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CORRELATED MULTIVARIATE SHOCK MODELS ASSOCIATED WITH A RENEWAL SEQUENCE AND ITS APPLICATION TO ANALYSIS OF BROWSING BEHAVIOR OF INTERNET USERS

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Abstract A correlated multivariate shock model is considered where a system is subject to a sequence of J different shocks triggered by a common renewal process. Let $(Y(k))_{k=1}^{\infty}$ be a sequence of independently and identically distributed (i.i.d.) nonnegative random variables associated with the renewal process. For the magnitudes of the k-th shock denoted by a random vector $\underline{X}(k)$, it is assumed that $[\underline{X}(k), Y(k)]$ $(k = 1, 2, \cdots)$ constitute a sequence of i.i.d. random vectors with respect to k while $\underline{X}(k)$ and $Y(k)$ may be correlated. The system fails as soon as the historical maximum of the magnitudes of any component of the random vector exceeds a prespecified level of that component. The Laplace transform of the probability density function of the system lifetime is derived, and its mean and variance are obtained explicitly. Furthermore, the probability of system failure due to the i-th component is obtained explicitly for all $i \in \mathcal{J} = \{1, \cdots, J\}$. The model is applied for analyzing the browsing behavior of Internet users.

Keywords: Applied probability, multivariate shock models, system lifetime, consumer browsing behavior

1. Introduction

A general shock model is studied by Shanthikumar and Sumita [3], where a system is subject to a sequence of random shocks generated by a renewal sequence. More specifically, the model is characterized by correlated pairs of nonnegative random variables $[X_j, Y_j]$ $(j = 1, 2, \cdots)$ where X_j is the magnitude of the jth shock and Y_j describes the time interval between two consecutive shocks. The variates $[X_j, Y_j]$ $(j = 1, 2, \cdots)$ are i.i.d. pairwise, while X_j and Y_j may be correlated. The underlying system fails as soon as the magnitude of a shock exceeds a prespecified level. The transform results, an exponential limit theorem and properties of the associated renewal processes of the system failure times are obtained with an application to a stochastic clearing system. The model is extended subsequently by Sumita and Shanthikumar [4] to incorporate the system lifetime based on the cumulative shock.

While the general shock model has widened the application areas much beyond the traditional Poisson shock model, it is still limited in that the model accepts only one type of shocks. In some applications, it is important to deal with multiple types of shocks generated by a common renewal sequence. In analyzing the browsing behavior of users of the Internet, for example, it is common to find a user moving from one website to another in order to gather information about a specific product of his/her interest. Assuming that dwell times at different websites constitute a renewal sequence, the first type of shocks may correspond to the values of information gathered from various websites concerning a product produced by Company $C1$, while the second type of shocks may describe those concerning a similar product produced by Company $C2$. The Internet search would be terminated when the user obtains enough information to decide which company's product should be purchased. The purpose of this paper is to extend the general shock model of Shanthikumar and Sumita [3] so as to incorporate such multiple different random shocks generated from a common renewal sequence. A preliminary version of this study is reported at IWAP2008 by Sumita and Zuo [5]. In this paper, however, the model analysis is elaborated further substantially. In particular, analysis of the probability of system failure due to the i-th component is totally new and numerical examples are also enriched. It may be worth noting that the proposed model can be interpreted as a fatal shock model, where the fatal shock is defined as the historical maximum of any component of a sequence of random vectors exceeding a pre-specified level for the component.

The structure of this paper is as follows. The correlated multivariate shock model is introduced in Section 2 and the system lifetime is analyzed in Section 3. In Section 4, the probability of system failure due to component i is evaluated explicitly. An application to analysis of the browsing behavior of users of the Internet is discussed in Section 5, and numerical examples are also presented. Finally, in Section 6, some concluding remarks are given.

2. Model Description

We consider a system where a sequence of J different types of shocks are triggered by a common renewal process characterized by a sequence of i.i.d. nonnegative random variables $(Y(k))_{k=1}^{\infty}$. Let $\underline{X}(k) = [X_1(k), \cdots, X_J(k)]$ be the random vector describing the magnitudes of J different shocks occurred at the k-th renewal epoch. Throughout the paper, we assume that all random variables are absolutely continuous with $\underline{X}(k) \in R_+^J$ and $Y(k) \in R_+$, where R_+^J is the set of J dimensional nonnegative vectors and R_+ denotes the set of nonnegative real numbers. For notational convenience, we define $\mathcal{J} = \{1, 2, , \cdots, J\}$ and its power set $\mathcal{B}(\mathcal{J}) = \{A : A \subset \mathcal{J}\}$. In addition, while $\underline{X}(k)$ and $Y(k)$ may be correlated, it is assumed that $[\underline{X}(k), Y(k)]$ $(k = 1, 2, \cdots)$ constitute a sequence of i.i.d. random vectors with respect to k. The joint distribution function and the joint probability density function of $[\underline{X}(k), Y(k)]$ are defined by

$$F_{\underline{X},Y}(\underline{x}, y) = P[X_1(k) < x_1, \cdots, X_J(k) < x_J, Y(k) \le y], \tag{2.1}$$

and

$$F_{\underline{X},Y}(\underline{x}, y) = \int_0^{x_J} \cdots \int_0^{x_1} \int_0^{y} f_{\underline{X},Y}(\underline{v}, w)\,dw\,d\underline{v}. \tag{2.2}$$

We note that the inequality associated with $\underline{X}(k)$ in $F_{\underline{X},Y}(\underline{x}, y)$ is taken to be strict. Since the historical maximum processes are of our main concern, equalities are attached to tail probabilities for random variables directly involving $\underline{X}(k)$ as a general rule in this paper. For notational convenience, the following functions are also introduced.

$$f_Y(y) = \int_0^{\infty} \cdots \int_0^{\infty} f_{\underline{X},Y}(\underline{x}, y)\,d\underline{x} \quad ; \quad f_{\underline{X}}(\underline{x}) = \int_0^{\infty} f_{\underline{X},Y}(\underline{x}, y)\,dy \tag{2.3}$$

$$G_{\underline{X}}(\underline{x}, y) = \int_0^{x_J} \cdots \int_0^{x_1} f_{\underline{X},Y}(\underline{v}, y)\,d\underline{v} \tag{2.4}$$

$$\overline{G}_{\underline{X}}(\underline{x}, y) = \int_{x_J}^{\infty} \cdots \int_{x_1}^{\infty} f_{\underline{X},Y}(\underline{v}, y)\,d\underline{v} \tag{2.5}$$

$$G_Y(\underline{x}, y) = \int_0^y f_{\underline{X},Y}(\underline{x}, \tau)d\tau \quad ; \quad \overline{G}_Y(\underline{x}, y) = \int_y^\infty f_{\underline{X},Y}(\underline{x}, \tau)d\tau. \tag{2.6}$$

For simplicity, with $\underline{x} = [x_1, \cdots, x_J]$, we write $f_{\underline{X}}(\underline{x}) = f_{\underline{X}}(x_1, \cdots, x_J)$, $G_{\underline{X}}(\underline{x}, y) = G_{\underline{X}}(x_1, \cdots, x_J, y)$, etc, interchangeably.

