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SUBEXPONENTIAL ASYMPTOTICS OF THE BMAP/GI/1 QUEUE

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Abstract This paper considers the stationary queue length and waiting time distributions in a FIFO BMAP/GI/1 queue with heavy-tailed service times and that with heavy-tailed batch sizes. In each case, we provide sufficient conditions under which the stationary queue length and waiting time distributions are subexponential. Furthermore, we obtain asymptotic relationships between the tail distributions of the stationary queue length and waiting time.

Keywords: Queue, subexponential asymptotics, independent sampling, BMAP (Batch Markovian Arrival Process), square-root insensitive, random sum

1. Introduction

The subexponential asymptotics of the waiting time distribution has been studied extensively in queues with heavy-tailed service times (or heavy-tailed equilibrium service times). Pakes derived the subexponential asymptotic formula for the actual waiting time distribution in the stationary GI/GI/1 queue [18]. See [5] also. Asmussen et al. extended the result in [18] to the MMPP/G/1 queue with state-dependent services [2]. Further Jelenković and Lazar obtained the subexponential asymptotic formula for the waiting time distribution in the Markov-modulated G/G/1 queue [10]. Takine studied the subexponential waiting time distribution in single-server queues with multiple Markovian arrival streams [23]. Contrarily, there are few studies on the heavy-tailed or subexponential asymptotics of the queue length distribution. Asmussen et al. studied the tail asymptotics of the queue length distribution in the GI/GI/1 queue, assuming that equilibrium service times are subexponential [3].

This paper considers the subexponential asymptotics of the queue length and waiting time distributions in stationary FIFO BMAP/GI/1 queues, where BMAP stands for batch Markovian arrival process [14]. To the best of our knowledge, the subexponential asymptotics in queues with batch arrivals has never been studied so far. In batch-arrival queues, the heavy-tailed asymptotics can emerge from heavy-tailed batch sizes, as well as heavy-tailed service times. Therefore we consider both cases: the BMAP/GI/1 queue with heavy-tailed service times and that with heavy-tailed batch sizes. The latter naturally arises when the heavy-tailed workload brought by arrivals is divided into small units of service, e.g., data transfer in IP networks.

After some preliminaries in section 2, we first study the queue length asymptotics when service times are heavy-tailed in section 3.1. Asmussen et al. derived the subexponential asymptotic formula for the queue length in the stationary GI/GI/1 queue [3], using the distributional form of Little's law (DLL) [9]. However, this approach is not readily applicable to the BMAP/GI/1 queue, because the conventional DLL does not hold even for queues with simple non-renewal arrivals such as the MAP/GI/1 queue (cf. [22, Theorem 2]). In addition, batch arrivals make our problem more complicated. Because the stationary queue length distribution in the BMAP/GI/1 queue is identical to the steady state solution of a certain Markov chain of M/G/1 type [21], we shall start with it. The subexponential asymptotics in structured Markov chains (including M/G/1 type) was studied in [4, 5, 10, 25]. However, those results have never been applied to queues so far because the relationship between heavy-tailed service times in queues and heavy-tailed increments in the corresponding structured Markov chains was not clear. Recently Jelenković et al. [11] provided some useful results, which enable us to characterize a heavy-tailed random sum of moderate- or light-tailed random variables (r.v.s). We slightly extend those results and examine the relationship between heavy-tailed service times in BMAP/GI/1 queues and heavy-tailed increments in the corresponding Markov chains of M/G/1 type. Furthermore, combining those with a recent study on Markov chains of M/G/1 type in [25], we establish a sufficient condition for the subexponential asymptotics of the stationary queue length distribution in the BMAP/GI/1 queue with heavy-tailed service times. As far as we know, this is the first result on the subexponential asymptotic tail of the stationary queue length distribution in queues with non-renewal arrivals.

We then consider the queue length asymptotics when batch sizes are heavy-tailed in section 3.2. Contrary to the case of heavy-tailed service times, heavy-tailed increments in the corresponding Markov chain of M/G/1 type can be characterized through a certain light-tailed random sum of heavy-tailed r.v.s. Thus we use some analytical tools in [10] and establish a sufficient condition under which the stationary queue length distribution is subexponential.

Next we study the waiting time asymptotics in section 4. When service times are heavy-tailed, a subexponential asymptotic formula for the waiting time distribution can be readily obtained from the existing results; what we have to do is to evaluate a certain light-tailed random sum of heavy-tailed r.v.s, and this can be done with the results in [10]. When batch sizes are heavy-tailed, however, the problem turns out to be a little more complicated, because the analysis involves the characterization of a certain heavy-tailed random sum of moderate- or light-tailed r.v.s. We utilize the result in [11] in a tactful manner for evaluating the random sum and derive a sufficient condition under which the waiting time distribution is subexponential. As a by-product, we also obtain asymptotic relationships between the tail distributions of the stationary queue length and waiting time in the BMAP/GI/1 queue with heavy-tailed service times and that with heavy-tailed batch sizes.

Throughout this paper, we use the following conventions. The (i,j)th element of any matrix $\boldsymbol{X}$ is denoted by $(\boldsymbol{X})_{i,j}$. For any nonnegative r.v. F, let $F(x) = \Pr[F \le x]$ and $\overline{F}(x) = 1 - F(x)$ for $x \ge 0$. We denote the nth-fold convolution of $F(x)$ with itself by $F^{(n)}(x)$ $(n = 1, 2, \dots)$. Thus $F^{(1)}(x) = F(x)$ $(x \ge 0)$ and for $n = 2, 3, \dots$, $F^{(n)}(x) = \int_0^x F^{(n-1)}(x-y)\mathrm{d}F(y)$ $(x \ge 0)$. Furthermore, for any nonnegative real-valued (resp. integer-valued) r.v. X (resp. Y) with positive finite mean, we denote its equilibrium (resp. discrete equilibrium) r.v. by X_{e} (resp. Y_{e}). Thus $\Pr[X_{\mathrm{e}} \le x] = \int_0^x \Pr[X > y]/\mathrm{E}[X]\mathrm{d}y$ for $x \ge 0$ (resp. $\Pr[Y_{\mathrm{e}} = k] = \Pr[Y > k]/\mathrm{E}[Y]$ for $k = 0, 1, \dots$). For convenience, we define $X_{\mathrm{e}} = 0$ (resp. $Y_{\mathrm{e}} = 0$) w.p.1 if $\mathrm{E}[X] = 0$ (resp. $\mathrm{E}[Y] = 0$). Finally, for any real-valued matrix function $\boldsymbol{R}(x)$ and any positive (scalar) function $g(x)$ $(x \ge 0)$, we write $\boldsymbol{R}(x) \overset{x}{\sim} \widetilde{\boldsymbol{R}} \cdot g(x)$ when $\lim_{x\to\infty} \boldsymbol{R}(x)/g(x) = \widetilde{\boldsymbol{R}}$ for some finite $\widetilde{\boldsymbol{R}}$ (which may have zero elements). Note that $\boldsymbol{R}(x)$ can be a scalar or vector function. In addition, for any real-valued function $f(x)$ and any positive function $g(x)$, we write $f(x) = o(g(x))$ to represent $\lim_{x\to\infty} f(x)/g(x) = 0$.

2. Model and Preliminaries

2.1. Model description

We consider a FIFO single-server queue with a buffer of infinite capacity, which is fed by a batch Markovian arrival process (BMAP) [14]. BMAP is driven by a continuous-time, time-homogeneous Markov chain $\{S(t);\ t \ge 0\}$ with finite state space $\mathcal{M} = \{1, 2, \dots, M\}$, which is called the underlying Markov chain hereafter. We assume that the underlying Markov chain $\{S(t);\ t \ge 0\}$ is irreducible.

The underlying Markov chain stays in state i ($i \in \mathcal{M}$) for an exponential interval of time with mean $\mu_i^{-1} > 0$ and then changes its state to state j ($j \in \mathcal{M}$) with probability $p_{i,j}$, where $\sum_{j \in \mathcal{M}} p_{i,j} = 1$ for all $i \in \mathcal{M}$. Given a state transition from state i to state j, k customers arrive in batch with probability $\zeta_{k,i,j}$, where $\sum_{k=0}^{\infty} \zeta_{k,i,j} = 1$ for all $i, j \in \mathcal{M}$. Without loss of generality, we assume [14] that $\zeta_{0,i,i} = 0$ for all $i \in \mathcal{M}$ and

$$\zeta_{0,i,j} = 1 \text{ if } p_{i,j} = 0. \tag{2.1}$$

For convenience, let $\boldsymbol{C}$ denote an $M \times M$ matrix such that $(\boldsymbol{C})_{i,j} = -\mu_i$ if $i = j \in \mathcal{M}$, and otherwise $(\boldsymbol{C})_{i,j} = \mu_i p_{i,j} \zeta_{0,i,j}$ ($i, j \in \mathcal{M}$). Also, let $\boldsymbol{D}_k$ ($k = 1, 2, \dots$) denote an $M \times M$ matrix such that $(\boldsymbol{D}_k)_{i,j} = \mu_i p_{i,j} \zeta_{k,i,j}$ ($i, j \in \mathcal{M}$). BMAP is then characterized by the set of $M \times M$ matrices $(\boldsymbol{C}, \boldsymbol{D}_1, \boldsymbol{D}_2, \dots)$. Let $\boldsymbol{D} = \sum_{k=1}^{\infty} \boldsymbol{D}_k$. By definition, we have for $i, j \in \mathcal{M}$,

$$(\boldsymbol{D})_{i,j} = \mu_i p_{i,j}, \tag{2.2}$$

$$(\boldsymbol{D}_k)_{i,j} = (\boldsymbol{D})_{i,j} \zeta_{k,i,j}, \qquad k = 1, 2, \dots. \tag{2.3}$$

Note here that $\boldsymbol{C} + \boldsymbol{D}$ is the irreducible infinitesimal generator of the underlying Markov chain. Let $\boldsymbol{\pi}$ denote the stationary probability vector of the underlying Markov chain. Because $\boldsymbol{C} + \boldsymbol{D}$ is irreducible, $\boldsymbol{\pi}$ is uniquely determined by $\boldsymbol{\pi}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{0}$ and $\boldsymbol{\pi e} = 1$, where $\boldsymbol{e}$ denotes an $M \times 1$ vector whose elements are all equal to one. When arrivals are simple, i.e., $\boldsymbol{D}_k = \boldsymbol{O}$ for all $k \ge 2$, the resulting arrival process is called a Markovian arrival process (MAP).

Let $N(t)$ ($t \ge 0$) denote the counting process of BMAP $(\boldsymbol{C}, \boldsymbol{D}_1, \boldsymbol{D}_2, \dots)$, where we assume $N(0) = 0$. We define $\boldsymbol{P}(z, t)$ ($|z| \le 1,\ t \ge 0$) as an $M \times M$ matrix such that $(\boldsymbol{P}(z,t))_{i,j} = \mathrm{E}\left[z^{N(t)} 1(S(t) = j) \,\middle|\, S(0) = i \right]$ for $i, j \in \mathcal{M}$, where $1(\chi)$ denotes an indicator function of event χ. We then have [14]

$$\boldsymbol{P}(z, t) = \exp\left[\left(\boldsymbol{C} + \sum_{k=1}^{\infty} z^k \boldsymbol{D}_k \right) t \right]. \tag{2.4}$$

Further the arrival rate λ of customers is given by

$$\lambda = \boldsymbol{\pi} \lim_{z \to 1-} \frac{\mathrm{d}}{\mathrm{d}z} \boldsymbol{P}(z, 1) \boldsymbol{e} = \boldsymbol{\pi} \sum_{k=1}^{\infty} k \boldsymbol{D}_k \boldsymbol{e}. \tag{2.5}$$

In this paper, we assume that service times are independent and identically distributed (i.i.d.) according to a distribution function $H(x)$ ($x \ge 0$) with finite mean h. Let H denote a generic r.v. representing i.i.d. service times, which is independent of arrivals. Clearly $H(x) = \Pr[H \le x]$ and $h = \mathrm{E}[H]$. Customers are served on a FIFO basis and ties are broken randomly. Throughout this paper, we assume

$$0 < \rho < 1, \tag{2.6}$$

where $\rho = \lambda h$. The first inequality excludes trivial cases of no arrivals and/or zero service times, and the second inequality ensures that the system is stable [13].

2.2. Subexponential and square-root insensitive distributions

This subsection summarizes definitions and some properties of subexponential and square-root insensitive distributions, both of which are subclasses of heavy-tailed distributions. Note that a nonnegative r.v. X and its distribution is called heavy-tailed if the moment generating function $\mathrm{E}[\exp(\theta X)]$ $(\theta > 0)$ does not exist. We first describe the class of long-tailed distributions, which is the largest operational class of heavy-tailed distributions and includes subexponential and square-root insensitive distributions.

Definition 2.1 ([7, 20]) *A distribution function $F(x)$ $(x \ge 0)$ and the corresponding nonnegative r.v. F are called long-tailed if $\overline{F}(x) > 0$ for all $x \ge 0$ and $\overline{F}(x+y) \stackrel{x}{\sim} \overline{F}(x)$ for all $y \ge 0$.*

We denote the class of long-tailed distributions by $\mathcal{L}$ and write $F(x) \in \mathcal{L}$ (resp. $F \in \mathcal{L}$) to represent that $F(x)$ (resp. F) is long-tailed.

Proposition 2.1 (Corollary 3.3 in [20]) *If $F_{\mathrm{e}} \in \mathcal{L}$, $\Pr[F > x] = o(\Pr[F_{\mathrm{e}} > x])$.*

Definition 2.2 ([6, 20]) *A distribution function $F(x)$ $(x \ge 0)$ and the corresponding nonnegative r.v. F are called subexponential if $\overline{F}(x) > 0$ for all $x \ge 0$ and $\overline{F^{(n)}}(x) \stackrel{x}{\sim} n\overline{F}(x)$ for all $n = 2, 3, \dots$.*

Let $\mathcal{S}$ denote the class of subexponential distributions. When $F(x)$ (resp. F) is subexponential, we write $F(x) \in \mathcal{S}$ (resp. $F \in \mathcal{S}$). Note that $\mathcal{S} \subset \mathcal{L}$ (see [19]).

