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SECOND-BEST CONGESTION PRICING AND USER HETEROGENEITY IN STATIC TRANSPORTATION NETWORKS

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Abstract This paper explores the second-best pricing problem of congested transportation networks with elastic demand where only a subset of links are charged. We derive a simple expression of the second-best pricing rule and provide a clear interpretation to the second-best pricing. Also examined is the issue of heterogeneity and anonymity of users. We extend the analysis of the second-best price to the case with heterogeneous user population, and study the impact of type-dependent/independent pricing schemes on social welfare.

Keywords: Transportation, economics, congestion pricing, second best, networks, anonymity

1. Introduction

In this paper, we study the second-best pricing problem of congested transportation networks with elastic demand. As modern methods of electronic toll collection are being implemented in many cities in the world, the discussion of road pricing is revitalized among transportation researchers.

It is well-known that the Pigouvian externality pricing results in Pareto optimal use of the road network. This pricing however requires charging users at all links. In real transportation networks, only a subset of links can be charged. Thus, we are forced to make do with a second-best solution. In other words, it is necessary to solve a constraint optimization. Marchand [6] investigates the second-best pricing in a transportation system with two parallel routes, one toll-way and one freeway. This problem now is considered to be a classic example of the subject. The externality pricing without considering its impact on the untolled road results in a suboptimally high use of the untolled road. Verhoef [10], [11] and Verhoef et al [12] further study this problem with various demand and cost parameters, and investigate the second-best price for general static networks.

A related issue is the heterogeneity of user population. Small [8] indicates that the opportunity cost of time varies depending on the user subgroup. Question arises if the charging should reflect the users' type. Arnott and Kraus [1] investigate this problem in the context of the bottleneck model, which is a dynamic model of rush-hour traffic congestion. They formally show that the type-dependent pricing does not outperform type-independent one if the congestion charge is time dependent and the first-best condition hold true. Yang and Huang [13] explore the issue of user heterogeneity and show that, if every link is tolled, type-independent toll pattern can supports a social optimum as an equilibrium, see also Yin and Yang [15]. Small and Yan [9] investigate the second-best pricing with two types of users in the classic two-route problem. They conclude that the heterogeneity in value of time plays an important role in evaluating pricing policies. For related issues on road pricing, readers are referred to a comprehensive survey by Button [3] and an extensive book by Yang and Huang [14].

The purpose of this paper is two-fold. First, we re-visit the second-best problem of general static networks considered by various researchers [9], [10], and [14]. We then derive under some technical condition a succinct expression of the second-best pricing rule for static networks with homogeneous users. This concise expression gives an clear interpretation of the second-best pricing that have never been provided in the literature. As an example, we apply this formula to the classic transport problem with two parallel routes, one tolled and the other not, discussed by [6]. The result provides an unequivocally clear graphical explanation to the second-best price to the classical example of uncharged alternative route.

Second, we consider the user heterogeneity in value of time and the impact of type dependent/independent pricing schemes on social welfare. To this end, we extend the analysis of the second-best price to the case with a heterogeneous user population. As Yang and Huang [13] show, the user anonymity does not play any negative role if all links can be tolled in a static network. Then, a natural question is: to what extent does the user anonymity hurt the social welfare if some links are untolled? The numerical experiments suggest that the magnitude of the welfare loss due to the anonymity is not very large.

The paper is organized as follows. In Section 2, we formally describe the model and the first-best outcome as a benchmark. In Section 3, we derive a simple expression for the second-best pricing. Section 4 is devoted to the issue of user heterogeneity and user anonymity. Finally, we give concluding remarks in Section 4.

2. A Static Model

In this section, we describe a static model of traffic network with only a subset of links being tolled.

Let $\mathcal{L} = \{1, 2, \ldots, L\}$ be the set of all links. A route r is characterized by a set of links. The set of all available routes is represented by $\mathcal{R}$. The origination-destination (OD) pair served by route r is denoted by $s(r)$. The set of all OD pairs is denoted by $\mathcal{S}$. We assume a continuum of users. The marginal value function of traffic for OD pair $s \in \mathcal{S}$ is given by $V_s'(z_s)$ where z_s is the traffic volume of the OD pair s. The delay $W_\ell(\cdot)$ at link ℓ is a function of link traffic volume y_ℓ. We assume that W_ℓ and V_s' are continuously differentiable, $V_s'(\cdot)$ is strictly decreasing, and $W_\ell(\cdot)$ is strictly increasing. The delay cost is linear with $c > 0$ per unit delay.

Let $\boldsymbol{H} = (H_{sr} \colon s \in \mathcal{S}, r \in \mathcal{R})$ with $H_{sr} = 1$ if $s = s(r)$ and $H_{sr} = 0$ otherwise, and $\boldsymbol{A} = (A_{\ell r} \colon \ell \in \mathcal{L}, r \in \mathcal{R})$ with $A_{\ell r} = 1$ if $\ell \in r$ and $A_{\ell r} = 0$ otherwise. Then, the route traffic $\boldsymbol{x} = (x_r \colon r \in \mathcal{R})$, link traffic $\boldsymbol{y} = (y_\ell \colon \ell \in \mathcal{L})$ and OD traffic $\boldsymbol{z} = (z_s \colon s \in \mathcal{S})$ are related by

$$\boldsymbol{Hx} = \boldsymbol{z}, \quad \boldsymbol{Ax} = \boldsymbol{y}. \tag{2.1}$$