The system fails as soon as the historical maximum of the magnitudes of any component of the random vector exceeds a prespecified level of that component. More specifically, let $N(t)$ be the counting process associated with the renewal sequence $(Y(k))_{k=1}^\infty$ and define the historical maximum process $\underline{M}(t)$ by

$$\underline{M}(t) = [M_1(t), \cdots, M_J(t)] \quad ; \quad M_i(t) = \max_{0 \le k \le N(t)} \{X_i(k)\}, \tag{2.7}$$

where $\underline{X}(0) = \underline{0}$ is employed for notational convenience. The system fails as soon as any one of the historical maximum processes $M_i(t)$, $i \in \mathcal{J}$, exceeds its prespecified level z_i . If only $M_i(t)$ exceeds z_i, then the i-th component causes the system failure. If multiple historical maximum processes exceed their prespecified levels simultaneously, the system failure is assumed to be triggered by the component having the largest value of them. For $\underline{z} = [z_1, \cdots, z_J] > \underline{0}$, the system lifetime $T_{\underline{z}}$ is then given by

$$T_{\underline{z}} = \inf\{t : M_i(t) \ge z_i, \quad \text{for some } i \in \mathcal{J}\}. \tag{2.8}$$

Of interest is the distribution of $T_{\underline{z}}$ and the probability $\rho_i(\underline{z})$ of the system failure being caused by the i-th component. In what follows, we analyze $T_{\underline{z}}$, deriving the transform results and its mean and variance, as well as $\rho_i(\underline{z})$ for all $i \in \mathcal{J}$.

3. Analysis of $T_{\underline{z}}$

Let the distribution functions of $\underline{M}(t)$ and $T_{\underline{z}}$ be defined by

$$V(\underline{z}, t) = P[\underline{M}(t) < \underline{z}] \quad ; \quad W_{\underline{z}}(t) = P[T_{\underline{z}} \le t]. \tag{3.1}$$

Laplace transforms with respect to t are denoted by a circumflex, i.e.,

$$\hat{V}(\underline{z}, s) = \int_0^\infty e^{-st} V(\underline{z}, t)dt \quad ; \quad \hat{w}_{\underline{z}}(s) = \int_0^\infty e^{-st} dW_{\underline{z}}(t). \tag{3.2}$$

One easily sees that there exists a dual relationship between $\underline{M}(t)$ and $T_{\underline{z}}$ specified by

$$V(\underline{z}, t) = P[\underline{M}(t) < \underline{z}] = P[T_{\underline{z}} > t] = \overline{W}_{\underline{z}}(t), \tag{3.3}$$

where $\overline{W}_{\underline{z}}(t) = 1 - W_{\underline{z}}(t)$ is the survival function of $T_{\underline{z}}$. In this section, we derive $\hat{w}_{\underline{z}}(s)$ explicitly based on Equation (3.3).

We assume that the system starts anew at time $t = 0$. For $k = 1, 2, \cdots$, the shock vector $\underline{X}(k)$ at the k-th renewal epoch is correlated only to the time interval $Y(k)$ since the $(k-1)$st renewal epoch and does not affect the future events. The following theorem then holds.

Theorem 3.1. *Let $\hat{\varphi}_Y(s)$ and $\hat{G}_{\underline{X}}(\underline{z}, s)$ be the Laplace transforms of $f_Y(t)$ in Equation (2.3) and $\hat{G}_{\underline{X}}(\underline{z}, s)$ in Equation (2.6) respectively, i.e. $\hat{\varphi}_Y(s) \overset{def}{=} \int_0^\infty e^{-st} f_Y(t)dt$ and $\hat{G}_{\underline{X}}(\underline{z}, s) \overset{def}{=} \int_0^\infty e^{-st} G_{\underline{X}}(\underline{z}, t)dt$. One then has*

$$\hat{V}(\underline{z}, s) = \frac{1 - \hat{\varphi}_Y(s)}{s\{1 - \hat{G}_{\underline{X}}(\underline{z}, s)\}}, \quad Re(s) \ge 0.$$

Proof. Since $V(\underline{z}, t)$ is the probability that the maximum value of $X_i(k)$ has not exceeded the level z_i for $k = 1, 2, \cdots, N(t)$ and $i \in \mathcal{J}$, by conditioning on the first renewal time $Y(1)$ and using the regenerative property of the paired process $[\underline{X}(k), Y(k)]$ at $Y(1)$, one sees that

$$V(\underline{z}, t) = \overline{F}_Y(t) + \int_0^t G_{\underline{X}}(\underline{z}, y) V(\underline{z}, t-y) dy. \tag{3.4}$$

By taking the Laplace transform of both sides of Equation (3.4) with respect to t, it can be seen that

$$\hat{V}(\underline{z}, s) \quad = \quad \frac{1 - \hat{\varphi}_Y(s)}{s} + \hat{G}_{\underline{X}}(\underline{z}, s) \hat{V}(\underline{z}, s).$$

This equation can be solved for $\hat{V}(\underline{z}, s)$, completing the proof. □

The system lifetime $T_{\underline{z}}$ has the dual relationship with $\underline{M}(t)$ given in Equation (3.3). The Laplace transform $\hat{w}_{\underline{z}}(s) = E[e^{-sT_{\underline{z}}}]$ is then easily found from Theorem 3.1.

Theorem 3.2.

$$\hat{w}_{\underline{z}}(s) = \frac{\hat{\varphi}_Y(s) - \hat{G}_{\underline{X}}(\underline{z}, s)}{1 - \hat{G}_{\underline{X}}(\underline{z}, s)}, \qquad Re(s) \geq 0.$$

Proof. From Equation (3.3), one finds that $\hat{V}(\underline{z}, s) = \frac{1 - \hat{w}_{\underline{z}}(s)}{s}$, so that $\hat{w}_{\underline{z}}(s) = 1 - s\hat{V}(\underline{z}, s)$. The theorem now follows from Theorem 3.1. □

By differentiating $\hat{w}_{\underline{z}}(s)$ at $s = 0$, the mean and the variance of $T_{\underline{z}}$ can be obtained.

Corollary 3.2.1.

$$a) \quad E[T_{\underline{z}}] = \frac{E[Y]}{1 - F_{\underline{X}}(\underline{z})}$$

$$b) \quad Var[T_{\underline{z}}] = \frac{E[Y^2]}{1 - F_{\underline{X}}(\underline{z})} + \frac{E[Y]}{(1 - F_{\underline{X}}(\underline{z}))^2} \left\{ 2F_{\underline{X}}(\underline{z}) E[Y | \underline{X} < \underline{z}] - E[Y] \right\}$$

The Laplace transform $\hat{w}_{\underline{z}}(s) = E[e^{-sT_{\underline{z}}}]$ has the following real-domain form.

$$w_{\underline{z}}(t) = f_Y(t) + \sum_{k=1}^{\infty} f_Y(t) * G_{\underline{X}}^{(k)}(\underline{z}, t) - \sum_{k=1}^{\infty} G_{\underline{X}}^{(k)}(\underline{z}, t), \tag{3.5}$$

where $G_{\underline{X}}^{(k+1)}(\underline{z}, t) = \int_0^t G_{\underline{X}}(\underline{z}, t-\tau) G_{\underline{X}}^{(k)}(\underline{z}, \tau) d\tau$ and the asterisk denotes similar convolution in t.

As the threshold levels z_i for $i \in \mathcal{J}$ tend to approach ∞, the system failure becomes a rare event. Accordingly, it may be expected that $T_{\underline{z}}/E[T_{\underline{z}}]$ converges in distribution to the exponential variate E of mean one. This type of exponential limit theorems is originated from Keilson [1,2] involving rare events in regenerative processes. Since the historical maximum is monotonically non-decreasing in time t, Keilson's theorem does not seem to be directly applicable here. However, Shanthikumar and Sumita [3] find the structural similarity between rare events in regenerative processes and those in historical maximum processes, proving a generalized version of the original theorem by Keilson [1,2].