Proposition 2.2 (Lemma 10 in [10]) *Let Y_i's $(i = 1, 2, \dots, n)$ denote independent nonnegative r.v.s such that for some r.v. $F \in \mathcal{S}$, $\Pr[Y_i > x] \stackrel{x}{\sim} \kappa_i \Pr[F > x]$, where $\kappa_i \ge 0$ for all $i = 1, 2, \dots, n$. Further let $\kappa_{\max} = \max\{\kappa_i; i = 1, 2, \dots, n\}$.*

(*a*) $\Pr[Y_1 + \cdots + Y_n > x] \stackrel{x}{\sim} (\sum_{i=1}^{n} \kappa_i) \Pr[F > x]$.

(*b*) *For any $\varepsilon > 0$, there exists some constant $K := K(\varepsilon, \kappa_{\max}) < \infty$ such that*

$$\frac{\Pr[Y_1 + \cdots + Y_n > x]}{\Pr[F > x]} \le K \cdot (1 + \varepsilon)^n, \qquad \forall x \ge 0,$$

where K is independent of n.

Proposition 2.3 (Lemma 2 in [18]) *A nonnegative r.v. Y is subexponential if $\Pr[Y > x] \stackrel{x}{\sim} \kappa \Pr[F > x]$ for some r.v. $F \in \mathcal{S}$ and some positive constant κ.*

Definition 2.3 (Definition 1 in [11]) *A distribution function $F(x)$ $(x \ge 0)$ and the corresponding nonnegative r.v. F are called square-root insensitive if $\overline{F}(x) > 0$ for all $x \ge 0$ and $\overline{F}(x - \sqrt{x}) \stackrel{x}{\sim} \overline{F}(x)$.*

We denote the class of square-root insensitive distributions by $\mathcal{L}^2$, taking account of Proposition A.1. Thus $F(x) \in \mathcal{L}^2$ (resp. $F \in \mathcal{L}^2$) represents that $F(x)$ (resp. F) is square-root insensitive. Note that $\mathcal{L}^2 \subset \mathcal{L}$. We summarize some important properties of class $\mathcal{L}^2$ in Appendix A, because class $\mathcal{L}^2$ was introduced recently in [11] and its properties are not generally known.

2.3. Independent sampling

We describe a result on independent sampling at heavy-tailed random times [11]. Let $\{B(t); t \geq 0\}$ denote a cumulative process associated with a (possibly delayed) regenerative process, where $B(0) = 0$. Let ν_n $(n = 0, 1, \dots)$ denote the length of the nth regenerative cycle. Note that ν_n's $(n = 1, 2, \dots)$ are i.i.d. and independent of ν_0. Let $\tau_n = \sum_{i=0}^{n} \nu_i$ $(n = 0, 1, \dots)$. We then define γ_n and γ_n^* $(n = 0, 1, \dots)$ as $\gamma_n = B(\tau_n) - B(\tau_{n-1})$ and $\gamma_n^* = \sup_{\tau_{n-1} \leq t \leq \tau_n} |B(t) - B(\tau_{n-1})|$, respectively, where $\tau_{-1} = 0$. By definition, $\gamma_n^* \geq 0$ and $\gamma_n^* \geq \gamma_n$ for all $n = 0, 1, \dots$. Further γ_n's (resp. γ_n^*'s) $(n = 1, 2, \dots)$ are i.i.d. and independent of γ_0 (resp. γ_0^*). We assume that all of ν_0, ν_1, γ_0^*, and γ_1^* are proper r.v.s. and that $0 < \mathrm{E}[\nu_1] < \infty$ and $0 < \mathrm{E}[\gamma_1] < \infty$.

The following lemma is considered as an extension of Proposition 3 in [11].

Lemma 2.1 *Suppose* $\mathrm{E}[\nu_1^2] < \infty$, $\gamma_1 \geq 0$ *w.p.1, and there exists some* $\phi > 0$ *such that* $\mathrm{E}[e^{\phi\sqrt{\gamma_n^*}}] < \infty$ *for* $n = 0, 1$. *Let* Y *denote a nonnegative r.v. independent of* $\{B(t); t \geq 0\}$. *If* Y *satisfies*

$$\Pr[Y > x] \overset{x}{\sim} \kappa \Pr[F > x], \tag{2.7}$$

for some r.v. $F \in \mathcal{L}^2$ *and some nonnegative constant* κ, *we have*

$$\Pr[B(Y) > bx] \overset{x}{\sim} \Pr[U(Y) > bx] \overset{x}{\sim} \kappa \Pr[F > x],$$

where $b = \mathrm{E}[\gamma_1]/\mathrm{E}[\nu_1] > 0$ *and* $U(t) = \sup_{0 \leq u \leq t} B(u)$. *In addition, if* $Y = F \in \mathcal{L}^2$,

$$\Pr[B(Y) > bx] \overset{x}{\sim} \Pr[U(Y) > bx] \overset{x}{\sim} \Pr[Y > x]. \tag{2.8}$$

Proof: The proof of Lemma 2.1 is given in Appendix C. □

Remark 2.1 *For* $\kappa > 0$, *Lemma 2.1 is an immediate consequence of Proposition 3 in [11], because* $Y \in \mathcal{L}^2$ *(see Lemma A.1). It is not the case, however, for* $\kappa = 0$, *because* Y *may not be square-root insensitive.*

Remark 2.2 *In the proof of Lemma 2.1, we apply the central limit theorem to* $\{B(t)\}$, *which requires* $\mathrm{E}[\nu_1^2] < \infty$ *and* $\mathrm{E}[\gamma_1^2] < \infty$. *The latter holds due to* $\gamma_1 \geq 0$ *and* $\mathrm{E}[\exp(\phi\sqrt{\gamma_1^*})] < \infty$. *In fact,* $\gamma_1^2 \leq (\gamma_1^*)^2$ *and therefore*

$$\frac{\phi^4}{4!}\mathrm{E}[\gamma_1^2] \leq \frac{\phi^4}{4!}\mathrm{E}[(\gamma_1^*)^2] \leq \sum_{k=0}^{\infty} \frac{\phi^k}{k!}\mathrm{E}[(\gamma_1^*)^{k/2}] = \mathrm{E}[e^{\phi\sqrt{\gamma_1^*}}] < \infty.$$

3. Queue Length Asymptotics

This section considers the queue length asymptotics in the FIFO BMAP/GI/1 queue. Let $L(t)$ $(t \geq 0)$ denote the queue length (including a customer in service, if any) at time t. We define $\boldsymbol{x}_k$ $(k = 0, 1, \dots)$ as a $1 \times M$ vector whose jth $(j \in \mathcal{M})$ element represents $\Pr[L = k, S = j]$, where L and S denote generic r.v.s representing $\{L(t); t \geq 0\}$ and $\{S(t); t \geq 0\}$, respectively, in steady state. In [21], Takine showed that $\{\boldsymbol{x}_k;\ k = 0, 1, \dots\}$ is identical to the steady state solution of a Markov chain of M/G/1 type [17], whose transition probability matrix $\boldsymbol{\Pi}$ is given by

$$\boldsymbol{\Pi} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.1}$$

where $\boldsymbol{A}_k$ $(k = 0, 1, \dots)$ denotes an $M \times M$ nonnegative matrix whose (i,j)th $(i,j \in \mathcal{M})$ element represents $\Pr[N(H) = k, S(H) = j \mid S(0) = i]$. Note that $\boldsymbol{A}_k$'s satisfy

$$\sum_{k=0}^{\infty} z^k \boldsymbol{A}_k = \int_0^{\infty} \boldsymbol{P}(z,x)\mathrm{d}H(x) = \int_0^{\infty} \exp\left[\left(\boldsymbol{C} + \sum_{k=1}^{\infty} z^k \boldsymbol{D}_k\right) x\right] \mathrm{d}H(x), \qquad (3.2)$$

which implies that $\sum_{k=0}^{\infty} \boldsymbol{A}_k$ is an irreducible and stochastic matrix such that $\boldsymbol{\pi} \sum_{k=0}^{\infty} \boldsymbol{A}_k = \boldsymbol{\pi}$, because $\boldsymbol{C} + \boldsymbol{D}$ is an irreducible and proper generator satisfying $\boldsymbol{\pi}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{0}$. Further it follows from (2.5), (2.6) and (3.2) that $\rho = \sum_{k=1}^{\infty} k\boldsymbol{\pi}\boldsymbol{A}_k\boldsymbol{e} < 1$. As a result, $\boldsymbol{\Pi}$ in (3.1) is irreducible and positive recurrent [1, Proposition 3.1 in Chapter XI].

Let $\boldsymbol{G}$ denote the minimal nonnegative solution of $\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{A}_k \boldsymbol{G}^k$. It is known [17] that if $\sum_{k=0}^{\infty} \boldsymbol{A}_k$ is irreducible and $\rho = \sum_{k=1}^{\infty} k\boldsymbol{\pi}\boldsymbol{A}_k\boldsymbol{e} < 1$, $\boldsymbol{G}$ is stochastic and equal to the limit $\boldsymbol{G}_\infty$ of an elementwise nondecreasing sequence $\{\boldsymbol{G}_n; n = 0, 1, \dots\}$, where

$$\boldsymbol{G}_0 = \boldsymbol{O}, \qquad \boldsymbol{G}_n = \sum_{k=0}^{\infty} \boldsymbol{A}_k \boldsymbol{G}_{n-1}^k \quad (n = 1, 2, \dots).$$

Note here that $\boldsymbol{D} \geq, \neq \boldsymbol{O}$ and $\exp[(\boldsymbol{C} + \boldsymbol{D})t] > \boldsymbol{O}$ for all $t > 0$, which leads to

$$\sum_{k=1}^{\infty} \boldsymbol{A}_k = \int_0^{\infty} \mathrm{d}H(x) \int_0^{x} \mathrm{d}y\, e^{(\boldsymbol{C}+\boldsymbol{D})y} \boldsymbol{D} e^{(\boldsymbol{C}+\boldsymbol{D})(x-y)} > \boldsymbol{O}.$$

Note also that $\boldsymbol{A}_0 = \int_0^{\infty} \mathrm{d}H(x) e^{\boldsymbol{C}x} \geq \boldsymbol{O}$, whose diagonal elements are all positive. Thus $\boldsymbol{G} \geq \boldsymbol{G}_2 = \boldsymbol{A}_0 + \sum_{k=1}^{\infty} \boldsymbol{A}_k \boldsymbol{A}_0^k > \boldsymbol{O}$. Let $\boldsymbol{g} > \boldsymbol{0}$ denote the unique stationary probability vector of $\boldsymbol{G}$, i.e., $\boldsymbol{g}\boldsymbol{G} = \boldsymbol{g}$ and $\boldsymbol{g}\boldsymbol{e} = 1$. We then have

$$\lim_{m \to \infty} \boldsymbol{G}^m = \boldsymbol{e}\boldsymbol{g}. \qquad (3.3)$$

Let $\overline{\boldsymbol{x}_k} = \sum_{l=k+1}^{\infty} \boldsymbol{x}_l$ for $k = 0, 1 \dots$. It is easy to see that $(\overline{\boldsymbol{x}_k})_j = \Pr[L > k, S = j]$ for $k = 0, 1, \dots$ and $j \in \mathcal{M}$.

Proposition 3.1 (Theorem 4 and Remark 12 in [25]) *Let* $\overline{\boldsymbol{A}_k} = \sum_{l=k+1}^{\infty} \boldsymbol{A}_l$ *for* $k = 0, 1, \dots$. *Suppose* (3.3) *holds and there exists a nonnegative integer-valued r.v.* F *with positive finite mean such that*
(*a*) $F_{\mathrm{e}} \in \mathcal{S}$, *and*
(*b*) $\overline{\boldsymbol{A}_k} \overset{k}{\sim} \boldsymbol{C}_A \Pr[F > k]/\mathrm{E}[F]$ *for some finite nonnegative matrix* $\boldsymbol{C}_A$ ($\boldsymbol{C}_A \neq \boldsymbol{O}$).
We then have $\overline{\boldsymbol{x}_k} \overset{k}{\sim} (1-\rho)^{-1}(\boldsymbol{\pi}\boldsymbol{C}_A\boldsymbol{e})\boldsymbol{\pi} \Pr[F_{\mathrm{e}} > k]$.

Remark 3.1 *The condition* (*3.3*) *is necessary for Proposition 3.1 (see Appendix A.6 in [24]), though Theorem 4 in [25] presented the subexponential asymptotic formula for the* $\overline{\boldsymbol{x}_k}$ *without this condition.*

Recall that $(\overline{\boldsymbol{A}_k})_{i,j} = \Pr[N(H) > k, S(H) = j \mid S(0) = i]$ $(i, j \in \mathcal{M})$ and that $N(H)$ is given by

$$N(H) = \sum_{l=1}^{N_G(H)} G_l, \qquad (3.4)$$

where $N_G(t)$ $(t \geq 0)$ denotes the number of batches arriving in interval $(0, t]$ and G_l $(l = 1, 2, \dots)$ denotes the number of customers in the lth arriving batch. Thus condition (b) of Proposition 3.1 requires us to characterize the tail distribution of $N(H)$ in (3.4). In what follows, we consider two cases $H \in \mathcal{L}$ and $G_l \in \mathcal{L}$ separately.

3.1. Queue with heavy-tailed service times

In this subsection, we consider the queue length asymptotics in the FIFO BMAP/GI/1 queue with heavy-tailed service times. More specifically, we utilize Lemma 2.1 to explore a sufficient condition under which condition (b) of Proposition 3.1 holds. As we will see, we regard the counting process $\{N(t); t \geq 0\}$ of arrivals (resp. the service time H) in the BMAP/GI/1 queue as $\{B(t)\}$ (resp. Y) in Lemma 2.1.

Assumption 3.1 *For some* $\phi > 0$, $\sum_{k=1}^{\infty} e^{\phi\sqrt{k}} \boldsymbol{D}_k < \infty$.

For simplicity, let $\mathrm{P}_i(\cdot) = \Pr[\,\cdot \mid S(0) = i]$ and $\mathrm{E}_i[\,\cdot\,] = \mathrm{E}[\,\cdot \mid S(0) = i]$.

Lemma 3.1 *If Assumption 3.1 holds and* $H \in \mathcal{L}^2$, $\mathrm{P}_i(N(H) > \lambda k) \overset{k}{\sim} \Pr[H > k]$ $(\forall i \in \mathcal{M})$.

