline{D}_i \geq y_i - y_{i-1}$ is as follows;

$$F_{D_i} * F_{\overline{D}_i}(a, \overline{a}) = \Pr(D_i \leq a \mid \overline{D}_i \geq \overline{a}). \tag{4.1}$$

Let ψ_i be the term in brackets in (2.2), and the expected revenue for class i is given by

$$\tilde{v}_i(x, y_i) = E[\psi_i(y_i) \mid \overline{D}_i \geq y_i - y_{i-1}^*] \tag{4.2}$$

and the optimal expected revenue is given by

$$\tilde{V}_i(x) = \max_{0 \leq y_i \leq x} \tilde{v}_i(x, y_i). \tag{4.3}$$

The part of condition in $F_{D_i} * F_{\overline{D}_i}$ depends on the protection level of higher class y_i. Thus, when we differentiate $\tilde{v}_i(x, y_i)$ with respect to y_i to find the optimal protection level, it turns out to be very complex expressions. Now, we treat with demand as discrete, and consider the situation that the airlines has already had $y_i - y_{i-1} - 1$ requests for class i, and the next passenger request arrives. If the passenger pays the fare in target period, then the expected revenue is $\tilde{v}_i(x, y_i)$. Otherwise, the expected revenue is $\tilde{v}_i(x, y_i - 1)$. Since the number of request $\overline{D}_i$ is greater than of equal to $y_i - y_{i-1}^*$ for each case, we will be able to drop the derivative of the part of $\overline{D}_i \geq y_i - y_{i-1}^*$ in future calculations. To hold the property of Lemma 1, we set the following assumption.

Assumption 4.1. $\Pr(D_i \geq y_i - y_{i-1}^* \mid \overline{D}_i \geq y_i - y_{i-1}^*)$ is non-increasing in y_i.

This assumption is similar in intent to "monotonic association property" in Brumelle et al. [2]. From this assumption, the main change from section 2 is only that F_i is replaced by $F_{D_i} * F_{\overline{D}_i}$. By the same argument as section 2, we obtain the optimal protection level for class i as follows;

$$\tilde{y}_i = \begin{cases} 0 & \text{if } \frac{\partial v_i(x,0)}{\partial y_i} < 0, \\ \max\{y_i : \tilde{H}_{y_i - y_{i-1}^*, y_i - y_{i-1}^*}(p_i - \frac{d}{dy_i}\tilde{V}_{i-1}(y_i - a_i)) < p_i - p_{i+2}\}, & \\ & \text{if } \frac{\partial v_i(x,C)}{\partial y_i} < 0 < \frac{\partial v_i(x,0)}{\partial y_i}, \\ C & \text{if } 0 < \frac{\partial v_i(x,C)}{\partial y_i}, \end{cases} \tag{4.4}$$

where

$$\tilde{H}_{s,t}(u_{a_i}) = \frac{1}{\overline{F}_{\overline{D}_i}(t)} \int_t^{\infty} \int_0^s u_{a_i} f_{D_i,\overline{D}_i}(a_i, \overline{a}_i) da_i d\overline{a}_i. \tag{4.5}$$

Corollary 4.1. For $N = 2$, the expected revenue is

$$\tilde{v}_1(C, y_1) = E[p_2 \min\{D_1^{'}, C - y_1\} + p_1 \min\{D_1, y_1 + (C - y_1 - D_1^{'})^+\} \mid \overline{D}_1 \geq y_1]. \tag{4.6}$$

Then, the optimal protection level for class 1 is given by

$$\tilde{y}_1 \equiv \max\left\{y : \Pr(D_1 \geq y \mid \overline{D}_1 \geq y) < \frac{p_2}{p_1}\right\}. \tag{4.7}$$

5. Numerical Examples

In this section we present numerical examples to explain the result obtained in sections 3 and 4. First, we compare the models with and without replenishment. Second, we make a comparison between the models with the demands depending on either the number of reservations or the number of confirmations.

5.1. The result of section 3

We assume that there are four fare classes, and that D_i and $D_i^{'}$ are independently normally distributed with means μ_i, $\mu_i^{'}$, and variances σ_i^2, $\sigma_i^{'2}$, (here after abbreviated by $N(\mu, \sigma)$) respectively. Table 1 shows the specific values of the parameters and Table 2 presents the demand data. Figure 3 draws the optimal protection levels for the replenishment model and non-replenishment model, and Figure 4 shows the improvement rate of maximum expected revenue. The horizontal axis presents the demand rate which comes from demand parameters for D_i in Table 2. For example, if the demand rate is 1.5, then D_i follows $N(1.5\mu_i, 1.5\sigma_i)$ for each i. We can see that the value of y_i^* is greater than $\overline{y}_i$ for each i. Also, the replenishment model shows better performances than the non-replenishment model as long as the demand is low.

Next, we investigate the influence of demand change in replenishment period. In Figure 5, we study how the maximum expected revenue changes as the demand rate increases.

5.2. The result of section 4

Suppose that D_i and $\overline{D}_i$ are bivariately normally distributed with means $\mu_i, \overline{\mu}_i$, variances σ_i^2, $\overline{\sigma}_i^2$, and correlation ρ_i. In this numerical example, we also use the data of Table 1 and 2. Note that ρ_i are equal to 0.9 for each i. Figure 6 shows a comparison between the independent and dependent models. Note that we assume that the demand rate is rate of demand parameters for D_i and $\overline{D}_i$. For each class, the value of $\tilde{y}_i$ is greater than y_i^*.