The limit theorem of [3] involves a sequence of non-negative random vectors $\underline{V}(k) = [X(k), Y(k)]$ where $X(k)$ and $Y(k)$ may be correlated but $\underline{V}(k)$'s are i.i.d. Then the state space $\mathcal{N} = R_+^2 = \{(x,y) : x \geq 0, y \geq 0\}$ is decomposed into $G(z)$ and $B(z)$ with $G(z) \neq \emptyset$, $B(z) \neq \emptyset$, $G(z) \cap B(z) = \emptyset$ and $G(z) \cup B(z) = \mathcal{N}$, and the following experiment is considered. If $\underline{V}(k) \in G(z)$, the experiment continues and $\underline{V}(k+1)$ is chosen. The experiment stops when a random vector falls in the region $B(z)$. The system failure time S_z is then defined as the sum of y-coordinates of all random vectors up to the stopping point. It is shown in Shanthikumar and Sumita [3] that, if $p_z = P[\underline{V} \in B(z)] \to 0$ as $z \to \infty$, then $S_z/E[S_z] \to E$ as $z \to \infty$. In this paper, one has $\underline{V}(k) = [\underline{X}(k), Y(k)]$, i.e. the first process becomes multivariate. This requires to redefine $\mathcal{N}, G(\underline{z}), B(\underline{z})$ and $p_{\underline{z}}$. However, the system failure time remains to be expressed as the sum of y-coordinates of all random vectors up to the stopping point in the random experiment. Since $p_{\underline{z}} = P[\underline{V}(k) \in B(\underline{z})] \to 0$ if $z_i \to \infty$ for all $i \in \mathcal{J}$, the following theorem can be shown along the line of the proof of Theorem 1.A4 in [3].

Theorem 3.3. *Let E be the exponential random variate of mean one and suppose* $0 < F_{\underline{X},Y}(\underline{x}, y) < 1$ *for* $\underline{0} < \underline{x} < \underline{\infty}$, $0 < y < \infty$, *and* $E[Y] < \infty$. *Then* $T_{\underline{z}}/E[T_{\underline{z}}] \xrightarrow{d} E$ *as* $\underline{z} \to \underline{\infty}$.

It is trivial that the almost sure dominance of $T_{\underline{z}_2}$ over $T_{\underline{z}_1}$ is present whenever $0 \leq \underline{z}_1 \leq \underline{z}_2$. We formally state this result.

Theorem 3.4.

$$0 \leq \underline{z}_1 \leq \underline{z}_2 \Rightarrow T_{\underline{z}_1} \leq_{a.s.} T_{\underline{z}_2}$$

4. Probability of System Failure Caused by the i-th Component

Given a threshold vector $\underline{z}$, we next turn our attention to evaluate the probability $\rho_i(\underline{z})$ of system failure caused by the i-th component for $i \in \mathcal{J}$. For this purpose, let τ_k be the k-th renewal epoch for $k = 1, 2, \cdots$ and define $\eta_{\underline{\mathcal{J}}}(\underline{z}, t, k)$ to describe the event that the system failure is avoided at the k-th renewal epoch with the marginal probability density of t at $t = \tau_k$. Since $[\underline{X}(k), Y(k)]$ constitute a sequence of i.i.d. random vectors, $\eta_{\underline{\mathcal{J}}}(\underline{z}, t, 1)$ represents the avoidance of system failure at any single renewal epoch. It can be seen that

$$\begin{aligned} \eta_{\underline{\mathcal{J}}}(\underline{z}, t) &\stackrel{\text{def}}{=} \eta_{\underline{\mathcal{J}}}(\underline{z}, t, 1) = \frac{d}{dt} F_{\underline{X},Y}(\underline{z}, t) \\ &= \int_0^{z_J} \cdots \int_0^{z_1} f_{\underline{X},Y}(\underline{x}, t) dx_1 \cdots dx_J. \end{aligned} \tag{4.1}$$

For $k \geq 2$, one sees that

$$\begin{aligned} \eta_{\underline{\mathcal{J}}}(\underline{z}, t, k) &\stackrel{\text{def}}{=} \frac{d}{dt} P[X_i(m) < z_i \text{ for all } i \in \mathcal{J} \text{ and } m = 1, \cdots, k, \tau_k \leq t] \\ &= \int_0^t \eta_{\underline{\mathcal{J}}}(\underline{z}, \tau, k-1) \eta_{\underline{\mathcal{J}}}(\underline{z}, t - \tau) d\tau. \end{aligned} \tag{4.2}$$

By taking Laplace transforms of Equations (4.1) and (4.2) with respect to t, one finds by induction that

$$\hat{\eta}_{\underline{\mathcal{J}}}(\underline{z}, s, k) = \int_0^\infty e^{-st} \eta_{\underline{\mathcal{J}}}(\underline{z}, t, k) dt = \{\hat{\eta}_{\underline{\mathcal{J}}}(\underline{z}, s)\}^k, \quad k = 1, 2, \cdots, \tag{4.3}$$

where

$$\hat{\eta}_{\underline{\mathcal{J}}}(\underline{z}, s) = \int_0^\infty e^{-st} \eta_{\underline{\mathcal{J}}}(\underline{z}, t) dt. \quad (4.4)$$

Let $\mathcal{F}_i(\mathcal{J})$ be the family of subsets of $\mathcal{J}$ containing i, that is,

$$\mathcal{F}_i(\mathcal{J}) \stackrel{\text{def}}{=} \{A : i \in A, A \subset \mathcal{J}\}, \quad (4.5)$$

and define $\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, k)$ to be the probability that the system failure is triggered by the i-th component and all the components in $A \in \mathcal{F}_i(\mathcal{J})$ exceed the corresponding threshold levels at the k-th renewal epoch while $X_j(k)$ for $j \in \mathcal{J}\backslash A$ remains below z_j, with the marginal probability density of t at $t = \tau_k$. More specifically, we define, for $k \geq 2$,

$$\begin{aligned} &\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, k) \\ \stackrel{\text{def}}{=} \; &\frac{d}{dt} P[X_i(m) < z_i \text{ for all } i \in \mathcal{J} \text{ and } m = 1, \cdots, k-1, and \\ &X_j(k) > z_j \text{ for } j \in A, X_j(k) < z_j \text{ for } j \in \mathcal{J}\backslash A, X_i(k) = \max_{j \in A}\{X_j(k)\}, \tau_k \leq t]. \end{aligned} \quad (4.6)$$

For $k = 1$, the first half of the conditions in the above probability would be ignored, i.e

$$\begin{aligned} &\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, 1) \\ \stackrel{\text{def}}{=} \; &\frac{d}{dt} P[X_j(1) > z_j \text{ for } j \in A, X_j(1) < z_j \text{ for } j \in \mathcal{J}\backslash A, X_i(1) = \max_{j \in A}\{X_j(1)\}, \tau_1 \leq t]. \end{aligned} \quad (4.7)$$

As before, since $[\underline{X}(k), Y(k)]$ are i.i.d. random vectors, $\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, 1)$ represents a system failure at any single renewal epoch with the probability density of the renewal lifetime being at t. It can be seen that, with $\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t) \stackrel{\text{def}}{=} \eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, 1)$, and for $k \geq 2$,

$$\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, k) = \int_0^t \eta_{\underline{\mathcal{J}}}(\underline{z}, \tau, k-1) \eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t-\tau) d\tau. \quad (4.8)$$

Adding $\eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, k)$ over $A \in \mathcal{F}_i(\mathcal{J})$, one obtains the probability that the system failure is triggered by the i-th component at the k-th renewal epoch with the marginal probability density of t at $t = \tau_k$. We define

$$\xi_i(\underline{z}, t) \stackrel{\text{def}}{=} \xi_i(\underline{z}, t, 1) \stackrel{\text{def}}{=} \sum_{A \in \mathcal{F}_i(\mathcal{J})} \eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t), \quad (4.9)$$

and

$$\xi_i(\underline{z}, t, k) \stackrel{\text{def}}{=} \sum_{A \in \mathcal{F}_i(\mathcal{J})} \eta_{i:\overline{A},\underline{\mathcal{J}\backslash A}}(\underline{z}, t, k). \quad (4.10)$$