Denoted by $\mathcal{B} = \{1, 2, \ldots, B\}$ is the set of toll booths. In this paper, we discuss the link pricing scheme. Suppose that toll booth b is located at link $\ell_b, b \in \mathcal{B}$. Let $\boldsymbol{B} = \left(\boldsymbol{A}_{\ell_1}^T, \boldsymbol{A}_{\ell_2}^T, \ldots, \boldsymbol{A}_{\ell_B}^T\right)$ where $\boldsymbol{A}_\ell$ is the ℓth row vector of $\boldsymbol{A}$. Note that, given the pricing vector $\boldsymbol{p} = (p_b : b \in \mathcal{B})$, the toll charged to users of route r is given by the rth element of $\boldsymbol{Bp}$.

A problem of finding the demand level and routes to maximize the net overall benefit or

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the social welfare is given by the following convex programming problem.

$$\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{s=1}^{S} V_s(z_s) - \sum_{\ell=1}^{L} c y_\ell W_\ell(y_\ell) \tag{2.2}$$

$$\text{s.t. } \boldsymbol{H}\boldsymbol{x} = \boldsymbol{z}, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \quad \boldsymbol{x} \geq \boldsymbol{0}. \tag{2.3}$$

A solution to (2.2) is called a first-best solution and is denoted by $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$. For notational convenience, let

$$\begin{aligned} W(\boldsymbol{y}) &\equiv (W_1(y_1), W_2(y_2), \ldots, W_L(y_L))^T, \\ V(\boldsymbol{z}) &\equiv (V_1(z_1), V_2(z_2), \ldots, V_S(z_S))^T, \\ M(\boldsymbol{z}) &\equiv V'(\boldsymbol{z})^T \boldsymbol{1}, \end{aligned} \tag{2.4}$$

where $\boldsymbol{1}$ is the vector with all elements equal to 1. The first order condition to (2.2) is given by

$$\begin{aligned} \boldsymbol{H}^T M(\boldsymbol{z}) - c\boldsymbol{A}^T W(\boldsymbol{y}) - c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{y} \leq \boldsymbol{0}, \\ \boldsymbol{x}^T \left(\boldsymbol{H}^T M(\boldsymbol{z}) - c\boldsymbol{A}^T W(\boldsymbol{y}) - c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{y}\right) = \boldsymbol{0}. \end{aligned} \tag{2.5}$$

On the other hand, given the pricing vector $\boldsymbol{p} = (p_b \colon b \in \mathcal{B})$, the equilibrium traffic is characterized by the fact that everyone uses the best route among those serving his OD pair and the effective cost (toll + the opportunity cost of time) is equal to the marginal value, see e.g. [4]. Thus, for every $r \in \mathcal{R}$,

$$x_r > 0 \Rightarrow \begin{cases} \min_{r': s(r')=s(r)} \left(\sum_b B_{r'b} p_b + \sum_{\ell \in r'} c W_\ell(y_\ell) \right) & = \sum_b B_{rb} p_b + \sum_{\ell \in r} c W_\ell(y_\ell) \\ & = V'_{s(r')}(z_{s(r')}) \end{cases}$$

This condition is neatly expressed as the following:

$$\begin{aligned} \boldsymbol{H}^T M(\boldsymbol{z}) - c\boldsymbol{A}^T W(\boldsymbol{y}) - \boldsymbol{B}\boldsymbol{p} \leq \boldsymbol{0}, \\ \boldsymbol{x}^T \left(\boldsymbol{H}^T M(\boldsymbol{z}) - c\boldsymbol{A}^T W(\boldsymbol{y}) - \boldsymbol{B}\boldsymbol{p}\right) = \boldsymbol{0}. \end{aligned} \tag{2.6}$$

The traffic flow $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ satisfying (2.3) and (2.6) is the Wardrop equilibrium. It is known that under this setting the problem of finding an equilibrium is equivalent to a certain convex programming problem, see Florian and Hearn [4]. Also known is that the equilibrium is unique in the link flow $\boldsymbol{y}$ and the OD flow $\boldsymbol{z}$ but not in the route flow $\boldsymbol{x}$, see Kelly [5].

When all links can be tolled, the link pricing can induce the optimality. To see this, we note that $\boldsymbol{B} = \boldsymbol{A}^T$ if all links are tolled independently. The optimality arises from a link pricing scheme $\boldsymbol{p}^* \equiv (p^*_\ell \colon \ell \in \mathcal{L})$ by setting

$$\boldsymbol{p}^* \equiv cW'(\boldsymbol{y}^*)\boldsymbol{y}^* \tag{2.7}$$

since (2.5) and (2.6) coincide under this setting. As in the standard literature, (2.7) represents the congestion externality.

3. Second Best Pricing

A problem arises when not all links can be charged. In this section we provide a clear economic interpretation to the second-best price. In particular, the classic two-route problem is re-examined and its second-best price is neatly explained in a graph. To this end, we derive a succinct expression of the second-best price.