Proof: For each $i \in \mathcal{M}$, we apply Lemma 2.1 to $N(H)$, assuming $S(0) = i$. In this specific application, let $\tau_0 = 0$ and we define τ_n $(n = 1, 2, \dots)$ as the nth point in time, at which the underlying Markov chain $\{S(t); t \geq 0\}$ enters state i from other states. It is easy to see that τ_n's $(n = 1, 2, \dots)$ are regenerative points for the counting process $\{N(t); t \geq 0\}$ of BMAP arrivals. Thus in the framework of section 2.3, the above can be described with $B(t) = N(t)$, $Y = F = H$, $\nu_0 = \tau_0 = 0$, $\nu_n = \tau_n - \tau_{n-1}$ $(n = 1, 2, \dots)$, $\gamma_0 = N(\tau_0) = 0$, and $\gamma_n = N(\tau_n) - N(\tau_{n-1})$ $(n = 1, 2, \dots)$. It is clear that ν_n's $(n = 1, 2, \dots)$ are i.i.d. and so are γ_n's $(n = 1, 2, \ldots)$. Also $\gamma_0^* = 0$ and $\gamma_n^* = \gamma_n \geq 0$ $(n = 1, 2, \ldots)$ because $N(t)$ is nondecreasing in t sample path wise. By (2.6) and the definition of BMAP, ν_1 and γ_1 are proper nonnegative r.v.s, and $\mathrm{E}_i[\nu_1^2] < \infty$, and $0 < \mathrm{E}_i[\gamma_1] < \infty$.

There remains to verify $\mathrm{E}_i[\exp(\phi\sqrt{\gamma_1})] < \infty$ and $\mathrm{E}_i[\gamma_1^2] < \infty$, in order to apply Lemma 2.1 to $N(H)$. From the renewal reward theorem [26, Theorem 2 in Chapter 2], we have

$$\frac{\mathrm{E}_i[\gamma_1]}{\mathrm{E}_i[\nu_1]} = \lambda > 0, \qquad \frac{\mathrm{E}_i[e^{\phi\sqrt{\gamma_1}}]}{\mathrm{E}_i[\nu_1]} = \boldsymbol{\pi} \sum_{k=1}^{\infty} e^{\phi\sqrt{k}} \boldsymbol{D}_k \boldsymbol{e} < \infty.$$

The former shows $0 < \mathrm{E}_i[\nu_1] < \infty$. Thus from the latter, we have $\mathrm{E}_i[\exp(\phi\sqrt{\gamma_1^*})] = \mathrm{E}_i[\exp(\phi\sqrt{\gamma_1})] < \infty$, which yields $\mathrm{E}_i[\gamma_1^2] < \infty$ due to $\gamma_1 > 0$ (see Remark 2.2). As a result, we apply (2.8) in Lemma 2.1 to $N(H)$ and obtain $\mathrm{P}_i(N(H) > \lambda k) \overset{k}{\sim} \mathrm{P}_i(H > k) = \Pr[H > k]$. □

Lemma 3.2 *Suppose Assumption 3.1 holds. If* $H \in \mathcal{L}^2$,

$$\mathrm{P}_i(N(H) > \lambda k, S(H) = j) \overset{k}{\sim} \Pr[H > k](\boldsymbol{\pi})_j, \qquad i, j \in \mathcal{M}. \tag{3.5}$$

Proof: The proof of this lemma is given in Appendix D. □

Remark 3.2 *Lemmas 3.1 and 3.2 yield* $\lim_{k\to\infty} \mathrm{P}_i(S(H) = j \mid N(H) > \lambda k) = (\boldsymbol{\pi})_j$.

Recall that $(\overline{\boldsymbol{A}_k})_{i,j} = \mathrm{P}_i(N(H) > k, S(H) = j)$ $(i, j \in \mathcal{M})$. Thus (3.5) is equivalent to

$$\overline{\boldsymbol{A}_k} \overset{k}{\sim} \boldsymbol{e\pi} \cdot \Pr[\lambda H > k] = \mathrm{E}[\lambda H] \boldsymbol{e\pi} \cdot \Pr[\lambda H > k]/\mathrm{E}[\lambda H]. \tag{3.6}$$

Let $\Theta = \lambda H$. It is easy to see that $\Pr[\Theta_{\mathrm{e}} > x] = \Pr[\lambda H_{\mathrm{e}} > x]$, which implies that if $H_{\mathrm{e}} \in \mathcal{S}$, $\Theta_{\mathrm{e}} \in \mathcal{S}$ for any $\lambda > 0$. Because $\boldsymbol{\pi e} = 1$ and $\mathrm{E}[\lambda H] = \rho$, combining Proposition 3.1 with (3.6) yields the following theorem.

Theorem 3.1 *Suppose Assumption 3.1 holds. If $H \in \mathcal{L}^2$ and $H_{\rm e} \in \mathcal{S}$,*

$$\overline{\boldsymbol{x}_k} \overset{k}{\sim} \frac{\rho}{1-\rho}\boldsymbol{\pi} \cdot \Pr[\lambda H_{\rm e} > k], \qquad \Pr[L > k] \overset{k}{\sim} \frac{\rho}{1-\rho} \cdot \Pr[\lambda H_{\rm e} > k], \tag{3.7}$$

which shows that the stationary queue length L is subexponential, i.e., $L \in \mathcal{S}$.

Note that if the arrival process is MAP (i.e., $\boldsymbol{D}_k = \boldsymbol{O}$ for all $k \ge 2$), Assumption 3.1 always holds.

Corollary 3.1 *Consider the stationary FIFO MAP/GI/1 queue. If $H \in \mathcal{L}^2$ and $H_{\rm e} \in \mathcal{S}$, (3.7) holds.*

3.2. Queue with heavy-tailed batch sizes

In this subsection, we consider the queue length asymptotics in the FIFO BMAP/GI/1 queue with heavy-tailed batch sizes. Let G denote a generic r.v. representing the number of customers in a randomly chosen batch in steady state. We then have $\Pr[G = k] = \lambda_G^{-1}\boldsymbol{\pi}\boldsymbol{D}_k\boldsymbol{e}$ ($k = 1, 2, \dots$) and

$$\mathrm{E}[G] = \lambda/\lambda_G, \tag{3.8}$$

where $\lambda_G = \boldsymbol{\pi}\boldsymbol{D}\boldsymbol{e} < \infty$ denotes the arrival rate of batches.

Assumption 3.2 *There exists some nonnegative matrix $\widetilde{\boldsymbol{D}}$ ($\widetilde{\boldsymbol{D}} \ne \boldsymbol{O}$) such that $\overline{\boldsymbol{D}_k} \overset{k}{\sim} \widetilde{\boldsymbol{D}} \Pr[G > k]$, where $\overline{\boldsymbol{D}_k} = \sum_{l=k+1}^{\infty} \boldsymbol{D}_l$ for $k = 0, 1, \dots$.*

Note that if Assumption 3.2 holds, we have

$$\boldsymbol{\pi}\widetilde{\boldsymbol{D}}\boldsymbol{e} = \lambda_G, \tag{3.9}$$

because $\Pr[G > k] = \lambda_G^{-1}\boldsymbol{\pi}\overline{\boldsymbol{D}_k}\boldsymbol{e}$.

We define $\boldsymbol{\Lambda}_k$ ($k = 0, 1, \dots$) as

$$\boldsymbol{\Lambda}_0 = \boldsymbol{I} + \theta^{-1}\boldsymbol{C}, \qquad \boldsymbol{\Lambda}_k = \theta^{-1}\boldsymbol{D}_k \quad (k = 1, 2, \dots),$$

where $\theta = \max_{j \in \mathcal{M}} |(\boldsymbol{C})_{j,j}|$. We then rewrite (3.2) as

$$\sum_{k=0}^{\infty} z^k \boldsymbol{A}_k = \int_0^{\infty} \sum_{n=0}^{\infty} e^{-\theta x}\frac{(\theta x)^n}{n!} \mathrm{d}H(x) \left[\sum_{k=0}^{\infty} z^k \boldsymbol{\Lambda}_k\right]^n. \tag{3.10}$$

Let $\boldsymbol{\Lambda}_k^{(n)}$'s ($n = 1, 2, \dots,\ k = 0, 1, \dots$) denote $M \times M$ matrices satisfying

$$\sum_{k=0}^{\infty} z^k \boldsymbol{\Lambda}_k^{(n)} = \left[\sum_{k=0}^{\infty} z^k \boldsymbol{\Lambda}_k\right]^n, \qquad n = 1, 2, \dots.$$

It then follows from (3.10) that

$$\overline{\boldsymbol{A}_k} = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\theta x}\frac{(\theta x)^n}{n!} \mathrm{d}H(x) \overline{\boldsymbol{\Lambda}_k^{(n)}}, \qquad k = 0, 1, \dots, \tag{3.11}$$

where $\overline{\boldsymbol{\Lambda}_k^{(n)}} = \sum_{l=k+1}^{\infty} \boldsymbol{\Lambda}_l^{(n)}$ ($n = 1, 2, \dots,\ k = 0, 1, \dots$). Thus we can examine the asymptotics of $\left\{\overline{\boldsymbol{A}_k}\right\}$ through $\left\{\overline{\boldsymbol{\Lambda}_k^{(n)}}\right\}$, where the (i, j)th element of $\overline{\boldsymbol{\Lambda}_k^{(n)}}$ represents the probability that the uniformized underlying Markov chain with parameter θ moves from state i to state j in n steps during which more than k customers arrive.

Lemma 3.3 *Suppose Assumption 3.2 holds. If $G \in \mathcal{S}$,*

$$\overline{\boldsymbol{\Lambda}_k^{(n)}} \overset{k}{\sim} \sum_{\nu=0}^{n-1} \boldsymbol{\Lambda}^{\nu} \widetilde{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}^{n-\nu-1} \cdot \Pr[G > k], \qquad \forall n = 1, 2, \dots, \tag{3.12}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{I} + \theta^{-1}(\boldsymbol{C} + \boldsymbol{D}), \quad \widetilde{\boldsymbol{\Lambda}} = \theta^{-1}\widetilde{\boldsymbol{D}}. \tag{3.13}$$

Proof: The proof of Lemma 3.3 is given in Appendix E. □

Lemma 3.4 *Suppose Assumption 3.2 holds. If $G \in \mathcal{S}$, for any $\varepsilon > 0$ there exists some positive constant $K := K(\varepsilon)$ such that*

$$\frac{\overline{\boldsymbol{\Lambda}_k^{(n)}}}{\Pr[G > k]} \le K \cdot (1 + \varepsilon)^n \boldsymbol{\Lambda}^n, \quad \forall k = 0, 1, \dots, \ \forall n = 1, 2, \dots.$$

Proof: The proof of Lemma 3.4 is given in Appendix F. □

Assumption 3.3 *The service time distribution function $H(x)$ ($x \ge 0$) is light-tailed, i.e., there exists some $\delta > 0$ such that $\int_0^\infty e^{\delta x} \mathrm{d}H(x) < \infty$.*

Lemma 3.5 *Suppose Assumptions 3.2 and 3.3 hold. If $G \in \mathcal{S}$, $\overline{\boldsymbol{A}_k} \overset{k}{\sim} \boldsymbol{C}_A \cdot \Pr[G > k]/\mathrm{E}[G]$, where*

$$\boldsymbol{C}_A = \mathrm{E}[G] \int_0^\infty \mathrm{d}H(x) \int_0^x e^{(\boldsymbol{C}+\boldsymbol{D})y} \widetilde{\boldsymbol{D}} e^{(\boldsymbol{C}+\boldsymbol{D})(x-y)} \mathrm{d}y.$$

Proof: We choose $\varepsilon > 0$ such that $\varepsilon\theta \le \delta$. Because $\boldsymbol{\Lambda}\boldsymbol{e} = \boldsymbol{e}$, it follows from (3.11) and Lemma 3.4 that

$$\frac{\overline{\boldsymbol{A}_k}\boldsymbol{e}}{\Pr[G > k]} \le K \int_0^\infty \sum_{n=1}^\infty e^{-\theta x} \frac{\{(1+\varepsilon)\theta x\}^n}{n!} \mathrm{d}H(x)\boldsymbol{e} \le K \int_0^\infty e^{\varepsilon\theta x} \mathrm{d}H(x)\boldsymbol{e} < \infty.$$

Thus (3.11), Lemma 3.3, and the dominated convergence theorem yield

$$\begin{aligned}
\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}_k}}{\Pr[G > k]} &= \int_0^\infty \sum_{n=1}^\infty e^{-\theta x} \frac{(\theta x)^n}{n!} \mathrm{d}H(x) \lim_{k\to\infty} \frac{\overline{\boldsymbol{\Lambda}_k^{(n)}}}{\Pr[G > k]} \\
&= \int_0^\infty \mathrm{d}H(x) \left(\sum_{n=1}^\infty e^{-\theta x} \frac{(\theta x)^n}{n!} \sum_{\nu=0}^{n-1} \boldsymbol{\Lambda}^{\nu} \widetilde{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}^{n-\nu-1} \right) \\
&= \int_0^\infty \mathrm{d}H(x) \int_0^x e^{(\boldsymbol{C}+\boldsymbol{D})y} \widetilde{\boldsymbol{D}} e^{(\boldsymbol{C}+\boldsymbol{D})(x-y)} \mathrm{d}y,
\end{aligned}$$

where we use (3.13) in the last equality. □

Lemma 3.5 implies that we can utilize Proposition 3.1. From (3.8) and (3.9), we have $\boldsymbol{\pi}\boldsymbol{C}_A\boldsymbol{e} = \rho$. Thus we obtain the following theorem.

Theorem 3.2 *Suppose Assumptions 3.2 and 3.3 hold. If $G \in \mathcal{S}$ and $G_{\mathrm{e}} \in \mathcal{S}$, we have*

$$\overline{\boldsymbol{x}_k} \overset{k}{\sim} \frac{\rho}{1-\rho} \boldsymbol{\pi} \cdot \Pr[G_{\mathrm{e}} > k], \qquad \Pr[L > k] \overset{k}{\sim} \frac{\rho}{1-\rho} \cdot \Pr[G_{\mathrm{e}} > k],$$

which shows $L \in \mathcal{S}$.