In the dependent model, we know that there are enough number of reservations. Thus, the more reservations are available, the more seats are allocated to prevent the bargain sale from replenishment as much as possible. When demand rate is larger than 1.4, the protection level for class 2 increases suddenly because there are enough demands for two high fare classes to fill the number of total seats.

Table 1: The data for numerical example

C	N	p_1	p_2	p_3	p_4
200	4	950	450	300	230

Table 2: Demand parameters

Demand/Class	i	1	2	3	4
Normal	$N(\mu_i, \sigma_i)$	N(17.3,6.2)	N(35.1,12.0)	N(48.6,18.5)	N(65.2,20.3)
Replenishment	$N(\mu_i', \sigma_i')$	N(8.2,3.0)	N(10.5,5.0)	—	—
Reservation	$N(\overline{\mu}_i, \overline{\sigma}_i)$	N(20.0,5.5)	N(40.2,12.5)	N(55.5,20.3)	—

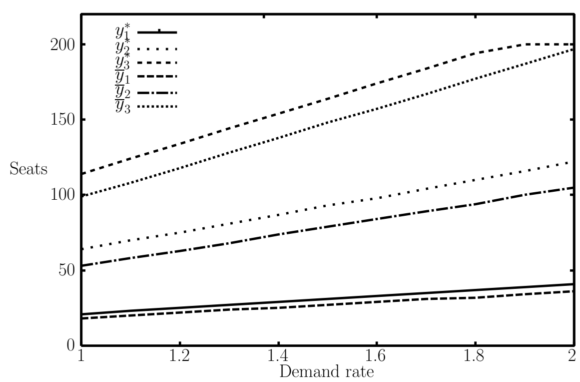

Figure 3: Comparison of y_i^* and $\overline{y}_i$ with respect to demand rate

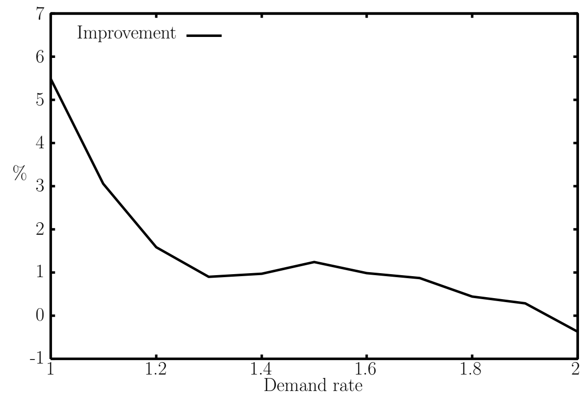

Figure 4: Improvement rate of $V(y_i^*)$ for $V(\overline{y}_i)$

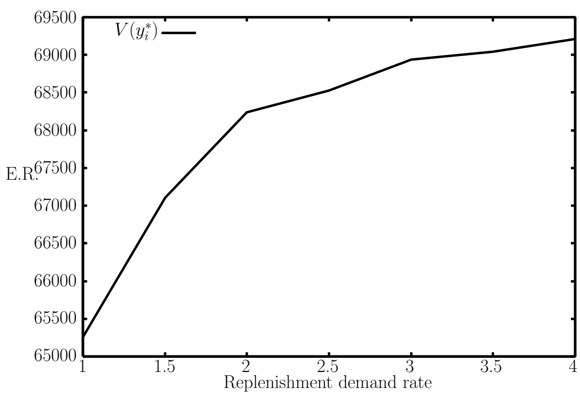

Figure 5: The value of $V(y_i^*)$ with respect to demand rate for replenishment period

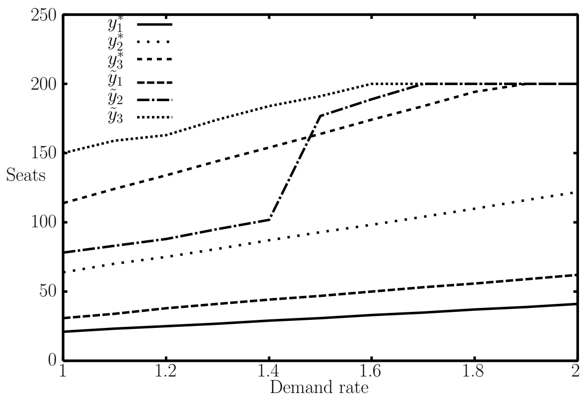

Figure 6: Comparison of y_i^* and $\tilde{y}_i$ with respect to demand rate

6. Conclusion

In this paper, we have proposed a seat inventory model where multiple fare classes are available and replenishment can be allowed for the lower fare classes. We also have derived an optimal booking policy, and showed the effectiveness of the expected total revenue in our model by comparing our model with the existing models. Extending our model could be fruitful directions for future research. First, it is of interest to consider the simultaneous determination of the number of seats and selling price for replenishment seats based on the number of the remaining seats. Second, for practical purposes the continuous seat model for the first or business classes should be considered. Third, the distribution of high fare demands should depend on the demand of lower fare class. These extensions would lead to additional insights and closer agreement with airlines practice.

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