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The second-best pricing is a solution to the following problem:

$$\max_{\boldsymbol{p},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \mathbf{1}^T V(\boldsymbol{z}) - c\boldsymbol{y}^T W(\boldsymbol{y}) \quad (3.1)$$
$$\text{s.t.} \quad (2.3), (2.6) \text{ and } \boldsymbol{x} \geq \mathbf{0}.$$

Suppose that $(\boldsymbol{p}^{**}, \boldsymbol{x}^{**}, \boldsymbol{y}^{**}, \boldsymbol{z}^{**})$ is a solution to (3.1). Assume a mild condition of non-degeneracy. That is, $x_r^{**} > 0$ if and only if $V_{s(r)}(z^{**}) = \sum_b B_{rb} p_b^{**} + \sum_{\ell \in r} W_\ell(y_\ell^{**})$. To investigate the structure of the solution, we drop all inactive routes r with $x_r^{**} = 0$, and redefine $\mathcal{R}$ to be the set of all active routes at the optimal solution. We adjust the dimension of $\boldsymbol{x}$ and other relevant vectors and matrices accordingly. Then, the problem of finding the second-best solution (3.1) is written as

$$\max_{\boldsymbol{p},\boldsymbol{x},\boldsymbol{y}.\boldsymbol{z}} \mathbf{1}^T V(\boldsymbol{z}) - c\boldsymbol{y}^T W(\boldsymbol{y})$$
$$\text{s.t.} \quad \boldsymbol{H}^T M(\boldsymbol{z}) - c\boldsymbol{A}^T W(\boldsymbol{y}) - \boldsymbol{B}\boldsymbol{p} = \mathbf{0} \quad (3.2)$$
$$\text{and } (2.3).$$

After some algebra (see Appendix 1), one has the following first order condition:

$$\left(c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{A} - \boldsymbol{H}^T M'(\boldsymbol{z})\boldsymbol{H}\right)\boldsymbol{\lambda} = c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{y} - \boldsymbol{B}\boldsymbol{p}, \quad (3.3)$$
$$\boldsymbol{B}^T \boldsymbol{\lambda} = \mathbf{0} \quad (3.4)$$

where $\boldsymbol{\lambda} \in \mathbb{R}^R$ is the Lagrange multiplier associated with constraint (3.2). While (3.3) and (3.4) for the second-best price may look mysterious, the literature (e.g. Marchand [6]) strongly suggests that the second-best price is the congestion externality arising from both priced and unpriced roads. Here, we provide an easy interpretation based on a concise expression of the second-best price under some technical assumption.

To derive a concise expression of the second-best price, we make the following technical assumption as in [7].

$$\text{rank}\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{H} \end{pmatrix} = R. \quad (3.5)$$

The column full rank condition (3.5) admittedly is strong. Specifically, it is likely to hold only for small networks. For example, the classic two-route problem satisfies this condition. The simple and well-known example known as Braess's paradox, see Section 3 of Braess et al [2], satisfies the rank condition. We note that the number of routes tends to grow exponentially as a function of the number of links. When the number of columns (= the number of routes) in (3.5) is larger than that of rows, the column full rank condition obviously cannot hold. Thus, we cannot expect the full rank condition to be valid for large networks. In Appendix 2, we provide a more complicated example of network satisfying the full rank condition. Under assumption (3.5), the matrix in the brackets of the left hand side of (3.3) is positive definite. Thus, from (3.3) and (3.4), one has

$$\boldsymbol{B}^T \left(c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{A} - \boldsymbol{H}^T M'(\boldsymbol{z})\boldsymbol{H}\right)^{-1} \boldsymbol{B}\boldsymbol{p}$$
$$= \boldsymbol{B}^T \left(c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{A} - \boldsymbol{H}^T M'(\boldsymbol{z})\boldsymbol{H}\right)^{-1} c\boldsymbol{A}^T W'(\boldsymbol{y})\boldsymbol{y}. \quad (3.6)$$

Furthermore, since (2.6) determines $\boldsymbol{y}$ and $\boldsymbol{z}$ uniquely for an arbitrarily given $\boldsymbol{p}$, (2.1) uniquely determines $\boldsymbol{x}$ under assumption (3.5). We thus define $x^\circ(\boldsymbol{p})$ to be the equilibrium

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route traffic under pricing $\boldsymbol{p}$. Similar notations are employed for link and OD traffic. The equilibrium traffic $x^{\circ}(\boldsymbol{p})$ is implicitly determined by

$$\boldsymbol{H}^T M(\boldsymbol{Hx}) = c\boldsymbol{A}^T W(\boldsymbol{Ax}) + \boldsymbol{Bp}. \tag{3.7}$$

Taking the derivative of (3.7) with respect to $\boldsymbol{p}$, one has

$$x^{\circ}_{\boldsymbol{p}} = -\left(c\boldsymbol{A}^T W'\boldsymbol{A} - \boldsymbol{H}^T M'\boldsymbol{H}\right)^{-1}\boldsymbol{B} \tag{3.8}$$

where $x^{\circ}_{\boldsymbol{p}} = x^{\circ}_{\boldsymbol{p}}(\boldsymbol{p})$ is the Jacobian of $x^{\circ}(\boldsymbol{p})$. We note that rank$\boldsymbol{B} = B$. Thus, $x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}$ is nonsingular. We hence have the following theorem from (3.6) and (3.8).