It then follows from Equations (4.8) through (4.10) that, for $k \geq 2$,

$$\xi_i(\underline{z}, t, k) = \int_0^t \eta_{\underline{\mathcal{J}}}(\underline{z}, \tau, k-1) \xi_i(\underline{z}, t-\tau) d\tau. \quad (4.11)$$

Let the Laplace transform of $\xi_i(\underline{z}, t)$ with respect to t be defined by

$$\hat{\xi}_i(\underline{z}, s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st}\xi_i(\underline{z}, t)dt. \tag{4.12}$$

From Equations (4.3) and (4.11), one then has, for $k \geq 2$,

$$\hat{\xi}_i(\underline{z}, s, k) \stackrel{\text{def}}{=} \int_0^\infty e^{-st}\xi_i(\underline{z}, t, k)dt = \{\hat{\eta}_{\underline{\mathcal{J}}}(z, s)\}^{k-1}\hat{\xi}_i(\underline{z}, s). \tag{4.13}$$

We note that this Laplace transform result is valid even for $k = 1$, yielding the definition $\hat{\xi}_i(\underline{z}, s, 1) = \hat{\xi}_i(\underline{z}, s)$. The corresponding Laplace transform generating function can then be obtained as

$$\hat{\hat{\xi}}_i(\underline{z}, s, u) \stackrel{\text{def}}{=} \sum_{k=1}^\infty \hat{\xi}_i(\underline{z}, s, k)u^k = \frac{u \cdot \hat{\xi}_i(\underline{z}, s)}{1 - u \cdot \hat{\eta}_{\underline{\mathcal{J}}}(\underline{z}, s)}. \tag{4.14}$$

We are now in a position to state the main theorem of this section.

Theorem 4.1. *Given a threshold level vector $\underline{z}$, let $\rho_i(\underline{z})$ be the probability that the system failure is eventually caused by the i-th component. Then one has*

$$\rho_i(\underline{z}) = \frac{\int_0^\infty \xi_i(\underline{z}, t)dt}{1 - \int_0^\infty \eta_{\underline{\mathcal{J}}}(\underline{z}, t)dt}. \tag{4.15}$$

Proof. Since $\rho_i(\underline{z}) = \hat{\hat{\xi}}_i(\underline{z}, 0, 1)$, the theorem follows immediately from Equation (4.14). □

Remark 4.2. *In e-commerce, the probability $\rho_i(\underline{z})$ that Product i is chosen to be purchased over Product j, $j \in \mathcal{J} \setminus \{i\}$, represents the strength of Product i against other competitive products. If the brand power of Product i is strong, customers would not require much information about Product i. This means that a smaller value of z_i is likely to convince customers to purchase. Given z_i, if the website of Product i is well organized, it is likely to enable customers to reach z_i sooner. Consequently, one may expect that $\rho_i(\underline{z})$ increases as z_i decreases or X_i increases stochastically. It is non-trivial to prove this conjecture based on Theorem 4.1. However, we will demonstrate this conjecture through numerical examples.*

In Theorem 4.1, the denominator of $\rho_i(\underline{z})$ can be computed rather easily from Equation (4.1). As can be seen from Equation (4.15), the numerator, however, requires the summation over $A \in \mathcal{F}_i(\mathcal{J})$ which grows exponentially as a function of J. Accordingly, it is not easy to compute the numerator when J is large. If the threshold level of each component is identical, i.e. $\underline{z} = z\underline{1}$ where $\underline{1}$ is the vector having all components equal to 1, the computation of the numerator can be simplified significantly. Namely, one has

$$\begin{aligned} \xi_i(z\underline{1}, t) &= \frac{d}{dt}P[X_i(1) = \max_{j\in\mathcal{J}}\{X_j(1)\} > z, \tau_1 \leq t] \\ &= \int_z^\infty dx_i \int_{\underline{0}\setminus i}^{x_i\underline{1}\setminus i} f_{\underline{X},Y}(\underline{x}, t)d\underline{x}, \end{aligned} \tag{4.16}$$

where $\underline{a}\backslash i \stackrel{\text{def}}{=} [a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_J]^T$.

When $\mathcal{J} = \{1, 2\}$, the summation over $A \in \mathcal{F}_i(\mathcal{J})$ can be written explicitly, enabling one to evaluate $\rho_i(z_1, z_2)$. More specifically, one has

$$\xi_1(z_1, z_2, t) = \eta_{\overline{1},\underline{2}}(z_1, z_2, t) + \eta_{1:\overline{1},\overline{2}}(z_1, z_2, t) \tag{4.17}$$

$$\xi_2(z_1, z_2, t) = \eta_{\underline{1},\overline{2}}(z_1, z_2, t) + \eta_{2:\overline{1},\overline{2}}(z_1, z_2, t) \tag{4.18}$$

where

$$\eta_{\overline{1},\underline{2}}(z_1, z_2, t) = \int_0^{z_2} \int_{z_1}^{\infty} f_{\underline{X},Y}(x_1, x_2, t) dx_1 dx_2 \quad ; \tag{4.19}$$

$$\eta_{\underline{1},\overline{2}}(z_1, z_2, t) = \int_{z_2}^{\infty} \int_0^{z_1} f_{\underline{X},Y}(x_1, x_2, t) dx_1 dx_2 \quad ; \tag{4.20}$$

and

$$\eta_{1:\overline{1},\overline{2}}(z_1, z_2, t) \stackrel{\text{def}}{=} \frac{d}{dt} P[X_1(1) \geq z_1, X_2(1) \geq z_2, X_1(1) > X_2(1), \tau_1 \leq t] \quad ; \tag{4.21}$$

$$\eta_{2:\overline{1},\overline{2}}(z_1, z_2, t) \stackrel{\text{def}}{=} \frac{d}{dt} P[X_1(1) \geq z_1, X_2(1) \geq z_2, X_1(1) < X_2(1), \tau_1 \leq t]. \tag{4.22}$$

When $z_1 > z_2$, Equations (4.21) and (4.22) are given by

$$\eta_{1:\overline{1},\overline{2}}(z_1, z_2, t) = \int_{z_1}^{\infty} \left[\int_{z_2}^{x_1} f_{\underline{X},Y}(x_1, x_2, t) dx_2 \right] dx_1 \quad ; \tag{4.23}$$

$$\eta_{2:\overline{1},\overline{2}}(z_1, z_2, t) = \int_{z_1}^{\infty} \left[\int_{z_1}^{x_2} f_{\underline{X},Y}(x_1, x_2, t) dx_1 \right] dx_2. \tag{4.24}$$

For $z_1 \leq z_2$, one has

$$\eta_{1:\overline{1},\overline{2}}(z_1, z_2, t) = \int_{z_2}^{\infty} \left[\int_{z_2}^{x_1} f_{\underline{X},Y}(x_1, x_2, t) dx_2 \right] dx_1 \quad ; \tag{4.25}$$

$$\eta_{2:\overline{1},\overline{2}}(z_1, z_2, t) = \int_{z_2}^{\infty} \left[\int_{z_1}^{x_2} f_{\underline{X},Y}(x_1, x_2, t) dx_1 \right] dx_2. \tag{4.26}$$

The results in Equations (4.17) through (4.26) will be used for numerical examples to be presented in the next section.