4. Waiting Time Asymptotics

This section considers the subexponential asymptotics of the waiting time and sojourn time distributions in the FIFO BMAP/GI/1 queue. Let V, W, and T denote generic r.v.s representing the amount of unfinished work in system, the actual waiting time, and the sojourn time, respectively, in the stationary FIFO BMAP/GI/1 queue. Also let S^+ denote a generic r.v. representing the state of the underlying Markov chain immediately after arrivals in steady state. We then define $\boldsymbol{v}(x)$, $\boldsymbol{w}(x)$, and $\boldsymbol{t}(x)$ ($x \ge 0$) as $1 \times M$ vectors whose jth ($j \in \mathcal{M}$) elements represent $\Pr[V \le x, S = j]$, $\Pr[W \le x, S^+ = j]$, and $\Pr[T \le x, S^+ = j]$, respectively.

Lemma 4.1 *$\boldsymbol{w}(x)$ and $\boldsymbol{t}(x)$ ($x \ge 0$) are given by*

$$\boldsymbol{w}(x) = \frac{1}{\lambda}\int_0^x \mathrm{d}\boldsymbol{v}(y) \sum_{k=0}^{\infty} \overline{\boldsymbol{D}_k} H^{(k)}(x-y), \quad \boldsymbol{t}(x) = \frac{1}{\lambda}\int_0^x \mathrm{d}\boldsymbol{v}(y) \sum_{k=0}^{\infty} \overline{\boldsymbol{D}_k} H^{(k+1)}(x-y), \qquad (4.1)$$

respectively, where $H^{(0)}(x) = 1$ for all $x \ge 0$.

Proof: Let $\boldsymbol{v}^*(s)$, $\boldsymbol{w}^*(s)$, and $H^*(s)$ denote the Laplace-Stieltjes transforms of $\boldsymbol{v}(x)$, $\boldsymbol{w}(x)$, and $H(x)$, respectively. Applying Theorem III.2 in [15] to the FIFO BMAP/GI/1 queue, we can readily obtain

$$\boldsymbol{w}^*(s) = \sum_{n=1}^{\infty} \frac{\boldsymbol{v}^*(s)\boldsymbol{D}_n}{\lambda} \sum_{k=1}^{n} \{H^*(s)\}^{k-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty} \boldsymbol{v}^*(s)\overline{\boldsymbol{D}_k}\{H^*(s)\}^k,$$

from which the result for $\boldsymbol{w}(x)$ follows. Noting $T = W + H$, we also have the result for $\boldsymbol{t}(x)$. □

Let $\overline{\boldsymbol{v}}(x)$, $\overline{\boldsymbol{w}}(x)$, and $\overline{\boldsymbol{t}}(x)$ ($x \ge 0$) denote $1 \times M$ vectors whose jth ($j \in \mathcal{M}$) elements are given by

$$(\overline{\boldsymbol{v}}(x))_j = \Pr[V > x, S = j], \ (\overline{\boldsymbol{w}}(x))_j = \Pr[W > x, S^+ = j], \ (\overline{\boldsymbol{t}}(x))_j = \Pr[T > x, S^+ = j],$$

respectively. Let V_i ($i \in \mathcal{M}$) denote a conditional r.v. representing the amount of unfinished work in system given the underlying Markov chain being in state i. Clearly, $\Pr[V_i > x] = (\overline{\boldsymbol{v}}(x))_i/(\boldsymbol{\pi})_i$ for $x \ge 0$. Let $\overline{\zeta_{k,i,j}} = \sum_{l=k+1}^{\infty} \zeta_{l,i,j}$ ($i, j \in \mathcal{M}$, $k = 0, 1, \dots$). It then follows from (2.3) and (4.1) that

$$(\overline{\boldsymbol{w}}(x))_j = \frac{1}{\lambda}\sum_{i\in\mathcal{M}} (\boldsymbol{\pi})_i(\boldsymbol{D})_{i,j} \sum_{k=0}^{\infty} \overline{\zeta_{k,i,j}} \cdot \Pr[V_i + H_1 + \cdots + H_k > x], \qquad (4.2)$$

$$(\overline{\boldsymbol{t}}(x))_j = \frac{1}{\lambda}\sum_{i\in\mathcal{M}} (\boldsymbol{\pi})_i(\boldsymbol{D})_{i,j} \sum_{k=0}^{\infty} \overline{\zeta_{k,i,j}} \cdot \Pr[V_i + H_1 + \cdots + H_{k+1} > x], \qquad (4.3)$$

where H_l ($l = 1, 2, \dots$) denotes the service time of the lth customer in a batch.

Let $G(i,j)$ ($i, j \in \mathcal{M}$) denote a conditional r.v. representing the number of customers in a batch given that the batch arrives with a transition of the underlying Markov chain from state i to state j. It then follows from (2.3) that $\Pr[G(i,j) = k] = \zeta_{k,i,j}$ ($i, j \in \mathcal{M}$) and therefore

$$(\boldsymbol{D})_{i,j} \Pr[G(i,j) = k] = (\boldsymbol{D}_k)_{i,j}, \qquad k = 1, 2, \dots, \qquad (4.4)$$

which leads to

$$(\boldsymbol{D})_{i,j}\mathrm{E}[G(i,j)] = \sum_{k=1}^{\infty} k(\boldsymbol{D}_k)_{i,j}, \qquad i,j \in \mathcal{M}. \tag{4.5}$$

From (2.1) and (2.2), we observe that if $(\boldsymbol{D})_{i,j} = 0$,

$$\Pr[G(i,j) = 0] = \zeta_{0,i,j} = 1 \text{ and } \Pr[G(i,j) > k] = \overline{\zeta_{k,i,j}} = 0 \ (k = 0,1,\dots). \tag{4.6}$$

Also (2.5) and (2.6) imply that the equilibrium r.v. $G_{\mathrm{e}}(i,j)$ of $G(i,j)$ $(i,j \in \mathcal{M})$ is well-defined. Thus for any $i,j \in \mathcal{M}$,

$$\overline{\zeta_{k,i,j}} = \mathrm{E}[G(i,j)] \Pr[G_{\mathrm{e}}(i,j) = k], \qquad k = 0,1,\dots. \tag{4.7}$$

From (4.7), we have

$$\begin{aligned}\sum_{k=0}^{\infty} \overline{\zeta_{k,i,j}} \cdot \Pr[H_1 + H_2 + \cdots + H_k > x] &= \mathrm{E}[G(i,j)] \sum_{k=0}^{\infty} \Pr[G_{\mathrm{e}}(i,j) = k] \overline{H^{(k)}}(x) \\ &= \mathrm{E}[G(i,j)] \Pr[\widehat{X}_{i,j} > x], \qquad i,j \in \mathcal{M}, \end{aligned} \tag{4.8}$$

where $\widehat{X}_{i,j}$ $(i,j \in \mathcal{M})$ is defined as

$$\widehat{X}_{i,j} = H_1 + H_2 + \cdots + H_{G_{\mathrm{e}}(i,j)}. \tag{4.9}$$

As a result, substituting (4.8) into (4.2) and (4.3) yields

$$(\overline{\boldsymbol{w}}(x))_j = \frac{1}{\lambda} \sum_{i \in \mathcal{M}} (\boldsymbol{\pi})_i (\boldsymbol{D})_{i,j} \mathrm{E}[G(i,j)] \Pr[V_i + \widehat{X}_{i,j} > x], \tag{4.10}$$

$$(\overline{\boldsymbol{t}}(x))_j = \frac{1}{\lambda} \sum_{i \in \mathcal{M}} (\boldsymbol{\pi})_i (\boldsymbol{D})_{i,j} \mathrm{E}[G(i,j)] \Pr[V_i + \widehat{X}_{i,j} + H > x]. \tag{4.11}$$

We now define $\overline{\boldsymbol{D}}(x)$ $(x \geq 0)$ as

$$\overline{\boldsymbol{D}}(x) = \sum_{k=1}^{\infty} \boldsymbol{D}_k \overline{H^{(k)}}(x). \tag{4.12}$$

Note that $\overline{\boldsymbol{D}}(0) = \boldsymbol{D}$. The following proposition is an adaptation of Theorem 1 in Takine [23].

Proposition 4.1 (Theorem 1 in [23]) *Suppose there exists a nonnegative r.v. F with positive finite mean such that*
(a) $F_{\mathrm{e}} \in \mathcal{S}$ and
(b) $\overline{\boldsymbol{D}}(x) \overset{x}{\sim} \boldsymbol{C}_D \Pr[F > x]/E[F]$ for some finite nonnegative matrix $\boldsymbol{C}_D$ ($\boldsymbol{C}_D \neq \boldsymbol{O}$).
We then have $\overline{\boldsymbol{v}}(x) \overset{x}{\sim} (1-\rho)^{-1}(\boldsymbol{\pi}\boldsymbol{C}_D\boldsymbol{e})\boldsymbol{\pi} \Pr[F_{\mathrm{e}} > x]$.

Substituting (4.4) into (4.12) yields

$$(\overline{\boldsymbol{D}}(x))_{i,j} = (\boldsymbol{D})_{i,j} \sum_{k=1}^{\infty} \Pr[G(i,j) = k] \overline{H^{(k)}}(x) = (\boldsymbol{D})_{i,j} \Pr[X_{i,j} > x], \qquad i,j \in \mathcal{M}, \tag{4.13}$$

where $X_{i,j}$ $(i,j \in \mathcal{M})$ is defined as

$$X_{i,j} = H_1 + H_2 + \cdots + H_{G(i,j)}. \tag{4.14}$$

Therefore (4.10), (4.11), and Proposition 4.1 imply that the waiting time and sojourn time asymptotics can be examined through the random sums $X_{i,j}$ and $\widehat{X}_{i,j}$. In what follows, we consider two cases $H_l \in \mathcal{L}$ and $G(i,j) \in \mathcal{L}$ separately.

4.1. Queue with heavy-tailed service times

We consider the FIFO BMAP/GI/1 queue with heavy-tailed service times, which satisfies the following assumption.

Assumption 4.1 *There exists some $\varepsilon > 0$ such that $\sum_{k=1}^{\infty}(1+\varepsilon)^k \boldsymbol{D}_k < \infty$.*

Remark 4.1 *Assumption 4.1 is a sufficient condition of Assumption 3.1.*

Proposition 4.2 (Proposition 2.9 in [20]) *Let $\{Z_n; n = 1, 2, \dots\}$ denote a sequence of i.i.d. subexponential r.v.s and N denote a nonnegative integer-valued r.v. independent of $\{Z_n\}$ such that $0 \le \mathrm{E}[N] < \infty$. If N is light-tailed, i.e., $\sum_{n=0}^{\infty}(1+\varepsilon)^n \Pr[N = n] < \infty$ for some $\varepsilon > 0$, we have $\Pr[Z_1 + Z_2 + \cdots + Z_N > x] \overset{x}{\sim} E[N]\Pr[Z_1 > x]$.*

Lemma 4.2 *Suppose Assumption 4.1 holds. If $H \in \mathcal{S}$, we have*

$$\overline{\boldsymbol{D}}(x) \overset{x}{\sim} \sum_{k=1}^{\infty} k\boldsymbol{D}_k \cdot \Pr[H > x]. \tag{4.15}$$

Proof: Assumption 4.1, (4.4), and (4.6) show that $G(i,j)$ $(i,j \in \mathcal{M})$ is light-tailed. Thus we apply Proposition 4.2 to the random sum $X_{i,j}$ in (4.14) and obtain $\Pr[X_{i,j} > x] \overset{x}{\sim} \mathrm{E}[G(i,j)]\Pr[H > x]$. (4.15) now follows from (4.5) and (4.13). □

Lemma 4.2 implies that $\boldsymbol{C}_D$ in Proposition 4.1 is given by $h\sum_{k=1}^{\infty} k\boldsymbol{D}_k$. Therefore the following lemma immediately follows from Proposition 4.1, Lemma 4.2 and (2.5).

Lemma 4.3 *Suppose Assumption 4.1 holds. If $H \in \mathcal{S}$ and $H_{\mathrm{e}} \in \mathcal{S}$,*

$$\overline{\boldsymbol{v}}(x) \overset{x}{\sim} \frac{\rho}{1-\rho}\boldsymbol{\pi} \cdot \Pr[H_{\mathrm{e}} > x], \qquad \Pr[V > x] \overset{x}{\sim} \frac{\rho}{1-\rho} \cdot \Pr[H_{\mathrm{e}} > x].$$

Lemma 4.4 *Suppose Assumption 4.1 holds. If $H \in \mathcal{S}$, $\widehat{X}_{i,j}$ in (4.9) satisfies*

$$\Pr[\widehat{X}_{i,j} > x] \overset{x}{\sim} \mathrm{E}[G_{\mathrm{e}}(i,j)]\Pr[H > x], \qquad i,j \in \mathcal{M}. \tag{4.16}$$

Proof: It follows from (4.7), Assumption 4.1, and $(\overline{\boldsymbol{D}_k})_{i,j} = (\boldsymbol{D})_{i,j}\overline{\zeta_{k,i,j}}$ that

$$\begin{aligned} &(\boldsymbol{D})_{i,j}\mathrm{E}[G(i,j)]\sum_{k=0}^{\infty}(1+\varepsilon)^k \Pr[G_{\mathrm{e}}(i,j) = k] \\ &= \sum_{k=0}^{\infty}(1+\varepsilon)^k (\overline{\boldsymbol{D}_k})_{i,j} = \frac{1}{\varepsilon}\left(\sum_{k=1}^{\infty}(1+\varepsilon)^k(\boldsymbol{D}_k)_{i,j} - \sum_{k=1}^{\infty}(\boldsymbol{D}_k)_{i,j}\right) < \infty, \end{aligned}$$

which implies $G_{\mathrm{e}}(i,j)$ is light-tailed. Consequently, Proposition 4.2 yields (4.16). □

Theorem 4.1 *Suppose Assumption 4.1 holds. If $H \in \mathcal{S}$ and $H_{\mathrm{e}} \in \mathcal{S}$,*

$$\overline{\boldsymbol{w}}(x) \overset{x}{\sim} \bar{\boldsymbol{t}}(x) \overset{x}{\sim} \frac{h}{1-\rho}\boldsymbol{\pi}\sum_{k=1}^{\infty} k\boldsymbol{D}_k \cdot \Pr[H_{\mathrm{e}} > x], \tag{4.17}$$

$$\Pr[W > x] \overset{x}{\sim} \Pr[T > x] \overset{x}{\sim} \frac{\rho}{1-\rho} \cdot \Pr[H_{\mathrm{e}} > x],$$

which shows that the actual waiting time W and sojourn time T are subexponential.