Theorem 1 *The second-best price* $\boldsymbol{p}^{**}$ *is given by*

$$\boldsymbol{p}^{**} = \left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1} x^{\circ T}_{\boldsymbol{p}}\left(c\boldsymbol{A}^T W' y^{\circ}\right). \tag{3.9}$$

where $x^{\circ}_{\boldsymbol{p}} = x^{\circ}_{\boldsymbol{p}}(\boldsymbol{p}^{**}), W' = W'(\boldsymbol{Ax}^{**})$ *and* $y^{\circ} = y^{\circ}(\boldsymbol{p}^{**})$.

The last factor $c\boldsymbol{A}^T W' y^{\circ}$ on the right hand side of (3.9) represents the standard congestion externality of routes. So, the second-best price $\boldsymbol{p}^{**}$ is a linear combination of route externalities. But what is the meaning of the coefficients $\left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1} x^{\circ T}_{\boldsymbol{p}}$? The literature [3] suggests that $\boldsymbol{p}^{**}$ should represent the congestion externalities. Here, we give a clear interpretation of the second best price (3.1). To understand the coefficients, pre-multiply $\boldsymbol{e}_r\boldsymbol{B}$ to the both sides of (3.9) where $\boldsymbol{e}_r$ is the rth unit vector of length R.

$$\begin{aligned}\begin{pmatrix}\text{the second-best price}\\ \text{for route-}r\text{ users}\end{pmatrix} &= \boldsymbol{e}_r\boldsymbol{Bp}^{**}\\ &= \boldsymbol{e}_r\boldsymbol{B}\left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1} x^{\circ T}_{\boldsymbol{p}}\left(c\boldsymbol{A}^T W' y^{\circ}\right).\end{aligned} \tag{3.10}$$

To understand (3.10), we examine the impact of an infinitesimal increment of route flow x_r. To be more specific, we suppose that the following happens at equilibrium:

(a) there is an increase of one (infinitesimal) unit of flow at all toll booths along route r,
(b) there is no change of flow at other toll booths, and
(c) due to (a) and (b) there may be changes of flow at links with no toll booth.

For (a) and (b) to happen at equilibrium, there should be changes in toll charges that induce (a) and (b) in equilibrium flow. A little pondering reveals that $\boldsymbol{e}_r\boldsymbol{B}\left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1}$ represents such changes of toll charges*, which in turn induce a change of flow in the whole network. The resulting change of route flow is given by $\boldsymbol{e}_r\boldsymbol{B}\left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1} x^{\circ T}_{\boldsymbol{p}}$. Since $c\boldsymbol{A}^T W' y$ represents the standard route externalities, we conclude from (3.10) that the second best price for the users of route r is the externality effect arising from (a), (b) and (c).

Since the first best solution is achieved when all links are tolled, the difference between the first best and the second pricing scheme lies in the presence of the externalities arising from (c).

For the following classical example of uncharged alternative route, Theorem 1 provides an unequivocally clear graphical explanation to the second-best price of the classic two-route problem.

*Let $\Delta\boldsymbol{p}$ be a change in price at toll booths. Then, $\Delta\boldsymbol{p}x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}$ gives the change in the route flow $\boldsymbol{x}$ due to the change in tolls $\Delta\boldsymbol{p}$. Thus, $\Delta\boldsymbol{x}\left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1}$ represents the change in tolls $\boldsymbol{p}$ that induces the change $\Delta\boldsymbol{x}$ in route flow. Note that $\Delta\boldsymbol{x} = \boldsymbol{e}_r\boldsymbol{B}$ is a flow change at the links with a toll booth resulting from the flow increase of route r by one (infinitesimal) unit. Hence, $\boldsymbol{e}_r\boldsymbol{B}\left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1}$ is the change in tolls that induces (a) and (b).

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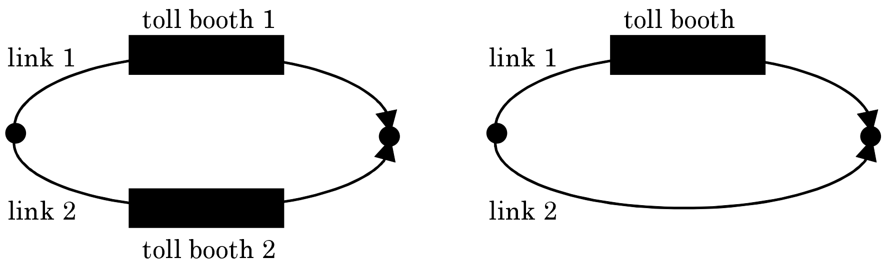

Figure 1: Charged alernative route v.s. uncharged alternative route

Example 1 *Consider networks as in Figures 1 with two routes. In the network on the left hand side each route has a toll booth, while in the network on the right hand side only route 1 can be charged. We note that these networks satisfy the column full rank condition (3.5). For the sake of exposition, we consider the following simple demand and link characteristics.*

$$V'(z) = -10z + 150, \quad W_1(y_1) = 10y_1, \quad W_2(y_2) = 10y_2, \quad c = 1.$$

If both routes can be charged, every flow pattern (x_1, x_2) *in the shaded area in Figure 2 can be induced as an equilibrium by an appropriate pricing* $\boldsymbol{p} = (p_1, p_2)$*. The first-best solution and the corresponding price are given by*

$$(x_1^*, x_2^*) = \left(\frac{15}{4}, \frac{15}{4}\right), \quad (p_1^*, p_2^*) = \left(\frac{75}{2}, \frac{75}{2}\right).$$