5. Application to Analysis of the Browsing Behavior of Users of the Internet

We suppose that a consumer visits various websites in order to gather information about two products of the same type. Let $X_1(k)$ be the value of information about the product $P1$ of Company $C1$ that the consumer gains from the k-th search with length of $Y(k)$, and $X_2(k)$ is defined similarly for the product $P2$ of Company $C2$. We assume that both $X_1(k)$ and $X_2(k)$ consist of two parts: a part independent of $Y(k)$ and another part proportional to $Y(k)$. The former parts for $X_1(k)$ and $X_2(k)$ are denoted by $\hat{X}_1(k)$ and $\hat{X}_2(k)$ respectively. They may represent the amount of information about $P1$ and that about $P2$ that can be gathered from websites, having information about both $P1$ and $P2$. The number of websites with $P1$ information only may not be necessarily the same as the number of websites with $P2$ information. If the ratio between the two numbers is given as $\alpha_1 : \alpha_2$, the amount of information about Pi gathered from such websites during search time $Y(k)$ may be written as $\alpha_i Y(k)$ for $i \in \{1, 2\}$. More formally, we define

$$X_1(k) \;=\; \hat{X}_1(k) + \alpha_1 Y(k) \quad ; \quad X_2(k) = \hat{X}_2(k) + \alpha_2 Y(k). \tag{5.1}$$

It is assumed that $\hat{X}_1(k)$, $\hat{X}_2(k)$ and $Y(k)$ constitute three independent renewal sequences with respect to k, but $X_1(k)$ and $X_2(k)$ are not independent because of sharing the common value of $Y(k)$.

Let $F_{\underline{X},Y}(x_1, x_2, y) = P[X_1(k) < x_1, X_2(k) < x_2, Y(k) \le y]$, and let the distribution functions of $\hat{X}_1$ and $\hat{X}_2$ be denoted by $F_{\hat{X}_1}(x)$ and $F_{\hat{X}_2}(x)$ respectively. From Equation (5.1), by conditioning on Y, one finds that

$$F_{\underline{X},Y}(x_1, x_2, y) = \int_0^{\min\{y, \frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}\}} F_{\hat{X}_1}(x_1 - \alpha_1 \tau) F_{\hat{X}_2}(x_2 - \alpha_2 \tau) f_Y(\tau) d\tau. \tag{5.2}$$

From Equation (2.2), it then follows that

$$f_{\underline{X},Y}(x_1, x_2, y) = f_{\hat{X}_1}(x_1 - \alpha_1 y) f_{\hat{X}_2}(x_2 - \alpha_2 y) f_Y(y) \cdot I\{0 \le y \le \min\{\frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}\}\}, \tag{5.3}$$

where $I\{ST\} = 1$ if statement ST is true, $I\{ST\} = 0$ otherwise. We assume that $\hat{X}_1(k)$, $\hat{X}_2(k)$ and $Y(k)$ are exponentially distributed with respective probability density functions given by

$$f_{\hat{X}_1}(\hat{x}_1) = \mu_1 e^{-\mu_1 \hat{x}_1}; \quad f_{\hat{X}_2}(\hat{x}_2) = \mu_2 e^{-\mu_2 \hat{x}_2}; \quad f_Y(y) \;=\; \lambda e^{-\lambda y}. \tag{5.4}$$

Suppose that the consumer will stop the search process whenever the desired information for either product, specified by z_1 or z_2, is obtained. Let $\underline{\gamma}^*$ and $\overline{\gamma}^*$ be defined as

$$\underline{\gamma}^* \stackrel{\text{def}}{=} \min\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\} \quad ; \quad \overline{\gamma}^* \stackrel{\text{def}}{=} \max\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\}, \tag{5.5}$$

and let A(s), B(s), C(s) and D(s) be given by

$$\begin{aligned} A(s) &= \frac{\lambda e^{-\mu_1 z_1}(1 - e^{-(s+\lambda-\mu_1\alpha_1)\underline{\gamma}^*})}{s + \lambda - \mu_1 \alpha_1}; \quad B(s) = \frac{\lambda e^{-\mu_2 z_2}(1 - e^{-(s+\lambda-\mu_2\alpha_2)\underline{\gamma}^*})}{s + \lambda - \mu_2 \alpha_2} \\ C(s) &= \frac{\lambda e^{-(\mu_1 z_1 + \mu_2 z_2)}(1 - e^{-(s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*})}{s + \lambda - \mu_1 \alpha_1 - \mu_2 \alpha_2}; \quad D(s) = \frac{\lambda e^{-(s+\lambda)\underline{\gamma}^*}}{s + \lambda}. \end{aligned} \tag{5.6}$$

Then from Theorem 3.2 and Equations (5.1) through (5.6), one has the following theorem.

Theorem 5.1. *Let $T_{\underline{z}}$ be the web search completion time as defined in Equation (2.8). Then its Laplace transform is given by*

$$\hat{w}_{\underline{z}}(s) = \left\{1 + \frac{\frac{s}{s+\lambda}}{A(s) + B(s) - C(s) + D(s)}\right\}^{-1}.$$

Proof. From Equations (5.2) through (5.4), one has

$$f_{\underline{X},Y}(x_1,x_2,y) = \mu_1 e^{-\mu_1(x_1-\alpha_1 y)}\mu_2 e^{-\mu_2(x_2-\alpha_2 y)}\lambda e^{-\lambda y} I\{0\le y\le \min(\frac{x_1}{\alpha_1},\frac{x_2}{\alpha_2})\}.$$

For notational convenience, let

$$\tilde{f}_{\underline{X},Y}(x_1,x_2,y) \stackrel{\text{def}}{=} \mu_1 e^{-\mu_1(x_1-\alpha_1 y)}\mu_2 e^{-\mu_2(x_2-\alpha_2 y)}\lambda e^{-\lambda y}. \tag{5.7}$$

It can be seen that

$$\begin{aligned} G_{\underline{X},Y}(z_1,z_2,y) &= \int_0^{z_2}\left\{\int_0^{z_1} f_{\underline{X},Y}(x_1,x_2,y)dx_1\right\}dx_2 \\ &= \int_0^{z_2}\left\{\int_0^{z_1} \tilde{f}_{\underline{X},Y}(x_1,x_2,y)I\{\alpha_1 y\le x_1\le \frac{\alpha_1}{\alpha_2}x_2\le z_1\}dx_1\right. \\ &\quad + \int_0^{z_1}\tilde{f}_{\underline{X},Y}(x_1,x_2,y)I\{\alpha_1 y\le x_1\le z_1\le \frac{\alpha_1}{\alpha_2}x_2\}dx_1 \\ &\quad \left. + \int_0^{z_1}\tilde{f}_{\underline{X},Y}(x_1,x_2,y)I\{\alpha_1 y\le \frac{\alpha_1}{\alpha_2}x_2\le x_1\le z_1\}dx_1\right\}dx_2, \end{aligned}$$

which leads to

$$\begin{aligned} G_{\underline{X},Y}(z_1,z_2,y) &= \int_0^{z_2}\left[\int_{\alpha_1 y}^{\frac{\alpha_1}{\alpha_2}x_2}\tilde{f}_{\underline{X},Y}(x_1,x_2,y)dx_1 I\{\alpha_1 y\le \frac{\alpha_1}{\alpha_2}x_2\le z_1\}\right. \\ &\quad + \int_{\alpha_1 y}^{z_1}\tilde{f}_{\underline{X},Y}(x_1,x_2,y)dx_1 I\{\alpha_1 y\le z_1\le \frac{\alpha_1}{\alpha_2}x_2\} \\ &\quad \left. + \int_{\frac{\alpha_1}{\alpha_2}x_2}^{z_1}\tilde{f}_{\underline{X},Y}(x_1,x_2,y)dx_1 I\{\alpha_1 y\le \frac{\alpha_1}{\alpha_2}x_2\le z_1\}\right]dx_2. \end{aligned}$$