Remark 4.2 (*4.17*) *was obtained in the MAP/G/1 queue (i.e.,* $\boldsymbol{D}_k = \boldsymbol{O}$ *for all* $k \ge 2$*) under the assumption* $H_{\rm e} \in \mathcal{S}$ *[23, Corollary 2].*

Proof: Lemma 4.3 implies $\Pr[V_i > x] \overset{x}{\sim} \rho(1-\rho)^{-1}\Pr[H_{\rm e} > x]$ ($i \in \mathcal{M}$). Also, it follows from Proposition 2.1 and Lemma 4.4 that $\Pr[H > x] = o(\Pr[H_{\rm e} > x])$ and $\Pr[\widehat{X}_{i,j} > x] = o(\Pr[H_{\rm e} > x])$. Thus applying Proposition 2.2 (a) to (4.10) and (4.11) yields

$$(\overline{\boldsymbol{w}}(x))_j \overset{x}{\sim} (\bar{\boldsymbol{t}}(x))_j \overset{x}{\sim} \frac{1}{\lambda}\sum_{i\in\mathcal{M}}(\boldsymbol{\pi})_i(\boldsymbol{D})_{i,j}\mathrm{E}[G(i,j)]\frac{\rho}{1-\rho}\cdot\Pr[H_{\rm e} > x].$$

Substituting (4.5) into the above equation, we obtain (4.17). The other statement immediately follows from (4.17). □

The following corollary is an immediate consequence of Theorems 3.1 and 4.1, Remark 4.1, and Lemma 4.3.

Corollary 4.1 *Suppose Assumption 4.1 holds. If* $H \in \mathcal{L}^2 \cap \mathcal{S}$ *and* $H_{\rm e} \in \mathcal{S}$*,*

$$\Pr[L > k, S = j] \overset{k}{\sim} \Pr[\lambda V > k, S = j], \qquad j \in \mathcal{M}, \tag{4.18}$$

$$\Pr[L > k] \overset{k}{\sim} \Pr[\lambda W > k] \overset{k}{\sim} \Pr[\lambda T > k]. \tag{4.19}$$

When the arrival process is MAP (i.e., $\boldsymbol{D}_k = \boldsymbol{O}$ for all $k \ge 2$), Assumption 4.1 always holds. Further $\overline{\boldsymbol{D}}(x) = \boldsymbol{D}\cdot\overline{H}(x)$ and therefore we can exclude the condition $H \in \mathcal{S}$.

Corollary 4.2 *Consider the stationary FIFO MAP/GI/1 queue. If* $H \in \mathcal{L}^2$ *and* $H_{\rm e} \in \mathcal{S}$*,* (*4.18*) *and* (*4.19*) *hold.*

Remark 4.3 (*4.19*) *shows that the tail of the queue length* L *behaves asymptotically like that of the sojourn time* T *multiplied by the arrival rate* λ*, which may be considered as an asymptotic Little's law. Asmussen et al. [3, Proposition 3.12] shows that* (*4.19*) *holds in the GI/GI/1 queue, assuming that* $H_{\rm e} \in \mathcal{S}$ *and* $H_{\rm e}$ *satisfies a certain technical condition.*

4.2. Queue with heavy-tailed batch sizes

This subsection considers the FIFO BMAP/GI/1 queue with heavy-tailed batch sizes.

Lemma 4.5 *Let* $\{Z_n; n = 1, 2, \dots\}$ *denote a sequence of i.i.d. nonnegative r.v.s with positive finite mean and* N *denote a nonnegative integer-valued r.v. independent of* $\{Z_n\}$*. If* $\mathrm{E}[\exp(\phi\sqrt{Z_1})] < \infty$ *for some* $\phi > 0$ *and* $\Pr[N > k] \overset{k}{\sim} \kappa\Pr[J > k]$ *for some r.v.* $J \in \mathcal{L}^2$ *and some nonnegative constant* κ*, we have*

$$\Pr[Z_1 + Z_2 + \cdots + Z_N > x] \overset{k}{\sim} \kappa\Pr[\mathrm{E}[Z_1]J > x]. \tag{4.20}$$

Proof: We apply Lemma 2.1 to the random sum $\sum_{n=1}^{N} Z_n$. Consider the cumulative process $\{B(t)\}$ associated with the regenerative process with unit cycle lengths, where $B(t)$ is defined as a (nondecreasing) step function with jumps Z_n's at time n ($n = 1, 2, \dots$). We then have $B(N) = Z_1 + Z_2 + \cdots + Z_N$. In the framework of section 2.3, $B(N)$ can be described with $Y = N$, $F = J$, $B(t) = Z_0 + Z_1 + \cdots + Z_{\lfloor t\rfloor}$, $Z_0 = 0$, $\nu_0 = 0$, $\nu_n = 1$ ($n = 1, 2, \dots$), and $\gamma_n = Z_n \ge 0$ ($n = 0, 1, \dots$). It is clear that $\mathrm{E}[\nu_1^2] = 1 < \infty$, $0 < \mathrm{E}[\gamma_1] = \mathrm{E}[Z_1] < \infty$, and $b = \mathrm{E}[\gamma_1]/\mathrm{E}[\nu_1] = \mathrm{E}[Z_1] > 0$. Note also that $\gamma_0^* = 0$ and $\gamma_n^* = Z_n$ for $n = 1, 2, \dots$ and therefore

$\mathrm{E}[\exp(\phi\sqrt{\gamma_1^*})] = \mathrm{E}[\exp(\phi\sqrt{Z_1})] < \infty$, from which and $\gamma_1 \ge 0$ we have $\mathrm{E}[\gamma_1^2] = \mathrm{E}[Z_1^2] < \infty$ (see Remark 2.2). As a result, applying Lemma 2.1 to the above setting, we obtain

$$\Pr[Z_1 + \cdots + Z_N > \mathrm{E}[Z_1]y] \overset{y}{\sim} \kappa \Pr[J > y]. \tag{4.21}$$

Setting $x = \mathrm{E}[Z_1]y$ in (4.21) then yields (4.20). □

Assumption 4.2 *There exists some $\phi > 0$ such that $\int_0^\infty e^{\phi\sqrt{x}} \mathrm{d}H(x) < \infty$.*

Remark 4.4 *Assumption 3.3 is a sufficient condition of Assumption 4.2.*

Lemma 4.6 *Suppose Assumptions 3.2 and 4.2 hold. If $G \in \mathcal{L}^2$,*

$$\overline{\boldsymbol{D}}(x) \overset{x}{\sim} \widetilde{\boldsymbol{D}} \cdot \Pr[hG > x]. \tag{4.22}$$

Proof: Recall that $\Pr[G(i,j) = k] = \zeta_{k,i,j}$ $(k = 0, 1, \ldots)$. It then follows from Assumptions 3.2, (4.4) and (4.6) that

$$\Pr[G(i,j) > k] \overset{k}{\sim} \widetilde{\zeta}_{i,j} \Pr[G > k], \qquad i, j \in \mathcal{M}, \tag{4.23}$$

where $\widetilde{\zeta}_{i,j}$ $(i, j \in \mathcal{M})$ is defined as $\widetilde{\zeta}_{i,j} = (\widetilde{\boldsymbol{D}})_{i,j}/(\boldsymbol{D})_{i,j}$ if $(\boldsymbol{D})_{i,j} > 0$, and otherwise $\widetilde{\zeta}_{i,j} = 0$. We also have

$$(\widetilde{\boldsymbol{D}})_{i,j} = (\boldsymbol{D})_{i,j} \widetilde{\zeta}_{i,j}, \qquad i, j \in \mathcal{M}, \tag{4.24}$$

because $(\widetilde{\boldsymbol{D}})_{i,j} = 0$ if $(\boldsymbol{D})_{i,j} = 0$ (see Assumption 3.2). Applying Lemma 4.5 to the random sum $X_{i,j}$ in (4.14) and using (4.23) yield $\Pr[X_{i,j} > x] \overset{x}{\sim} \widetilde{\zeta}_{i,j} \Pr[hG > x]$, from which, (4.13), and (4.24), we obtain (4.22). □

Let $\Gamma = hG$. We then have $\Pr[\Gamma_{\mathrm{e}} > x] = \Pr[hG_{\mathrm{e}} > x]$, which implies that $\Gamma_{\mathrm{e}} \in \mathcal{S}$ if $G_{\mathrm{e}} \in \mathcal{S}$. Also, Lemma 4.6 shows that $\boldsymbol{C}_D$ in Proposition 4.1 is given by $h\mathrm{E}[G]\widetilde{\boldsymbol{D}}$. Thus the following lemma immediately follows from Proposition 4.1, Lemma 4.6, (3.8), and (3.9).

Lemma 4.7 *Suppose Assumptions 3.2 and 4.2 hold. If $G \in \mathcal{L}^2$ and $G_{\mathrm{e}} \in \mathcal{S}$,*

$$\overline{\boldsymbol{v}}(x) \overset{x}{\sim} \frac{\rho}{1-\rho}\boldsymbol{\pi} \cdot \Pr[hG_{\mathrm{e}} > x], \qquad \Pr[V > x] \overset{x}{\sim} \frac{\rho}{1-\rho} \cdot \Pr[hG_{\mathrm{e}} > x].$$

Theorem 4.2 *Suppose Assumptions 3.2 and 4.2 hold. If $G \in \mathcal{L}^2$ and $G_{\mathrm{e}} \in \mathcal{S}$,*

$$\overline{\boldsymbol{w}}(x) \overset{x}{\sim} \overline{\boldsymbol{t}}(x) \overset{x}{\sim} \left(\frac{h}{1-\rho}\boldsymbol{\pi} \sum_{k=1}^{\infty} k\boldsymbol{D}_k + \frac{1}{\lambda_G}\boldsymbol{\pi}\widetilde{\boldsymbol{D}} \right) \cdot \Pr[hG_{\mathrm{e}} > x], \tag{4.25}$$

$$\Pr[W > x] \overset{x}{\sim} \Pr[T > x] \overset{x}{\sim} \frac{1}{1-\rho} \Pr[hG_{\mathrm{e}} > x], \tag{4.26}$$

which shows that $W \in \mathcal{S}$ and $T \in \mathcal{S}$.

Proof: We prove only (4.25), which leads immediately to (4.26). Lemma 4.7 implies

$$\Pr[V_i > x] \overset{x}{\sim} \frac{\rho}{1-\rho} \Pr[hG_{\mathrm{e}} > x], \qquad i \in \mathcal{M}. \tag{4.27}$$

It follows from (3.8), (4.6), and (4.23) that $\Pr[G_{\rm e}(i,j) > x] \stackrel{x}{\sim} (\lambda/\lambda_G) g_{i,j}\widetilde{\zeta}_{i,j} \cdot \Pr[G_{\rm e} > x]$ $(i,j \in \mathcal{M})$, where $g_{i,j} = 1/{\rm E}[G(i,j)]$ if $(\boldsymbol{D})_{i,j} > 0$, and otherwise $g_{i,j} = 0$. Note that $G_{\rm e} \in \mathcal{L}^2$ because $G \in \mathcal{L}^2$ (see Lemma A.2). Thus applying Lemma 4.5 to $\widehat{X}_{i,j}$ in (4.9), we obtain

$$\Pr[\widehat{X}_{i,j} > x] \stackrel{x}{\sim} \frac{\lambda}{\lambda_G} g_{i,j}\widetilde{\zeta}_{i,j} \cdot \Pr[hG_{\rm e} > x], \qquad i,j \in \mathcal{M}. \tag{4.28}$$

Because $hG_{\rm e} \in \mathcal{S}$, applying Proposition 2.2 (a) to (4.10) and using (4.27) and (4.28) yield

$$\begin{aligned}(\overline{\boldsymbol{w}}(x))_j &\stackrel{x}{\sim} \frac{1}{\lambda}\sum_{i\in\mathcal{M}} (\boldsymbol{\pi})_i (\boldsymbol{D})_{i,j} {\rm E}[G(i,j)] \left[\frac{\rho}{1-\rho} + \frac{\lambda}{\lambda_G} g_{i,j}\widetilde{\zeta}_{i,j}\right] \Pr[hG_{\rm e} > x] \\ &= \frac{1}{\lambda}\sum_{i\in\mathcal{M}} (\boldsymbol{\pi})_i \left[\frac{\rho}{1-\rho}\sum_{k=1}^{\infty} k(\boldsymbol{D}_k)_{i,j} + \frac{\lambda}{\lambda_G}(\widetilde{\boldsymbol{D}})_{i,j}\right] \Pr[hG_{\rm e} > x],\end{aligned} \tag{4.29}$$

where the last equality follows from (4.5), (4.24), and the fact that $(\widetilde{\boldsymbol{D}})_{i,j} = 0$ if $(\boldsymbol{D})_{i,j} = 0$. Consequently, (4.29) leads to the result for $\overline{\boldsymbol{w}}(x)$ in (4.25).

The rest is to show $\overline{\boldsymbol{t}}(x) \stackrel{x}{\sim} \overline{\boldsymbol{w}}(x)$. Because $G \in \mathcal{L}^2$, Lemma A.2 implies $hG_{\rm e} \in \mathcal{L}^2$, and thus it follows from Assumption 4.2 and Proposition A.2 that $\Pr[H > x] = o(\Pr[hG_{\rm e} > x])$. As a result, Proposition 2.2 (a) and $hG_{\rm e} \in \mathcal{S}$ show that H on the right hand side of (4.11) has no contribution to the limit $\lim_{x\to\infty} \overline{\boldsymbol{t}}(x)/\Pr[hG_{\rm e} > x]$, i.e., $\overline{\boldsymbol{t}}(x) \stackrel{x}{\sim} \overline{\boldsymbol{w}}(x)$. □

From Theorems 3.2 and 4.2, Remark 4.4, and Lemma 4.7, we readily obtain the following.