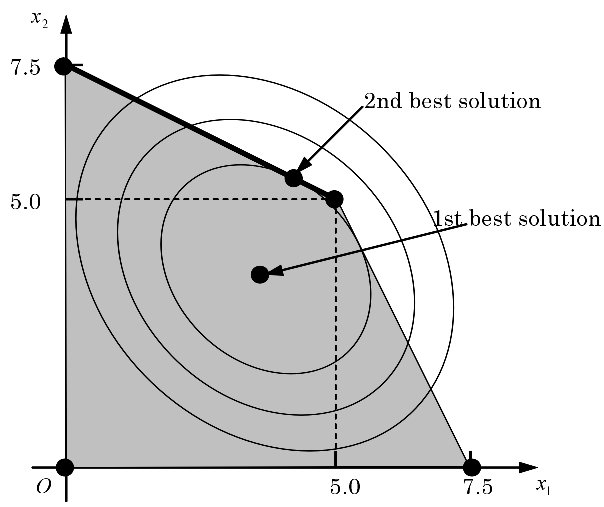

Figure 2: First best solution v.s. second best solution

When route 2 is uncharged, however, the set of inducible equilibria is restricted to the solid line (the line connecting $(0, 7.5)$ *and* $(5, 5)$*) in Figure 2. The second-best price (3.9) is*

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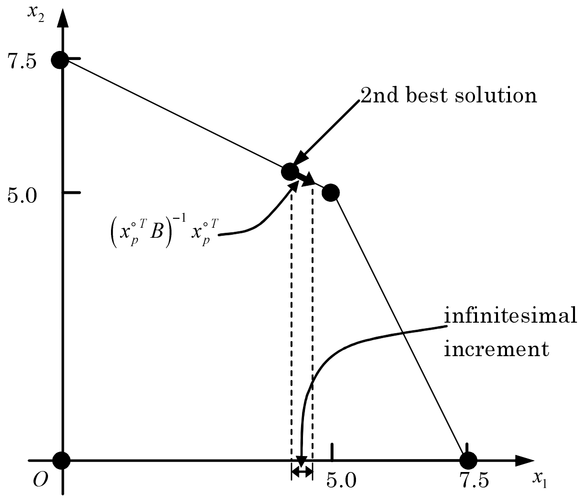

Figure 3: Impacts of an infinitesimal increment of x_1

calculated as follows. We first evaluate $x^\circ_{\boldsymbol{p}}$ *as*

$$\begin{aligned} x^\circ_{\boldsymbol{p}} &= -\left(c\boldsymbol{A}^T W'\boldsymbol{A} - \boldsymbol{H}^T M'\boldsymbol{H}\right)^{-1}\boldsymbol{B} \\ &= -\left\{1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}10 & 0\\ 0 & 10\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1\\ 1\end{pmatrix}(-10)\begin{pmatrix}1, & 1\end{pmatrix}\right\}^{-1}\begin{pmatrix}1\\ 0\end{pmatrix} \\ &= -\frac{1}{30}\begin{pmatrix}2\\ -1\end{pmatrix}. \end{aligned}$$

The impacts of an infinitesimal increment of x_1 *on the flow vector, see Fig. 3 is given by*

$$\begin{aligned} \left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1}x^{\circ T}_{\boldsymbol{p}} &= \left\{-\frac{1}{30}\begin{pmatrix}2, & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\right\}^{-1}\left\{-\frac{1}{30}\begin{pmatrix}2, & -1\end{pmatrix}\right\} \\ &= \begin{pmatrix}1, & -\frac{1}{2}\end{pmatrix}. \end{aligned}$$

Thus, the second-best price is the sum of the standard link externalities weighted by the flow changes due to an infinitesimal increment of x_1*. That is,*

$$\begin{aligned} p^{**} &= \left(x^{\circ T}_{\boldsymbol{p}}\boldsymbol{B}\right)^{-1}x^{\circ T}_{\boldsymbol{p}}\left(c\boldsymbol{A}^T W' y^\circ\right) \\ &= \begin{pmatrix}1, & -\frac{1}{2}\end{pmatrix}\left\{1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}10 & 0\\ 0 & 10\end{pmatrix}\begin{pmatrix}\frac{45}{11}\\ \frac{60}{11}\end{pmatrix}\right\} \\ &= 1\cdot\frac{450}{11} - \frac{1}{2}\cdot\frac{600}{11} \\ &= \frac{150}{11} \end{aligned} \tag{3.11}$$

To interpret (3.11), we note that an infinitesimal increment of flow of route 1 at the second-best solution has a negative externality of $\frac{450}{11}$ *on route 1. For the flow of route 1 to increase an infinitesimal unit at equilibrium, the flow of route 2 has to decrease by* $\frac{1}{2}$ *infinitesimal unit, see Figure 3. An infinitesimal increment of flow of route 2 has a negative externality of* $\frac{600}{11}$ *on route 2. Thus, the overall externality of an infinitesimal increment of flow of route 1 should be given by (3.11), which is the second-best pricing.*

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4. Heterogeneity and Anonymity of Users