Since the first term and the third term in the last part of the above equation can be combined as a single integral from $\alpha_1 y$ to z_1, one finds that

$$\begin{aligned} &G_{\underline{X},Y}(z_1,z_2,y) \\ &= \int_0^{z_2}\left[\int_{\alpha_1 y}^{z_1}\tilde{f}_{\underline{X},Y}(x_1,x_2,y)dx_1\left\{I\{\alpha_1 y\le z_1\le \frac{\alpha_1}{\alpha_2}x_2\}+I\{\alpha_1 y\le \frac{\alpha_1}{\alpha_2}x_2\le z_1\}\right\}\right]dx_2. \end{aligned}$$

Substituting Equation (5.7) into the above equation, one has

$$\begin{aligned} G_{\underline{X},Y}(z_1,z_2,y) &= \lambda e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)y}[e^{-\mu_1\alpha_1 y}-e^{-\mu_1 z_1}] \\ &\quad \times \int_0^{z_2}\mu_2 e^{-\mu_2 x_2}\left\{I\{\alpha_2 y\le \frac{\alpha_2}{\alpha_1}z_1\le x_2\}+I\{\alpha_2 y\le x_2\le \frac{\alpha_2}{\alpha_1}z_1\}\right\}dx_2. \end{aligned}$$

By repeating this procedure with respect to x_2, one concludes that

$$\begin{aligned} G_{\underline{X},Y}(z_1,z_2,y) &= \left\{e^{-(\mu_1 z_1+\mu_2 z_2)}\lambda e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)y}-e^{-\mu_1 z_1}\lambda e^{-(\lambda-\mu_1\alpha_1)y}\right. \\ &\quad \left. -e^{-\mu_2 z_2}\lambda e^{-(\lambda-\mu_2\alpha_2)y}+\lambda e^{-\lambda y}\right\}\cdot I\{0\le y\le \underline{\gamma}^*\}, \end{aligned} \tag{5.8}$$

where $\underline{\gamma}^* \stackrel{\text{def}}{=} \min\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\}$ is as in Equation (5.5). By taking the Laplace transform of both sides of Equation (5.8) with respect to y, it follows that

$$\begin{aligned}
&\hat{G}_{\underline{X},Y}(z_1,z_2,s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sy} G_{\underline{X},Y}(z_1,z_2,y)dy \\
= \;& \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2}(1-e^{-(s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}) - \frac{\lambda e^{-\mu_1 z_1}}{s+\lambda-\mu_1\alpha_1}(1-e^{-(s+\lambda-\mu_1\alpha_1)\underline{\gamma}^*}) \\
& -\frac{\lambda e^{-\mu_2 z_2}}{s+\lambda-\mu_2\alpha_2}(1-e^{-(s+\lambda-\mu_2\alpha_2)\underline{\gamma}^*}) + \frac{\lambda}{s+\lambda}(1-e^{-(s+\lambda)\underline{\gamma}^*}).
\end{aligned}$$

Since

$$\varphi_Y(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sy}\lambda e^{-\lambda y}dy = \frac{\lambda}{s+\lambda},$$

one sees that,

$$\begin{aligned}
&\varphi_Y(s) - \hat{G}_{\underline{X},Y}(z_1,z_2,s) \\
= \;& \frac{\lambda e^{-\mu_1 z_1}}{s+\lambda-\mu_1\alpha_1}(1-e^{-(s+\lambda-\mu_1\alpha_1)\underline{\gamma}^*}) + \frac{\lambda e^{-\mu_2 z_2}}{s+\lambda-\mu_2\alpha_2}(1-e^{-(s+\lambda-\mu_2\alpha_2)\underline{\gamma}^*}) \\
& -\frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2}(1-e^{-(s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}) + \frac{\lambda}{s+\lambda}e^{-(s+\lambda)\underline{\gamma}^*},
\end{aligned}$$

and

$$\begin{aligned}
&1 - \hat{G}_{\underline{X},Y}(z_1,z_2,s) \\
= \;& \frac{\lambda e^{-\mu_1 z_1}}{s+\lambda-\mu_1\alpha_1}(1-e^{-(s+\lambda-\mu_1\alpha_1)\underline{\gamma}^*}) + \frac{\lambda e^{-\mu_2 z_2}}{s+\lambda-\mu_2\alpha_2}(1-e^{-(s+\lambda-\mu_2\alpha_2)\underline{\gamma}^*}) \\
& -\frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2}(1-e^{-(s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}) + \frac{s+\lambda e^{-(s+\lambda)\underline{\gamma}^*}}{s+\lambda}.
\end{aligned}$$

With $A(s), B(s), C(s)$ and $D(s)$ as defined in Equation (5.6), it can be seen that

$$\hat{w}_{\underline{z}}(s) = \frac{\hat{\varphi}_Y(s) - \hat{G}_{\underline{X}}(\underline{z},s)}{1-\hat{G}_{\underline{X}}(\underline{z},s)} = \left\{1 + \frac{\frac{s}{s+\lambda}}{A(s)+B(s)-C(s)+D(s)}\right\}^{-1},$$

completing the proof. □

For this example, $E[T_{\underline{z}}]$ can be evaluated explicitly from Corollary 3.2.1, as depicted in Figure 1. We note that the monotonicity of $E[T_{\underline{z}}]$ in $\underline{z}$ can be observed.

The probability of Product i being purchased can be derived directly from Equations (4.16) through (4.25) and Theorem 4.1. Let $\underline{\gamma}^*$, $\overline{\gamma}^*$ and A(s), B(s), C(s) and D(s) be as in Equations (5.5) and (5.6) respectively. One then has the following theorem. Proof is rather mechanical and is omitted here.

Theorem 5.2. *The probability of Product i being purchased can be evaluated through the four cases below:*

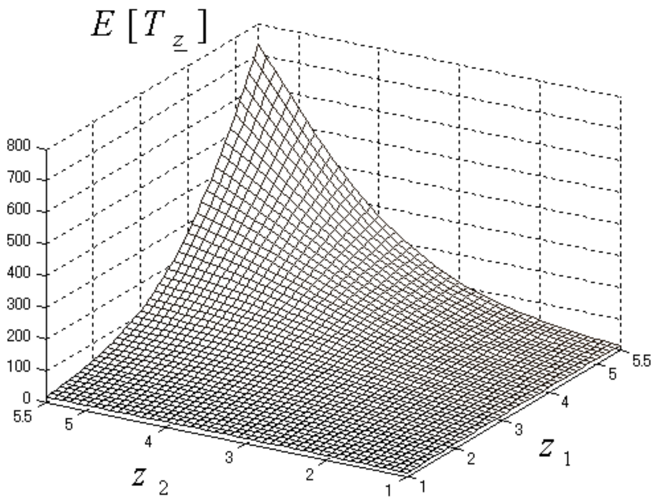

Figure 1: Mean search time ($\mu_1 = 2.7$, $\mu_2 = 2.7$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, $\lambda = 6$)