Corollary 4.3 *Suppose Assumptions 3.2 and 3.3 hold. If $G \in \mathcal{L}^2 \cap \mathcal{S}$ and $G_{\rm e} \in \mathcal{S}$,*

$$\begin{gathered}\Pr[hL > k, S = j] \stackrel{k}{\sim} \Pr[V > k, S = j], \qquad j \in \mathcal{M}, \\ \Pr[hL > k] \stackrel{k}{\sim} \rho\Pr[W > k] \stackrel{k}{\sim} \rho\Pr[T > k].\end{gathered}$$

5. Concluding Remarks

In this paper, we considered the tail asymptotics of the queue length and waiting time distributions in the stationary FIFO BMAP/GI/1 queue. In particular, we considered two cases: heavy-tailed service times and heavy-tailed batch sizes. In each case, we derived sufficient conditions under which the stationary queue length and waiting time distributions are subexponential. Both these distributions are also square-root insensitive due to Lemmas A.1 and A.2. Further, we obtained the asymptotic relationship between the queue length and waiting time distributions in each case. To the best of our knowledge, this is the first paper that reports on the queue length and waiting time asymptotics in queues with batch arrivals.

We conclude this paper by some comments on the inclusion relationship between $F \in \mathcal{L}^2$ and $F_{\rm e} \in \mathcal{S}$, which appeared in our subexponential asymptotic conditions on the service times and batch sizes (see Theorems 3.1 and 4.2). Note first that $F_{\rm e} \in \mathcal{S}$ does not imply $F \in \mathcal{L}$, and vice versa [7, 20]. Thus $F \in \mathcal{L}^2$ $(\subset \mathcal{L})$ and $F_{\rm e} \in \mathcal{S}$ are substantially different conditions. Nevertheless, there exists an intersection between class $\mathcal{L}^2$ and a rich subclass $\mathcal{S}^*$ of $\mathcal{S}$, in which $F, F_{\rm e} \in \mathcal{S}$ [7, 8, 12]. It is known that $F \in \mathcal{L}^2$ implies that its tail distribution $\overline{F}(x)$ is heavier than $e^{-\varepsilon\sqrt{x}}$ for any $\varepsilon > 0$ (see Proposition A.2). We can confirm that $\mathcal{L}^2$ includes typical distributions in $\mathcal{S}^*$, e.g., Pareto, heavy-tailed Weibull, lognormal, Benktander-type-I and type-II, Burr, and loggamma distributions, if they have positive finite mean and heavier tails than $e^{-\varepsilon\sqrt{x}}$. This fact also implies that $(\mathcal{L} - \mathcal{L}^2) \cap \mathcal{S}$ is not empty. For example, heavy-tailed Weibull with shape parameter β ($1/2 \le \beta < 1$) is in class $\mathcal{S}^*(\subset \mathcal{S})$ but not square-root insensitive. On the other hand, we can construct square-root insensitive distributions whose equilibrium distributions are not subexponential. For example, consider a continuous, nonnegative r.v. F such that the hazard rate function $q_{F_{\rm e}}(x)$ of $F_{\rm e}$ is given by

$$q_{F_{\rm e}}(x) = \frac{\Pr[F > x]}{\mathrm{E}[F]\Pr[F_{\rm e} > x]} = \frac{1}{n\sqrt{x_n}}, \qquad x_{n-1} < x < x_n,$$

where $\{x_n;\ n = 0, 1, \dots\}$ satisfies $x_0 = 0$, $x_1 = 1$, and $x_n - x_{n-1} = 2n\sqrt{x_n}/\Pr[F_{\rm e} > x_{n-1}]$ ($n = 2, 3, \dots$). We can show that $F \in \mathcal{L}^2$ but $F_{\rm e} \notin \mathcal{S}$ in a way similar to the argument at p. 343 in [19].

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A. Properties of Square-root Insensitive Distributions

This appendix summarizes some properties of the square-root insensitive class.

Proposition A.1 (Lemma 1 in [11]) *F is square-root insensitive if and only if $\sqrt{F} \in \mathcal{L}$.*

Proposition A.2 (Lemma 2 in [11]) *For any $F \in \mathcal{L}^2$,*

$$\lim_{x\to\infty} \frac{\exp(-\varepsilon\sqrt{x})}{\Pr[F > x]} = 0, \qquad \forall\varepsilon > 0.$$

Proposition A.3 (Remark 1 in [11]) *$F \in \mathcal{L}^2$ if and only if $\Pr[F > x - \xi\sqrt{x}] \overset{x}{\sim} \Pr[F > x]$ for all $\xi \in (-\infty, \infty)$.*

Proposition A.3 was stated without proof in [11]. Thus for completeness, we provide its proof in Appendix B.

Lemma A.1 *A nonnegative r.v. Y is square-root insensitive if $\Pr[Y > x] \overset{x}{\sim} \kappa \Pr[F > x]$ for some r.v. $F \in \mathcal{L}^2$ and some positive constant κ.*

Proof: The lemma follows from

$$\lim_{x\to\infty} \frac{\Pr[Y > x - \sqrt{x}]}{\Pr[Y > x]} = \lim_{x\to\infty} \frac{\Pr[Y > x - \sqrt{x}]}{\Pr[F > x - \sqrt{x}]} \frac{\Pr[F > x - \sqrt{x}]}{\Pr[F > x]} \frac{\Pr[F > x]}{\Pr[Y > x]} = \kappa \cdot 1 \cdot \kappa^{-1}.$$

□

Lemma A.2 *If $F \in \mathcal{L}^2$ and $\mathrm{E}[F] < \infty$, $aF_{\rm e} \in \mathcal{L}^2$ for any positive constant a.*

Proof: Using l'Hospital's rule, we have

$$\lim_{x\to\infty}\frac{\Pr[aF_{\rm e} > x-\sqrt{x}]}{\Pr[aF_{\rm e} > x]} = \lim_{x\to\infty}\frac{\overline{F}_{\rm e}(x/a-\sqrt{x}/a)}{\overline{F}_{\rm e}(x/a)} = \lim_{x\to\infty}\frac{\overline{F}(x/a-\sqrt{x}/a)}{\overline{F}(x/a)}\left(1-\frac{1}{2\sqrt{x}}\right) = 1,$$

where we use Proposition A.3 in the last equality. □

B. Proof of Proposition A.3

From Definition 2.3, the if part is obvious. Thus we assume $F \in \mathcal{L}^2$ and prove the only-if part. Note that for any real number ξ, there exists a nonnegative integer k such that $-2^k \le \xi \le 2^k$, and hence

$$\frac{\Pr[F > x+2^k\sqrt{x}]}{\Pr[F > x]} \le \frac{\Pr[F > x-\xi\sqrt{x}]}{\Pr[F > x]} \le \frac{\Pr[F > x-2^k\sqrt{x}]}{\Pr[F > x]}.$$

Therefore it suffices to show that for any nonnegative integer k,

$$\Pr[F > x+2^k\sqrt{x}] \overset{x}{\sim} \Pr[F > x], \qquad \Pr[F > x-2^k\sqrt{x}] \overset{x}{\sim} \Pr[F > x]. \tag{B.1}$$

We first prove the first limit in (B.1) for $k=0$. For $x \ge 0$, let y_0 denote a real number such that $y_0 \ge x$ and $x = y_0 - \sqrt{y_0} \ge 0$. Note that given $x \ge 0$, y_0 is uniquely determined. We then have

$$1 \ge \frac{\Pr[F > x+\sqrt{x}]}{\Pr[F > x]} \ge \frac{\Pr[F > x+\sqrt{y_0}]}{\Pr[F > x]} = \frac{\Pr[F > y_0]}{\Pr[F > y_0-\sqrt{y_0}]}.$$

Because $y_0 \to \infty$ as $x \to \infty$ and $F \in \mathcal{L}^2$, we obtain $\Pr[F > x+\sqrt{x}] \overset{x}{\sim} \Pr[F > x]$, which shows that the first limit in (B.1) holds for $k=0$. On the other hand, it is obvious from the definition of $\mathcal{L}^2$ that the second limit in (B.1) holds for $k=0$.

We now assume that (B.1) holds for some $k = n \ge 0$. It then follows that

$$\Pr[F > x+2^n\sqrt{x}] \overset{x}{\sim} \Pr[F > x-2^n\sqrt{x}]. \tag{B.2}$$

For $x \ge 0$, let y_n denote a real number such that $y_n \ge x$ and $x = y_n - 2^n\sqrt{y_n} \ge 0$. Note that given $x \ge 0$, y_n is uniquely determined and that $y_n \to \infty$ as $x \to \infty$. We then have

$$1 \ge \frac{\Pr[F > x+2^{n+1}\sqrt{x}]}{\Pr[F > x]} \ge \frac{\Pr[F > x+2^{n+1}\sqrt{y_n}]}{\Pr[F > x]} = \frac{\Pr[F > y_n+2^n\sqrt{y_n}]}{\Pr[F > y_n-2^n\sqrt{y_n}]},$$

from which and (B.2), we obtain $\Pr[F > x+2^{n+1}\sqrt{x}] \overset{x}{\sim} \Pr[F > x]$. Therefore the first limit in (B.1) holds for $k=n+1$.

From the above discussion, we have

$$\Pr[F > x-2^n\sqrt{x}] \overset{x}{\sim} \Pr[F > x+2^{n+1}\sqrt{x}]. \tag{B.3}$$

Solving $x = z_{n+1} + 2^{n+1}\sqrt{z_{n+1}}$ ($z_{n+1} \ge 0$) with respect to $\sqrt{z_{n+1}}$ yields $\sqrt{z_{n+1}} = -2^n + \sqrt{2^{2n}+x} \ge -2^n + \sqrt{x}$, from which it follows that $3\cdot 2^n\sqrt{z_{n+1}} \ge 2^n\{-3\cdot 2^n + (2+1)\sqrt{x}\} \ge 2^{n+1}\sqrt{x}$ for $x \ge (3\cdot 2^n)^2$. Thus we obtain for $x \ge (3\cdot 2^n)^2$,

$$1 \le \frac{\Pr[F > x-2^{n+1}\sqrt{x}]}{\Pr[F > x]} \le \frac{\Pr[F > x-3\cdot 2^n\sqrt{z_{n+1}}]}{\Pr[F > x]} = \frac{\Pr[F > z_{n+1}-2^n\sqrt{z_{n+1}}]}{\Pr[F > z_{n+1}+2^{n+1}\sqrt{z_{n+1}}]}.$$

Noting $z_{n+1} \to \infty$ as $x \to \infty$ and using (B.3), we obtain $\Pr[F > x-2^{n+1}\sqrt{x}] \overset{x}{\sim} \Pr[F > x]$, which shows that the second limit in (B.1) holds for $k=n+1$. □

C. Proof of Lemma 2.1

Because $\Pr[U(Y) > bx] \ge \Pr[B(Y) > bx]$, it suffices to show

$$\limsup_{x\to\infty} \frac{\Pr[U(Y) > bx]}{\Pr[F > x]} \le \kappa, \qquad \liminf_{x\to\infty} \frac{\Pr[B(Y) > bx]}{\Pr[F > x]} \ge \kappa. \tag{C.1}$$

In what follows, we prove the first and second inequalities in (C.1) separately, using the following results.

Let $Q_X(x)$ ($x \ge 0$) denote the integrated hazard function of a nonnegative r.v. X, i.e., $Q_X(x) = -\log(\Pr[X > x])$. X belongs to class $\mathcal{SC}$ (*subexponential concave*) if (i) $Q_X(x)$ is eventually concave, (ii) $\lim_{x\to\infty} Q_X(x)/\log x = \infty$, and (iii) there exists $x_0 > 0$, $0 < \alpha < 1$, and $0 < \beta < 1$ such that $[Q_X(x) - Q_X(u)]/Q_X(x) \le \alpha(x-u)/x$ for all $x \ge x_0$ and $\beta x \le u \le x$ (see [11, 16]).

Proposition C.1 (Proposition 1 in [11]) *Consider the cumulative process $\{B(t)\}$ introduced in section 2.3. If $\mathrm{E}[\nu_1^2] < \infty$, $\gamma_1 \ge 0$ w.p.1 and $\mathrm{E}[\exp(Q_X(\gamma_i^*))] < \infty$ ($i = 0,1$) for some nonnegative r.v. $X \in \mathcal{SC}$, there exist positive constants C and c such that*

$$\Pr\left[\sup_{0\le t\le x} \{B(t) - bt\} > u\right] \le C\left(e^{-cu^2/x} + e^{-cx} + xe^{-cQ_X(u)}\right), \qquad \forall x \ge 0,\ \forall u \ge 0.$$

Proposition C.2 *For any $F \in \mathcal{L}$ and $\varepsilon > 0$, there exists some $x_0 := x_0(\varepsilon) > 0$ such that*

$$\Pr[F > x - u] \le \Pr[F > x]e^{\varepsilon(u+1)}, \tag{C.2}$$

for all $x > u + x_0$ and $u \ge 0$, where x_0 is independent of u.

Proof: The proof of Proposition C.2 is given in Appendix G. □

C.1. Proof of the first inequality in (C.1)

Let δ and ξ denote fixed real numbers such that $0 < \delta < 1$ and $\xi > 1$, respectively, and assume $x > \xi^2/(1-\delta)^2$. We then have $0 < \delta x < x - \xi\sqrt{x}$ and therefore

$$\begin{aligned}\Pr[U(Y) > bx] &= \Pr[U(Y) > bx, Y > x - \xi\sqrt{x}] \\ &\quad + \Pr[U(Y) > bx, \delta x < Y \le x - \xi\sqrt{x}] + \Pr[U(Y) > bx, Y \le \delta x] \\ &\le \Pr[Y > x - \xi\sqrt{x}] \\ &\quad + \Pr[U(Y) > bx, \delta x < Y \le x - \xi\sqrt{x}] + \Pr[U(\delta x) > bx]. \end{aligned} \tag{C.3}$$

Note here that

$$\lim_{x\to\infty} \frac{\Pr[Y > x - \xi\sqrt{x}]}{\Pr[F > x]} = \lim_{x\to\infty} \frac{\Pr[Y > x - \xi\sqrt{x}]}{\Pr[F > x - \xi\sqrt{x}]} \cdot \frac{\Pr[F > x - \xi\sqrt{x}]}{\Pr[F > x]} = \kappa \cdot 1 = \kappa,$$

where we use (2.7) and Proposition A.3 in the second equality. Thus it suffices to show that the second and third terms on the right hand side of (C.3) are $o(\Pr[F > x])$.