In the previous sections, users are assumed to be homogeneous apart from their OD pairs. In particular, all users have an identical factor c of the opportunity cost of time. In this section, we investigate the impact of users' heterogeneity in opportunity cost of time on the second-best pricing policy and the associated problem of user anonymity. In the real traffic network, discrimination of users by types of users is implemented in a crude manner. Specifically, the tolls depend on the type/capacity of vehicle. For example, in Japan, vehicles are classified into five categories: two types of passenger automobiles and three types of cargo trucks where the type depends on the weight/capacity of the vehicles. The highway tolls depend on this classification. Clearly, the true value of user's time is private information. Thus, the crude toll discrimination based on the vehicle classification may be the best it can go. It is known that, when all links are tolled, the anonymous pricing scheme is as good as the type-dependent pricing scheme in terms of social welfare, see Yang and Huang [13], [14], and Yin and Yang [15]. In this section, we formulate the second best problem with heterogeneous users, and find through a numerical example that even when not all links are tolled, the welfare loss due to the anonymous pricing scheme is not be very large.

Suppose that the user population is segmented into T types depending on the value of time. Type t is characterized by the delay cost c_t. Let $\mathcal{T} = \{1, 2, \ldots, T\}$. The route flow vector $\boldsymbol{x}$ is denoted by

$$\boldsymbol{x} = (\boldsymbol{x}_t \colon t \in \mathcal{T}), \quad \boldsymbol{x}_t = (x_{tr} \colon r \in \mathcal{R}).$$

We employ similar notations for the link flow ($\boldsymbol{y}$ and $\boldsymbol{y}_t$) and the OD flow ($\boldsymbol{z}$ and $\boldsymbol{z}_t$). Let $\hat{\boldsymbol{H}}$ be the block-diagonal matrix with T diagonal blocks, each of which is $\boldsymbol{H}$. Define $\hat{\boldsymbol{A}}$ similarly. Then, the traffic equation is expressed as

$$\hat{\boldsymbol{H}}\boldsymbol{x} = \boldsymbol{z}, \quad \hat{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{y}. \tag{4.1}$$

Let

$$\begin{aligned} \hat{V}(\boldsymbol{z}) &= \left(\hat{V}_1(\boldsymbol{z}_1), \hat{V}_2(\boldsymbol{z}_2), \ldots, \hat{V}_T(\boldsymbol{z}_T)\right)^T, \\ \hat{V}_t(\boldsymbol{z}_t) &= \left(\hat{V}_{t1}(z_{t1}), \hat{V}_{t2}(z_{t2}), \ldots, \hat{V}_{tS}(z_{tS})\right)^T \end{aligned}$$

where $\hat{V}_{ts}(\cdot)$ is the value function of the traffic associated with type $t \in \mathcal{T}$ and OD pair $s \in \mathcal{S}$. Define the vectors of marginal value functions and delay functions as

$$\begin{aligned} \hat{M}(\boldsymbol{z}) &= \left(\hat{M}_1(\boldsymbol{z}_1), \hat{M}_2(\boldsymbol{z}_2), \ldots, \hat{M}_T(\boldsymbol{z}_T)\right)^T, \quad \hat{M}_t(\boldsymbol{z}_t) = (\hat{V}'_{ts}(z_{ts}) \colon s \in \mathcal{S})^T, \\ \hat{W}(\boldsymbol{y}) &= (W(\overline{\boldsymbol{y}}), W(\overline{\boldsymbol{y}}), \ldots, W(\overline{\boldsymbol{y}})), \quad \overline{\boldsymbol{y}} = \sum_{t \in \mathcal{T}} \boldsymbol{y}_t \end{aligned}$$

where $W(\cdot)$ is defined as in (2.4). Let $\hat{\boldsymbol{C}}$ be the diagonal matrix with diagonal blocks $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_T$ where $\boldsymbol{C}_t$ is the diagonal matrix of order L with all diagonal elements being c_t. Then, the problem of finding the first-best solution can be written as

$$\begin{aligned} &\max_{\boldsymbol{x}} \mathbf{1}^T \hat{V}(\hat{\boldsymbol{H}}\boldsymbol{x}) - (\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x})^T \hat{W}(\hat{\boldsymbol{A}}\boldsymbol{x}) \\ &\text{s.t. } \boldsymbol{x} \geq \mathbf{0}. \end{aligned} \tag{4.2}$$

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We denote the first-best solution by $\boldsymbol{x}^*$. The corresponding first order condition is given by

$$\begin{aligned} &\boldsymbol{x} \geq \boldsymbol{0}, \\ &\hat{\boldsymbol{H}}^T \hat{M}(\hat{\boldsymbol{H}}\boldsymbol{x}) - (\hat{\boldsymbol{C}}\hat{\boldsymbol{A}})^T(\hat{W}(\hat{\boldsymbol{A}}\boldsymbol{x})) - \hat{\boldsymbol{A}}^T \hat{W}'(\hat{\boldsymbol{A}}\boldsymbol{x})(\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x}) \leq \boldsymbol{0}, \\ &\boldsymbol{x}^T \left(\hat{\boldsymbol{H}}^T \hat{M}(\hat{\boldsymbol{H}}\boldsymbol{x}) - (\hat{\boldsymbol{C}}\hat{\boldsymbol{A}})^T(\hat{W}(\hat{\boldsymbol{A}}\boldsymbol{x})) - \hat{\boldsymbol{A}}^T \hat{W}'(\hat{\boldsymbol{A}}\boldsymbol{x})(\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x})\right) = 0. \end{aligned} \tag{4.3}$$