Case 1: $z_1 > z_2$, $\alpha_1 > \alpha_2$

$$
\begin{aligned}
\rho_1(\underline{z}) \;=\;& \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\
\times\;& \Bigg\{\Bigg\{[e^{-\lambda\frac{z_1}{\alpha_1}} - e^{-\lambda\frac{z_2}{\alpha_2}}] - \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}]\Bigg\} I\left\{\frac{z_1}{\alpha_1} \le \frac{z_2}{\alpha_2}\right\} \\
&+\frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[1-e^{-(\lambda-\mu_1\alpha_1)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \\
&+[e^{-(\mu_1 z_1+\mu_2 z_2)} - \frac{\mu_1}{\mu_1+\mu_2}e^{-(\mu_1 z_1+\mu_2 z_1)}]\frac{\lambda[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\
&+\frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}] I\left\{\frac{z_1}{\alpha_1} \le \frac{z_2}{\alpha_2}\right\} \\
&-\frac{\mu_1}{\mu_1+\mu_2}\cdot\frac{\lambda[e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\frac{z_2}{\alpha_2}}]}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1} I\left\{\frac{z_1}{\alpha_1} \le \frac{z_2}{\alpha_2}\right\} \\
&-\frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[e^{-(\lambda-\mu_1\alpha_1)\frac{z_2}{\alpha_2}} - e^{-(\lambda-\mu_1\alpha_1)\frac{z_1}{\alpha_2}}] I\left\{\frac{z_2}{\alpha_2} \le \frac{z_1}{\alpha_1}\right\} \\
&-\frac{\mu_1\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\mu_1+\mu_2}\cdot\frac{[e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_2}{\alpha_2}} - e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} I\left\{\frac{z_2}{\alpha_2} \le \frac{z_1}{\alpha_1}\right\} \\
&+e^{-\lambda\overline{\gamma}^*} + \frac{\lambda e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\overline{\gamma}^*}}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\rho_2(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\
&\times \left\{ \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_2\alpha_2)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \right. \\
&\quad + \left\{ [e^{-\lambda\frac{z_2}{\alpha_2}} - e^{-\lambda\frac{z_1}{\alpha_1}}] - \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[e^{-(\lambda-\mu_1\alpha_1)\frac{z_2}{\alpha_2}} - e^{-(\lambda-\mu_1\alpha_1)\frac{z_1}{\alpha_1}}] \right\} \\
&\quad \left. + \frac{\mu_1\lambda e^{-(\mu_1+\mu_2)z_1}}{\mu_1+\mu_2}\cdot\frac{1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} + \frac{\mu_1\lambda}{\mu_1+\mu_2}\cdot\frac{e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\frac{z_1}{\alpha_1}}}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1} \right\}
\end{aligned}
$$

Case 2: $z_1 \leq z_2,\ \alpha_1 > \alpha_2,$

$$
\begin{aligned}
\rho_1(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\
&\times \left\{ [e^{-\lambda\frac{z_1}{\alpha_1}} - e^{-\lambda\frac{z_2}{\alpha_2}}] - \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}] \right. \\
&\quad + \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[1-e^{-(\lambda-\mu_1\alpha_1)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \\
&\quad \left. + \frac{\lambda\mu_2\alpha_2}{\mu_1\alpha_1+\mu_2\alpha_2}\cdot\left\{ \frac{e^{-(\mu_1\frac{\alpha_1}{\alpha_2}z_2+\mu_2 z_2)}[1-e^{(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} + \frac{1}{\lambda}e^{-\lambda\frac{z_2}{\alpha_2}} \right\} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\rho_2(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\
&\times \left\{ \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_2\alpha_2)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \right. \\
&\quad + \left\{ [e^{-\lambda\frac{z_2}{\alpha_2}} - e^{-\lambda\frac{z_1}{\alpha_1}}] - \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[e^{-(\lambda-\mu_1\alpha_1)\frac{z_2}{\alpha_2}} - e^{-(\lambda-\mu_1\alpha_1)\frac{z_1}{\alpha_1}}] \right\} \\
&\quad + \frac{\lambda[e^{-\mu_1 z_1} - \frac{\mu_2}{\mu_1+\mu_2}e^{-(\mu_1+\mu_2)z_1}][1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\
&\quad + \frac{\lambda e^{-\mu_2 z_2}[e^{-\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_1}}]}{\lambda-\mu_2\alpha_2} \\
&\quad - \frac{\mu_2}{\mu_1+\mu_2}\cdot\frac{\lambda e^{-(\mu_1+\mu_2)z_2}[e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_2}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\
&\quad \left. + \frac{\mu_1}{\mu_1+\mu_2}\cdot\frac{\lambda e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\frac{z_2}{\alpha_1}}}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1} \right\}
\end{aligned}
$$

Case 3: $z_1 > z_2,\ \alpha_1 \le \alpha_2,$

$$\begin{aligned}\rho_1(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\ &\times \left\{ \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[1-e^{-(\lambda-\mu_1\alpha_1)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \right. \\ &+ \left\{ [e^{-\lambda\frac{z_1}{\alpha_1}} - e^{-\lambda\frac{z_2}{\alpha_2}}] - \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}] \right\} \\ &\left. + \frac{\mu_1\lambda e^{-(\mu_1+\mu_2)z_2}}{\mu_1+\mu_2}\cdot\frac{1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} + \frac{\mu_1\lambda}{\mu_1+\mu_2}\cdot\frac{e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\frac{z_1}{\alpha_1}}}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1} \right\}\end{aligned}$$

$$\begin{aligned}\rho_2(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\ &\times \left\{ \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[1-e^{-(\lambda-\mu_1\alpha_1)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \right. \\ &+ [e^{-(\mu_1 z_1+\mu_2 z_2)} - \frac{\mu_1}{\mu_1+\mu_2}e^{-(\mu_1 z_1+\mu_2 z_1)}]\frac{\lambda[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\ &+ \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[e^{-(\lambda-\mu_1\alpha_1)\frac{z_2}{\alpha_2}} - e^{-(\lambda-\mu_1\alpha_1)\frac{z_1}{\alpha_2}}] \\ &- \frac{\mu_1\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\mu_1+\mu_2}\cdot\frac{[e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_2}{\alpha_2}} - e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\ &\left. + e^{-\lambda\underline{\gamma}^*} + \frac{\lambda e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\overline{\gamma}^*}}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1} \right\}\end{aligned}$$

Case 4: $z_1 \le z_2,\ \alpha_1 \le \alpha_2,$

$$\begin{aligned}\rho_1(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\ &\times \left\{ \frac{\lambda e^{-\mu_1 z_1}}{\lambda-\mu_1\alpha_1}[1-e^{-(\lambda-\mu_1\alpha_1)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \right. \\ &+ \left\{ [e^{-\lambda\frac{z_2}{\alpha_2}} - e^{-\lambda\frac{z_2}{\alpha_2}}] - \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_1\alpha_1}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_1\alpha_1)\frac{z_1}{\alpha_1}}] \right\} \\ &+ \frac{\lambda[e^{-\mu_1 z_1} - \frac{\mu_2}{\mu_1+\mu_2}e^{-(\mu_1+\mu_2)z_1}][1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\ &+ \frac{\lambda e^{-\mu_2 z_2}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_1}}]}{\lambda-\mu_2\alpha_2} \\ &- \frac{\mu_1}{\mu_1+\mu_2}\cdot\frac{\lambda e^{-(\mu_1+\mu_2)z_1}[e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_2}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} \\ &\left. + \frac{\mu_1}{\mu_1+\mu_2}\cdot\frac{\lambda e^{-(\lambda-\mu_2\alpha_2+\mu_2\alpha_1)\frac{z_2}{\alpha_1}}}{\lambda-\mu_2\alpha_2+\mu_2\alpha_1} \right\}\end{aligned}$$