We start with the third term. Note first that

$$\begin{aligned}\Pr[U(\delta x) > bx] &= \Pr\left[\sup_{0\le t\le \delta x} B(t) > bx\right] = \Pr\left[\sup_{0\le t\le \delta x} B(t) - \sup_{0\le t\le \delta x} bt > bx - b\delta x\right] \\ &\le \Pr\left[\sup_{0\le t\le \delta x} (B(t) - bt) > (1-\delta)bx\right]. \end{aligned} \tag{C.4}$$

Let $Q(x) = \phi\sqrt{x}$ for $x \ge 0$, where $\phi > 0$. The assumption of the existence of ϕ in Lemma 2.1 is then rewritten to be $\mathrm{E}[\exp(Q(\gamma_n^*))] < \infty$. Because $Q(x) = \phi\sqrt{x}$ is considered as the integrated hazard function of a r.v. that belongs to class $\mathcal{SC}$ (see p. 101 in [11]), we can apply Proposition C.1 to (C.4), which ensures the existence of positive constants C and c such that

$$\Pr[U(\delta x) > bx] \le C\left(e^{-c\{(1-\delta)b\}^2 x/\delta} + e^{-c\delta x} + \delta x e^{-c\phi\sqrt{(1-\delta)bx}}\right).$$

Therefore we have

$$\Pr[U(\delta x) > bx] \le C'\left(e^{-c'x} + xe^{-c'\sqrt{x}}\right),$$

where $C' = 2C$ and $c' = c\min\left\{\{(1-\delta)b\}^2/\delta,\ \delta,\ \phi\sqrt{(1-\delta)b}\right\}$. Proposition A.2 yields $e^{-c'x} = o(\Pr[F > x])$ and $xe^{-c'\sqrt{x}} = e^{-c'\sqrt{x}+2\log\sqrt{x}} = e^{-c'\sqrt{x}+o(\sqrt{x})} = o(\Pr[F > x])$. Consequently, we conclude $\Pr[U(\delta x) > bx] = o(\Pr[F > x])$.

Next we consider the second term on the right hand side of (C.3). Because $U(u) - bu = \sup_{0\le t\le u} B(t) - bu \le \sup_{0\le t\le u}(B(t) - bt)$ for $u \ge 0$, we have

$$\Pr[U(Y) > bx, \delta x < Y \le x - \xi\sqrt{x}] \le \int_{\delta x}^{x-\xi\sqrt{x}} \Pr\left[\sup_{0\le t\le u}(B(t) - bt) > b(x-u)\right] \mathrm{d}Y(u),$$

where $Y(x) = \Pr[Y \le x]$ for $x \ge 0$. Applying Proposition C.1 to the right hand side of the above inequality, we have for some positive constants C and c,

$$\begin{aligned}
&\Pr[U(Y) > bx, \delta x < Y \le x - \xi\sqrt{x}] \\
&\qquad \le \int_{\delta x}^{x-\xi\sqrt{x}} C\left(e^{-c\{b(x-u)\}^2/u} + e^{-cu} + ue^{-c\phi\sqrt{b(x-u)}}\right)\mathrm{d}Y(u) \\
&\qquad = C\{f_1(x) + f_2(x) + f_3(x)\},
\end{aligned}$$

where $f_i(x)$'s $(i = 1, 2, 3)$ are defined as

$$\begin{aligned}
f_1(x) &= \int_{\delta x}^{x-\xi\sqrt{x}} e^{-c_1(x-u)^2/u}\mathrm{d}Y(u), \qquad f_2(x) = \int_{\delta x}^{x-\xi\sqrt{x}} e^{-cu}\mathrm{d}Y(u), \\
f_3(x) &= \int_{\delta x}^{x-\xi\sqrt{x}} ue^{-c_3\sqrt{x-u}}\mathrm{d}Y(u),
\end{aligned}$$

respectively, with $c_1 = cb^2$ and $c_3 = c\phi\sqrt{b}$. Proposition A.2 implies that $f_2(x) \le e^{-c\delta x} = o(\Pr[F > x])$. Therefore, it suffices to show $f_1(x) = o(\Pr[F > x])$ and $f_3(x) = o(\Pr[F > x])$.

We first consider $f_3(x)$. Letting $\hat{c} = c_3/2$, we have

$$f_3(x) \le x\int_{\delta x}^{x-\xi\sqrt{x}} e^{-c_3\sqrt{x-u}}\mathrm{d}Y(u) \le xe^{-\hat{c}x^{\frac{1}{4}}} \int_{\delta x}^{x-\xi\sqrt{x}} e^{-\hat{c}\sqrt{x-u}}\mathrm{d}Y(u), \tag{C.5}$$

where the second inequality holds because $e^{-c_3\sqrt{x-u}} = e^{-\hat{c}\sqrt{x-u}}e^{-\hat{c}\sqrt{x-u}} \le e^{-\hat{c}x^{\frac{1}{4}}}e^{-\hat{c}\sqrt{x-u}}$ for $0 \le u \le x - \xi\sqrt{x}$ and $\xi > 1$. Let $\mathit{\Pi}_3$ denote a nonnegative r.v. independent of Y, whose distribution function is given by $\Pr[\mathit{\Pi}_3 \le x] = 1 - \exp(-\hat{c}\sqrt{x})$ $(x \ge 0)$. We then have from (C.5),

$$f_3(x) \le xe^{-\hat{c}x^{\frac{1}{4}}} \int_{\delta x}^{x-\xi\sqrt{x}} \Pr[\mathit{\Pi}_3 > x - u]\mathrm{d}Y(u) \le xe^{-\hat{c}x^{\frac{1}{4}}}\Pr[Y + \mathit{\Pi}_3 > x]. \tag{C.6}$$

Further we obtain

$$\begin{aligned}\Pr[Y+\Pi_3>x] &\le \Pr[\Pi_3>x/4]+\Pr[Y>x-\Pi_3,\Pi_3\le x/4]\\ &= \Pr[\Pi_3>x/4]+\int_0^{x/4}\Pr[\sqrt{Y}>\sqrt{x-u}]\,\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u\\ &\le \Pr[\Pi_3>x/4]+\int_0^{x/4}\Pr[\sqrt{Y}>\sqrt{x}-\sqrt{u}]\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u, \qquad (C.7)\end{aligned}$$

where we use $\sqrt{x-u}\ge\sqrt{x}-\sqrt{u}$ for $0\le u\le x$ in the last inequality. Using Proposition A.2, the first term of the right hand side of (C.7) can be evaluated as $\Pr[\Pi_3>x/4]=\exp[-\hat{c}\sqrt{x}/2]=o(\Pr[F>x])$. Applying (2.7) and $\sqrt{x}-\sqrt{u}\ge\sqrt{x}/2$ $(0\le\forall u\le x/4)$ to the second term of the right hand side of (C.7), we obtain

$$\begin{aligned}&\limsup_{x\to\infty}\frac{\displaystyle\int_0^{x/4}\Pr[\sqrt{Y}>\sqrt{x}-\sqrt{u}]\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u}{\Pr[F>x]}\\ &\quad= \limsup_{x\to\infty}\int_0^{x/4}\frac{\Pr[\sqrt{Y}>\sqrt{x}-\sqrt{u}]}{\Pr[\sqrt{F}>\sqrt{x}-\sqrt{u}]}\frac{\Pr[\sqrt{F}>\sqrt{x}-\sqrt{u}]}{\Pr[\sqrt{F}>\sqrt{x}]}\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u\\ &\quad\le \limsup_{x\to\infty}\sup_{y\ge\sqrt{x}/2}\frac{\Pr[\sqrt{Y}>y]}{\Pr[\sqrt{F}>y]}\int_0^{x/4}\frac{\Pr[\sqrt{F}>\sqrt{x}-\sqrt{u}]}{\Pr[\sqrt{F}>\sqrt{x}]}\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u\\ &\quad= \kappa\limsup_{x\to\infty}\int_0^{x/4}\frac{\Pr[\sqrt{F}>\sqrt{x}-\sqrt{u}]}{\Pr[\sqrt{F}>\sqrt{x}]}\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u. \qquad (C.8)\end{aligned}$$

Noting $\sqrt{F}\in\mathcal{L}$, it follows from Proposition C.2 that for any $0<\varepsilon<\hat{c}$,

$$\limsup_{x\to\infty}\int_0^{x/4}\frac{\Pr[\sqrt{F}>\sqrt{x}-\sqrt{u}]}{\Pr[\sqrt{F}>\sqrt{x}]}\frac{\hat{c}}{2\sqrt{u}}e^{-\hat{c}\sqrt{u}}\mathrm{d}u<e^{\varepsilon}\int_0^{\infty}\frac{\hat{c}}{2\sqrt{u}}e^{-(\hat{c}-\varepsilon)\sqrt{u}}\mathrm{d}u<\infty. \qquad (C.9)$$

As a result, the left hand side of (C.7) can be evaluated in the following way.

$$\limsup_{x\to\infty}\frac{\Pr[Y+\Pi_3>x]}{\Pr[F>x]}<\infty. \qquad (C.10)$$

Applying (C.10) and $\lim_{x\to\infty}x\exp(-\widehat{c}x^{1/4})=0$ to (C.6) yields $f_3(x)=o(\Pr[F>x])$.

Finally we consider $f_1(x)$. Note that for $0<u\le x-\xi\sqrt{x}$, $e^{-c_1(x-u)^2/u}\le e^{-c_1(x-u)^2/x}=e^{-\tilde{c}(x-u)^2/x}e^{-\tilde{c}(x-u)^2/x}\le e^{-\tilde{c}\xi^2}e^{-\tilde{c}(x-u)^2/x}$, where $\tilde{c}=c_1/2$. Thus from the definition of $f_1(x)$, we have

$$f_1(x)\le e^{-\tilde{c}\xi^2}\int_{\delta x}^{x-\xi\sqrt{x}}e^{-\tilde{c}(x-u)^2/x}\mathrm{d}Y(u). \qquad (C.11)$$

Let Π_1 denote a nonnegative r.v. independent of Y, whose distribution function is given by $\Pr[\Pi_1\le x]=1-e^{-\tilde{c}x^2}$ $(x\ge0)$. It then follows that $e^{-\tilde{c}(x-u)^2/x}=\Pr[\sqrt{x}\Pi_1>x-u]$. Substituting this into (C.11), we have

$$f_1(x)\le e^{-\tilde{c}\xi^2}\int_{\delta x}^{x-\xi\sqrt{x}}\Pr[\sqrt{x}\Pi_1>x-u]\mathrm{d}Y(u)\le e^{-\tilde{c}\xi^2}\Pr[Y+\sqrt{x}\Pi_1>x]. \qquad (C.12)$$

Further we have

$$\begin{aligned}\Pr[Y + \sqrt{x}\mathit{\Pi}_1 > x] &\le \Pr[\mathit{\Pi}_1 > \sqrt{x}/2] + \Pr[Y > x - \mathit{\Pi}_1\sqrt{x}, \mathit{\Pi}_1 \le \sqrt{x}/2] \\ &\le o(\Pr[F > x]) + \int_0^{\sqrt{x}/2} \Pr[\sqrt{Y} > \sqrt{x} - u]\, 2\tilde{c}ue^{-\tilde{c}u^2}\mathrm{d}u, \quad \text{(C.13)}\end{aligned}$$

where we use Proposition A.2 and $\sqrt{x - u\sqrt{x}} \ge \sqrt{x} - u$ for all $u \le \sqrt{x}$ in the second inequality. According to reasoning similar to the derivation of (C.8) and (C.9) with $\sqrt{x} - u \ge \sqrt{x}/2$ ($\forall u \le \sqrt{x}/2$), we obtain for any $\varepsilon > 0$,

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\displaystyle\int_0^{\sqrt{x}/2} \Pr[\sqrt{Y} > \sqrt{x} - u]\, 2\tilde{c}ue^{-\tilde{c}u^2}\mathrm{d}u}{\Pr[F > x]} \\ &\le \kappa \limsup_{x\to\infty} \int_0^{\sqrt{x}/2} \frac{\Pr[\sqrt{F} > \sqrt{x} - u]}{\Pr[\sqrt{F} > \sqrt{x}]} 2\tilde{c}ue^{-\tilde{c}u^2}\mathrm{d}u \le \kappa e^{\varepsilon} \int_0^{\infty} 2\tilde{c}ue^{-\tilde{c}u^2+\varepsilon u}\mathrm{d}u < \infty,\end{aligned}$$

from which and (C.13) it follows that

$$\limsup_{x\to\infty} \frac{\Pr[Y + \sqrt{x}\mathit{\Pi}_1 > x]}{\Pr[F > x]} < \infty. \quad \text{(C.14)}$$

Therefore combining (C.14) with (C.12) and letting $\xi \to \infty$ yield $f_1(x) = o(\Pr[F > x])$. □

C.2. Proof of the second inequality in (C.1)

Let ξ denote a fixed positive real number and assume $x > \xi^2$. We then have

$$\begin{aligned}\Pr[B(Y) > bx] &\ge \int_{x+\xi\sqrt{x}}^{\infty} \Pr[B(u) > bx]\mathrm{d}Y(u) \ge \inf_{u > x+\xi\sqrt{x}} \Pr[B(u) > bx] \cdot \Pr[Y > x + \xi\sqrt{x}] \\ &= \inf_{u > x+\xi\sqrt{x}} \Pr\left[\frac{B(u) - bu}{\sqrt{u}} > \frac{b(x - u)}{\sqrt{u}}\right] \Pr[Y > x + \xi\sqrt{x}]. \quad \text{(C.15)}\end{aligned}$$