Let $\mathcal{B} = \{1, 2, \ldots, B\}$ be the set of all toll booths. Each toll booth is associated the a link-type pair. Set $B_{(t,r),b} = 1$ if toll booth b is located at a link along route r and collects charges from users of type t, and $B_{(t,r),b} = 0$ otherwise. Define $R \times B$ matrix $\boldsymbol{B}_t = (B_{(t,r),b} \colon r \in \mathcal{R}, b \in \mathcal{B})$ for $t \in \mathcal{T}$. Let

$$\hat{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_T \end{pmatrix}$$

and denote the price vector by $\boldsymbol{p} = (p_b \colon b \in \mathcal{B})$. Following the argument similar to the derivation of (2.6), we see the equilibrium condition as

$$\boldsymbol{x} \geq \boldsymbol{0}\ ,\ G(\boldsymbol{x}, \boldsymbol{p}) \leq \boldsymbol{0}\ ,\ \boldsymbol{x}^T G(\boldsymbol{x}, \boldsymbol{p}) = 0 \tag{4.4}$$

where

$$G(\boldsymbol{x}, \boldsymbol{p}) \equiv \hat{\boldsymbol{H}}^T \hat{M}(\hat{\boldsymbol{H}}\boldsymbol{x}) - (\hat{\boldsymbol{C}}\hat{\boldsymbol{A}})^T \hat{W}(\hat{\boldsymbol{A}}\boldsymbol{x}) - \hat{\boldsymbol{B}}\boldsymbol{p}. \tag{4.5}$$

The problem of finding a second-best price is

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{p}}\ &\boldsymbol{1}^T \hat{V}(\hat{\boldsymbol{H}}\boldsymbol{x}) - (\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x})^T \hat{W}(\hat{\boldsymbol{A}}\boldsymbol{x}) \\ &\text{s.t.}\ \ (4.4). \end{aligned} \tag{4.6}$$

As we did in the previous section, we remove inactive routes at the second-best, and rewrite the problem (4.6) as

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{p}}\ &\boldsymbol{1}^T \hat{V}(\hat{\boldsymbol{H}}\boldsymbol{x}) - (\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x})^T \hat{W}(\hat{\boldsymbol{A}}\boldsymbol{x}) \\ &\text{s.t.}\ \ G(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{0}. \end{aligned} \tag{4.7}$$

Introducing Lagrange multipliers, we have the following first order condition:

$$\left(\hat{\boldsymbol{A}}^T \hat{\boldsymbol{C}} \hat{W}'(\hat{\boldsymbol{A}}\boldsymbol{x})\hat{\boldsymbol{A}} - \hat{\boldsymbol{H}}^T \hat{M}'(\hat{\boldsymbol{H}}\boldsymbol{x})\hat{\boldsymbol{H}}\right)^T \boldsymbol{\lambda} = \hat{\boldsymbol{A}}\hat{W}'(\hat{\boldsymbol{A}}\boldsymbol{x})(\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x}) - \hat{\boldsymbol{B}}\boldsymbol{p}, \tag{4.8}$$

$$\hat{\boldsymbol{B}}^T \boldsymbol{\lambda} = \boldsymbol{0}. \tag{4.9}$$

We will exploit this condition to demonstrate that an anonymous charging scheme hurts the social welfare.

Suppose just for a moment that all routes can be charged and the price can be type-dependent. That is, there are $T \times R$ tolls, each of which corresponds to a type-route pair. It is easy to see that this pricing scheme is represented by $\hat{\boldsymbol{B}} = \boldsymbol{I}$ where $\boldsymbol{I}$ here an identity matrix of order $T \times R$. From (4.9), one sees $\boldsymbol{\lambda} = \boldsymbol{0}$. Thus, from (4.8), the solution to (4.7) is given by

$$\boldsymbol{p} = \hat{\boldsymbol{A}}^T \hat{W}'(\hat{\boldsymbol{A}}\boldsymbol{x})\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x}. \tag{4.10}$$

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Suppose that every link has a toll booth which charges a each type of users independently. This pricing scheme is represented by $\hat{\boldsymbol{B}} = \hat{\boldsymbol{A}}^T$ Let the price $\boldsymbol{p} = \big(p_{(t,r)} : t \in \mathcal{T}, r \in \mathcal{R}\big)$ be

$$\boldsymbol{p} = \hat{W}'(\hat{\boldsymbol{A}}\boldsymbol{x})\hat{\boldsymbol{C}}\hat{\boldsymbol{A}}\boldsymbol{x}. \tag{4.11}$$

Note that under this pricing the equilibrium condition (4.4) coincides with the first order condition (4.3) to the problem of finding the first-best. Thus, pricing (4.11) achieves the first-best solution. From the definition of $\hat{W}'(\cdot)$, one sees that the link pricing $\boldsymbol{p}$ in (4.11) does not discriminate users based on their types. Hence if all links are tolled, then there is no gain in social welfare by making the charge type-dependent as seen in Yang and Huang [13].

In general, an anonymous charging scheme hurts the social welfare. We next examine a numerical example to see how bad anonymous charging scheme may be.