$$
\begin{aligned}
\rho_2(\underline{z}) &= \frac{1}{A(0)+B(0)-C(0)+e^{-\lambda\underline{\gamma}^*}} \\
&\times \left\{ [e^{-\lambda\frac{z_1}{\alpha_1}} - e^{-\lambda\frac{z_2}{\alpha_2}}] I\left\{\frac{z_1}{\alpha_1} \le \frac{z_2}{\alpha_2}\right\} \right. \\
&\quad - \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[e^{-(\lambda-\mu_2\alpha_2)\frac{z_1}{\alpha_1}} - e^{-(\lambda-\mu_2\alpha_2)\frac{z_2}{\alpha_2}}] I\left\{\frac{z_1}{\alpha_1} \le \frac{z_2}{\alpha_2}\right\} \\
&\quad + \frac{\lambda e^{-\mu_2 z_2}}{\lambda-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_2\alpha_2)\underline{\gamma}^*}] - \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2}[1-e^{-(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\underline{\gamma}^*}] \\
&\quad \left. - \frac{\lambda\mu_1\alpha_1}{\mu_1\alpha_1+\mu_2\alpha_2}\left\{ \frac{e^{-(\mu_1\frac{\alpha_1}{\alpha_2}z_2+\mu_2 z_2)}[1-e^{(\lambda-\mu_1\alpha_1-\mu_2\alpha_2)\frac{z_1}{\alpha_1}}]}{\lambda-\mu_1\alpha_1-\mu_2\alpha_2} + \frac{1}{\lambda}e^{-\lambda\frac{z_2}{\alpha_2}} \right\} \right\}
\end{aligned}
$$

We are now in a position to demonstrate numerical examples based on Theorem 5.2. The basic set of the underlying parameter values is given in Table 1.

Table 1 : Basic set of parameter values

parameter	λ	z_1	z_2	α_1	α_2	μ_1	μ_2
value	6.0	3.2	3.2	0.2	0.1	2.7	2.7

Figure 2 depicts $\rho_1(\underline{z})$ and $\rho_2(\underline{z})$ as functions of μ_2 and z_2 where μ_2 is varied from 0.5 to

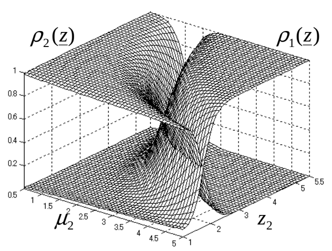

Figure 2: $\mu_1 = 2.7$ $z_1 = 3.2$

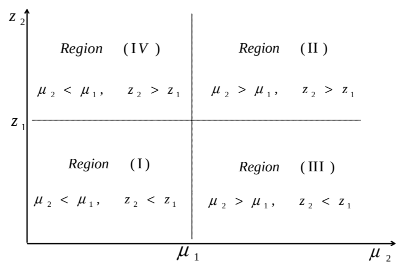

Figure 3: Parameter range decomposition

5.0, and z_2 is varied from 1.0 to 5.5. We recall that the exponential variate $E_1(\mu_1)$ of mean μ_1^{-1} is stochastically larger than the exponential variate $E_2(\mu_2)$ of mean μ_2^{-1} if and only if $\mu_1 < \mu_2$, i.e.

$$P[E_1(\mu_1) > x] = e^{-\mu_1 x} > e^{-\mu_2 x} = P[E_2(\mu_2) > x] \Longleftrightarrow \mu_1 < \mu_2.$$

Keeping this in mind, one can observe that the conjecture stated in Remark 4.2 holds true in these numerical examples, that is, $\rho_2(\underline{z})$ increases as both z_2 and μ_2 decrease. In order to see this point more clearly, we decompose the parameter range into four regions as shown in Figure 3. The corresponding graphs of $\rho_1(\underline{z})$ and $\rho_2(\underline{z})$ are redrawn for each region, as given in Figures 4 through 7. We note that $\rho_2(\underline{z})$ dominates $\rho_1(\underline{z})$ in Region (I) with $\mu_2 < \mu_1$, $z_2 < z_1$, while this dominance is reversed in Region (II) with $\mu_2 > \mu_1$, $z_2 > z_1$, as expected. In Region (III) with $\mu_2 > \mu_1$, $z_2 < z_1$, it can be seen that $\rho_2(\underline{z})$ is greater than $\rho_1(\underline{z})$ for relatively large μ_2 and small z_2. This is so because the advantage of $P2$ in z_2 smaller than

z_1 overwhelms the disadvantage of $P2$ in μ_2 larger than μ_1. However, this dominance is reversed as both μ_2 and z_2 increase, resulting in crossing of the graphs of $\rho_1(\underline{z})$ and $\rho_2(\underline{z})$. Similar behaviors of $\rho_1(\underline{z})$ and $\rho_2(\underline{z})$ can be observed in the opposite manner in Region (IV), where $\mu_2 < \mu_1$ and $z_2 > z_1$.

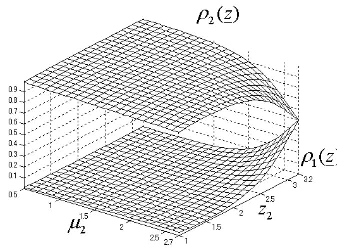

Figure 4: Region (I)
$\mu_2 < \mu_1 = 2.7$, $z_2 < z_1 = 3.2$

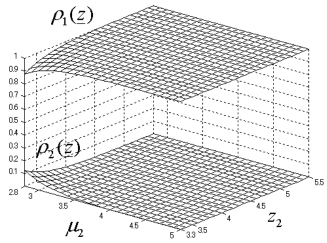

Figure 5: Region (II)
$\mu_2 > \mu_1 = 2.7$, $z_2 > z_1 = 3.2$

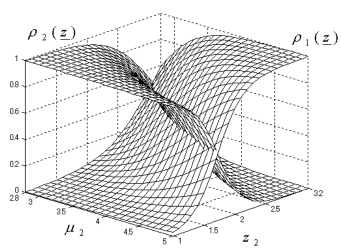

Figure 6: Region (III)
$\mu_2 > \mu_1 = 2.7$, $z_2 < z_1 = 3.2$

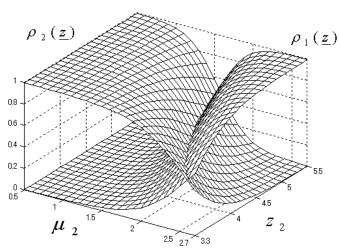

Figure 7: Region (IV)
$\mu_2 < \mu_1 = 2.7$, $z_2 > z_1 = 3.2$

6. Concluding Remarks

In this paper, the general shock model of Shanthikumar and Sumita [3] is extended so as to incorporate multiple types of shocks generated from a common renewal sequence. More specifically, a correlated multivariate shock model is considered where a system is subject to a sequence of J different shocks triggered by a common renewal process. Let $(Y(k))_{k=1}^{\infty}$ be a sequence of independently and identically distributed (i.i.d.) nonnegative random variables associated with the renewal process. For the magnitudes of the k-th shock denoted by a random vector $\underline{X}(k)$, it is assumed that $[\underline{X}(k), Y(k)]$ $(k = 1, 2, \cdots)$ constitute a sequence of i.i.d. random vectors with respect to k while $\underline{X}(k)$ and $Y(k)$ may be correlated. The system fails as soon as the historical maximum of the magnitudes of any component of the random vector exceeds a prespecified level of that component. The Laplace transform of the probability density function of the system lifetime is derived, and its mean and variance are obtained explicitly. Furthermore, the probability of system failure due to the i-th component is obtained explicitly for all $i \in \mathcal{J} = \{1, \cdots, J\}$. The model is applied for analyzing the browsing behavior of Internet users.

The model proposed in this paper relies upon the information search completion time determined by the historical maximum of the value of information gathered by a customer. In some situations, however, the customer may make a decision based on the cumulative value of information gathered by time t. While such cumulative shock models with a single type of shocks have been studied by Sumita and Shanthikumar [4], the multivariate version has not been studied yet. This research is in progress and will be reported elsewhere.

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