For any $x > 0$, $b(x - u)/\sqrt{u}$ is a nonincreasing function of $u > 0$ because $b > 0$. It thus follows from (C.15) that for any $x > \xi^2$,

$$\begin{aligned}\Pr[B(Y) > bx] &\ge \inf_{u > x+\xi\sqrt{x}} \Pr\left[\frac{B(u) - bu}{\sqrt{u}} > \frac{-b\xi\sqrt{x}}{\sqrt{x + \xi\sqrt{x}}}\right] \Pr[Y > x + \xi\sqrt{x}] \\ &= \inf_{u > x+\xi\sqrt{x}} \Pr\left[\frac{B(u) - bu}{\sqrt{u}} > \frac{-b\xi}{\sqrt{1 + \xi/\sqrt{x}}}\right] \Pr[Y > x + \xi\sqrt{x}] \\ &\ge \inf_{u > x+\xi\sqrt{x}} \Pr\left[\frac{B(u) - bu}{\sqrt{u}} > \frac{-b\xi}{\sqrt{2}}\right] \Pr[Y > x + \xi\sqrt{x}], \quad \text{(C.16)}\end{aligned}$$

where we use $\sqrt{1 + \xi/\sqrt{x}} < \sqrt{2}$ in the last inequality. Note here that when $\mathrm{Var}[\nu_1] < \infty$ and $\mathrm{Var}[\gamma_1] < \infty$, there exists some $\sigma > 0$ such that

$$\lim_{t\to\infty} \Pr\left[\frac{B(t) - bt}{\sigma\sqrt{t}} \le x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\mathrm{d}y, \qquad x \in (-\infty, \infty), \quad \text{(C.17)}$$

where $b = \mathrm{E}[\gamma_1]/\mathrm{E}[\nu_1]$ [1, Theorem 3.2, Chapter VI]. Applying (C.17) and Proposition A.3 to (C.16), we have

$$\begin{aligned}\liminf_{x\to\infty} \frac{\Pr[B(Y) > bx]}{\Pr[F > x]} &\geq \liminf_{x\to\infty} \inf_{u > x+\xi\sqrt{x}} \Pr\left[\frac{B(u) - bu}{\sigma\sqrt{u}} > \frac{-b\xi}{\sigma\sqrt{2}}\right] \\ &\quad \cdot \frac{\Pr[F > x + \xi\sqrt{x}]}{\Pr[F > x]} \frac{\Pr[Y > x + \xi\sqrt{x}]}{\Pr[F > x + \xi\sqrt{x}]} = \frac{1}{\sqrt{2\pi}} \int_{\frac{-b\xi}{\sigma\sqrt{2}}}^{\infty} e^{-\frac{y^2}{2}} \mathrm{d}y \cdot 1 \cdot \kappa.\end{aligned}$$

Consequently, letting $\xi \to \infty$ yields the second equation in (C.1). □

D. Proof of Lemma 3.2

As in Appendix C.1, let δ and ξ denote fixed real numbers such that $0 < \delta < 1$ and $\xi > 1$, respectively, and assume $x > \xi^2/(1-\delta)^2$. We also fix $i \in \mathcal{M}$ arbitrarily. It then follows that

$$\begin{aligned}\mathrm{P}_i(N(H) > \lambda x, S(H) = j) &= \mathrm{P}_i(N(H) > \lambda x, S(H) = j, H > x - \xi\sqrt{x}) \\ &\quad + \mathrm{P}_i(N(H) > \lambda x, S(H) = j, \delta x < H \leq x - \xi\sqrt{x}) + \mathrm{P}_i(N(H) > \lambda x, S(H) = j, H \leq \delta x) \\ &\leq \mathrm{P}_i(H > x - \xi\sqrt{x}, S(H) = j) + \mathrm{P}_i(N(H) > \lambda x, \delta x < H \leq x - \xi\sqrt{x}) + \mathrm{P}_i(N(\delta x) > \lambda x).\end{aligned} \tag{D.1}$$

In the same way as in Appendix C.1, we can show that

$$\mathrm{P}_i(N(H) > \lambda x, \delta x < H \leq x - \xi\sqrt{x}) = o(\Pr[H > x]), \qquad \mathrm{P}_i(N(\delta x) > \lambda x) = o(\Pr[H > x]).$$

As for the first term on the right hand side of (D.1), Proposition A.3 yields

$$\begin{aligned}\mathrm{P}_i(H > x - \xi\sqrt{x}, S(H) = j) &= \mathrm{P}_i(H > x - \xi\sqrt{x})\mathrm{P}_i(S(H) = j \mid H > x - \xi\sqrt{x}) \\ &= \Pr[H > x - \xi\sqrt{x}]\mathrm{P}_i(S(H) = j \mid H > x - \xi\sqrt{x}) \\ &\overset{x}{\sim} \Pr[H > x](\boldsymbol{\pi})_j,\end{aligned}$$

for any $j \in \mathcal{M}$ because $H \in \mathcal{L}^2$ and $\lim_{t\to\infty} \mathrm{P}_i(S(t) = j) = (\boldsymbol{\pi})_j$ ($\forall j \in \mathcal{M}$). As a result,

$$\limsup_{x\to\infty} \frac{\mathrm{P}_i(N(H) > \lambda x, S(H) = j)}{\Pr[H > x]} \leq (\boldsymbol{\pi})_j, \qquad j \in \mathcal{M}. \tag{D.2}$$

On the other hand, it follows from (D.2), Lemma 3.1, and $\sum_{\nu\in\mathcal{M}} \mathrm{P}_i(S(H) = \nu \mid N(H) > \lambda x) = 1$ ($\forall x \geq 0$) that

$$\begin{aligned}&\liminf_{x\to\infty} \frac{\mathrm{P}_i(N(H) > \lambda x, S(H) = j)}{\Pr[H > x]} \\ &= \liminf_{x\to\infty} \frac{\mathrm{P}_i(N(H) > \lambda x, S(H) = j)}{\mathrm{P}_i(N(H) > \lambda x)} \frac{\mathrm{P}_i(N(H) > \lambda x)}{\Pr[H > x]} = \liminf_{x\to\infty} \frac{\mathrm{P}_i(N(H) > \lambda x, S(H) = j)}{\mathrm{P}_i(N(H) > \lambda x)} \\ &= \liminf_{x\to\infty} \mathrm{P}_i(S(H) = j \mid N(H) > \lambda x) = \liminf_{x\to\infty} \left(1 - \sum_{\substack{\nu\in\mathcal{M} \\ \nu\neq j}} \mathrm{P}_i(S(H) = \nu \mid N(H) > \lambda x)\right) \\ &\geq 1 - \sum_{\substack{\nu\in\mathcal{M} \\ \nu\neq j}} \limsup_{x\to\infty} \mathrm{P}_i(S(H) = \nu \mid N(H) > \lambda x) = (\boldsymbol{\pi})_j,\end{aligned} \tag{D.3}$$

where the last equality is due to Remark 3.2. Lemma 3.2 now follows from (D.2) and (D.3). □

E. Proof of Lemma 3.3

It follows from $\overline{\boldsymbol{\Lambda}_k^{(1)}} = \theta^{-1}\overline{\boldsymbol{D}_k}$ $(k = 0, 1, \dots)$, Assumption 3.2 and (3.13) that

$$\overline{\boldsymbol{\Lambda}_k^{(1)}} \overset{k}{\sim} \widetilde{\boldsymbol{\Lambda}} \Pr[G > k], \tag{E.1}$$

which shows that (3.12) holds for $n = 1$. In what follows, we fix $i, j \in \mathcal{M}$ arbitrarily and consider the case of $n \geq 2$. Let $\mathcal{M}_{i,j}^{(n-1)+} = \{(l_1, \dots, l_{n-1}) \in \mathcal{M}^{n-1}; (\boldsymbol{\Lambda})_{i,l_1}(\boldsymbol{\Lambda})_{l_1,l_2} \cdots (\boldsymbol{\Lambda})_{l_{n-1},j} > 0\}$, where $\mathcal{M}^m = \{(l_1, \dots, l_m); l_i \in \mathcal{M} \ (i = 1, 2, \dots, m)\}$ for $m = 1, 2, \dots$. For each $(\nu, \eta) \in \mathcal{M}^2$ such that $(\boldsymbol{\Lambda})_{\nu,\eta} > 0$, there exists a nonnegative r.v. $\Lambda_{\nu,\eta}$ satisfying

$$\Pr[\Lambda_{\nu,\eta} = k] = \frac{(\boldsymbol{\Lambda}_k)_{\nu,\eta}}{(\boldsymbol{\Lambda})_{\nu,\eta}}, \quad k = 0, 1, \dots, \tag{E.2}$$

because $(\boldsymbol{\Lambda}_k)_{\nu,\eta} \geq 0$ $(k = 0, 1, \dots)$ and $(\boldsymbol{\Lambda})_{\nu,\eta} = \sum_{k=0}^{\infty} (\boldsymbol{\Lambda}_k)_{\nu,\eta}$. Further because $\mathbf{\Lambda}_k = \mathbf{\Lambda}_k^{(1)}$ $(k = 0, 1, \dots)$, it follows from (E.1) and (E.2) that for any $(\nu, \eta) \in \mathcal{M}^2$ such that $(\boldsymbol{\Lambda})_{\nu,\eta} > 0$,

$$\Pr[\Lambda_{\nu,\eta} > k] = \frac{\left(\overline{\mathbf{\Lambda}_k^{(1)}}\right)_{\nu,\eta}}{(\mathbf{\Lambda})_{\nu,\eta}} \overset{k}{\sim} \frac{(\widetilde{\mathbf{\Lambda}})_{\nu,\eta}}{(\mathbf{\Lambda})_{\nu,\eta}} \cdot \Pr[G > k].$$

Thus using Proposition 2.2 (a), we have

$$\begin{aligned}
\left(\overline{\boldsymbol{\Lambda}_k^{(n)}}\right)_{i,j} &= \sum_{(l_1,\dots,l_{n-1}) \in \mathcal{M}^{n-1}} \sum_{k_1 + \dots + k_n > k} (\boldsymbol{\Lambda}_{k_1})_{i,l_1} (\boldsymbol{\Lambda}_{k_2})_{l_1,l_2} \cdots (\boldsymbol{\Lambda}_{k_n})_{l_{n-1},j} \\
&= \sum_{(l_1,\dots,l_{n-1}) \in \mathcal{M}_{i,j}^{(n-1)+}} (\boldsymbol{\Lambda})_{i,l_1} (\boldsymbol{\Lambda})_{l_1,l_2} \cdots (\boldsymbol{\Lambda})_{l_{n-1},j} \Pr\left[\sum_{m=1}^{n} \Lambda_{l_{m-1},l_m} > k\right] \\
&\overset{k}{\sim} \sum_{(l_1,\dots,l_n) \in \mathcal{M}_{i,j}^{(n-1)+}} (\boldsymbol{\Lambda})_{i,l_1} (\boldsymbol{\Lambda})_{l_1,l_2} \cdots (\boldsymbol{\Lambda})_{l_{n-1},j} \sum_{m=1}^{n} \frac{(\widetilde{\boldsymbol{\Lambda}})_{l_{m-1},l_m}}{(\boldsymbol{\Lambda})_{l_{m-1},l_m}} \Pr[G > k], \quad \text{(E.3)}
\end{aligned}$$

where $l_0 = i$ and $l_n = j$. Lemma 3.3 follows from (E.3) because $(\widetilde{\mathbf{\Lambda}})_{i,j} = 0$ if $(\mathbf{\Lambda})_{i,j} = 0$. □

F. Proof of Lemma 3.4

Note first that for any $i, j \in \mathcal{M}$,

$$\left(\overline{\boldsymbol{\Lambda}_k^{(n)}}\right)_{i,j} = \sum_{(l_1,\dots,l_{n-1}) \in \mathcal{M}^{n-1}} \sum_{k_1 + \dots + k_n > k} (\boldsymbol{\Lambda}_{k_1})_{i,l_1} (\boldsymbol{\Lambda}_{k_2})_{l_1,l_2} \cdots (\boldsymbol{\Lambda}_{k_n})_{l_{n-1},j}, \quad \forall k = 0, 1, \dots. \tag{F.1}$$

Because $\mathcal{M}$ is a finite set, Proposition 2.2 (b) implies that for any $\varepsilon > 0$ there exists a positive constant $K := K(\varepsilon)$ such that for all $i, j, l_1, \dots, l_{n-1} \in \mathcal{M}$ and $k = 0, 1, \dots$,

$$\frac{\displaystyle\sum_{k_1 + \dots + k_n > k} (\boldsymbol{\Lambda}_{k_1})_{i,l_1} (\boldsymbol{\Lambda}_{k_2})_{l_1,l_2} \cdots (\boldsymbol{\Lambda}_{k_n})_{l_{n-1},j}}{\Pr[G > k]} \leq K \cdot (1 + \varepsilon)^n (\boldsymbol{\Lambda})_{i,l_1} (\boldsymbol{\Lambda})_{l_1,l_2} \cdots (\boldsymbol{\Lambda})_{l_{n-1},j},$$

where K is independent of n. Substituting the above inequality into (F.1), we have for any $i, j \in \mathcal{M}$,

$$\frac{\left(\overline{\boldsymbol{\Lambda}_k^{(n)}}\right)_{i,j}}{\Pr[G > k]} \leq K \cdot (1 + \varepsilon)^n (\boldsymbol{\Lambda}^n)_{i,j}, \qquad \forall k = 0, 1, \dots.$$

□

G. Proof of Proposition C.2

It is obvious that if $u = 0$, (C.2) holds for all $x \geq 0$. Thus we consider the case of $u > 0$. Note first that for any $\varepsilon > 0$ there exists some $x_0 := x_0(\varepsilon) > 0$ such that $\Pr[F > y] \leq \Pr[F > y + 1]e^{\varepsilon}$ for all $y > x_0$, because $F \in \mathcal{L}$. Note also that $0 < u/\lceil u \rceil \leq 1$ for all $u > 0$. We then have

$$\frac{\Pr[F > y]}{\Pr[F > y + u/\lceil u \rceil]} \leq \frac{\Pr[F > y]}{\Pr[F > y + 1]} \leq e^{\varepsilon}, \qquad y > x_0,\ u > 0,$$

from which it follows that for all $x > u + x_0$ and $u > 0$,

$$\frac{\Pr[F > x - u]}{\Pr[F > x]} = \prod_{i=0}^{\lceil u \rceil - 1} \frac{\Pr[F > x - (i+1) \cdot u/\lceil u \rceil]}{\Pr[F > x - i \cdot u/\lceil u \rceil]} \leq e^{\lceil u \rceil \varepsilon} \leq e^{(u+1)\varepsilon}.$$

□

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