Table 1: Impact of User Anonymity

	1st best solution	2nd best solution: type-independent	2nd best solution: type-dependent
p_1	—	101.8	56.08
p_2	—	101.8	97.91
x_{11}	6.629	0	21.47
x_{12}	78.27	132.7	125.5
x_{21}	0	25.45	13.74
x_{22}	21.84	0	0
social welfare	11509	7964	8234

Example 2 *Consider the two-route problem as in the right hand side of Figure 1 with one toll booth. Suppose that there are two types of users. The model parameters are given as follows.*

$$\begin{array}{ll} V_1'(z_1) = -\frac{1}{2}z_1 + 200, & V_2'(z_2) = -3z_2 + 280, \\ c_1 = 1, & c_2 = 2, \\ W_1(y_1) = 2y_1, & W_2(y_2) = y_2 + 1 \end{array}$$

The first-best solution is listed in the first column of Table 1 for reference purposes. The second-best price p_i for type i users is given in the second and third rows. To investigate further the impact of the type-dependent pricing on social welfare, we evaluate the second-best pricing and the corresponding social welfare for some range of c_2. Figures 4 and 5 plot the equilibrium flow under type-independent and type-dependent pricing scheme, respectively. In both cases, the flow of high type (type-2) users decline as c_2 increases. The behavior of these two pricing schemes as the parameter c_2 changes its value from 1 to 3 is summarized as follows.

- For $c_2 = 1 (= c_1)$, there is no user heterogeneity. Thus, the type dependent pricing is of no use in this case.
- For $c_2 = 1.2$, users of each type compartmentalize themselves. That is, for both pricing schemes, the tollway (link 1) is used only by high type (type 2) users and link 2 only by low type.
- For $1.4 \le c_2 \le 2.0$, the type-dependent pricing scheme induces a mixed traffic of high and low-type users on the tollway, while the type-independent pricing scheme still segregates two types of users. In other words, type-independent toll is too high for low-type users.

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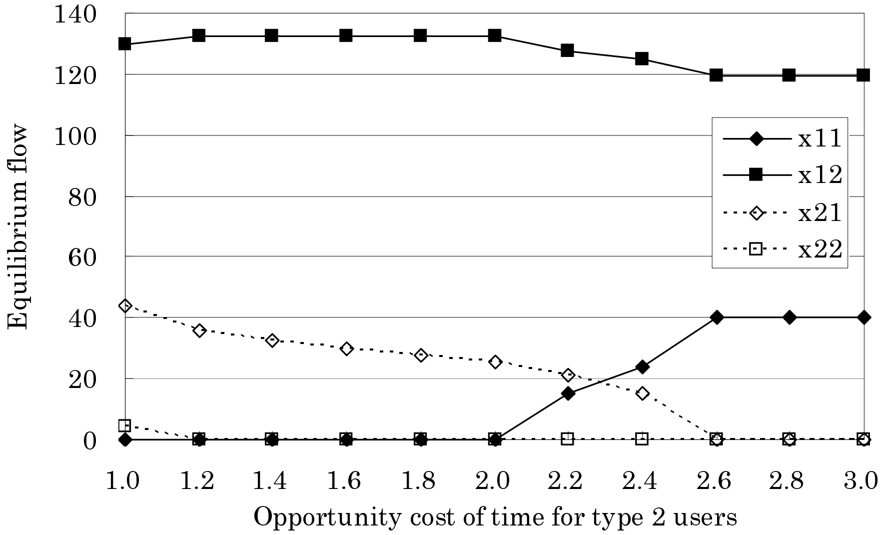

Figure 4: Second best flow — type-independent pricing

- For $2.2 \leq c_2 \leq 2.4$, both types of users coexist on the tollway for both pricing schemes.
- For $c_2 \geq 2.6$, the time value of high-type users is so high that they cannot bear the delay.

Table 1 shows that type-dependent pricing does outperform anonymous pricing in terms of social welfare. How bad is the anonymous pricing? We explore this question numerically here. As it turns out, the welfare loss resulting from the anonymous pricing may not be very large.

Example 3 *Consider the model as in Example 2. Figure 6 plots the social welfare as the cost c_2 of time for type-2 users varies from 1 to 3. The type-dependent pricing outperforms the type-independent pricing as expected. The difference however is marginal. Specifically, the maximal difference in the social welfare between the type dependent and independent pricing schemes is 3.4%.*

5. Conclusion

This paper investigate a second-best problem of a static transportation network where only a subset of links can be tolled. We derived a concise expression of the second best pricing under some technical condition (the column full rank condition), and provide a clear interpretation to the second-best price. More specifically, we show that the second best route toll is a linear combination of route externality and provide a clear interpretation of the coefficients of the linear combination. The technical condition admittedly is strong, however, the interpretation of the second best pricing available under this assumption is unequivocally clear and has never been given in the literature. It will be a challenge in the future to unveil the meaning of the second best price in the absence of the technical condition. It may be the case that the interpretation given in this paper is valid even when the technical condition does not hold.

Also considered is the issue of user heterogeneity in value of time and user anonymity. It is known that if all the links can be tolled, no gain is attained by making the price type-dependent. In a more general setting, the type-dependent pricing scheme is superior to the type-independent pricing. A numerical example however suggests that the advantage of the type-dependent pricing is small.

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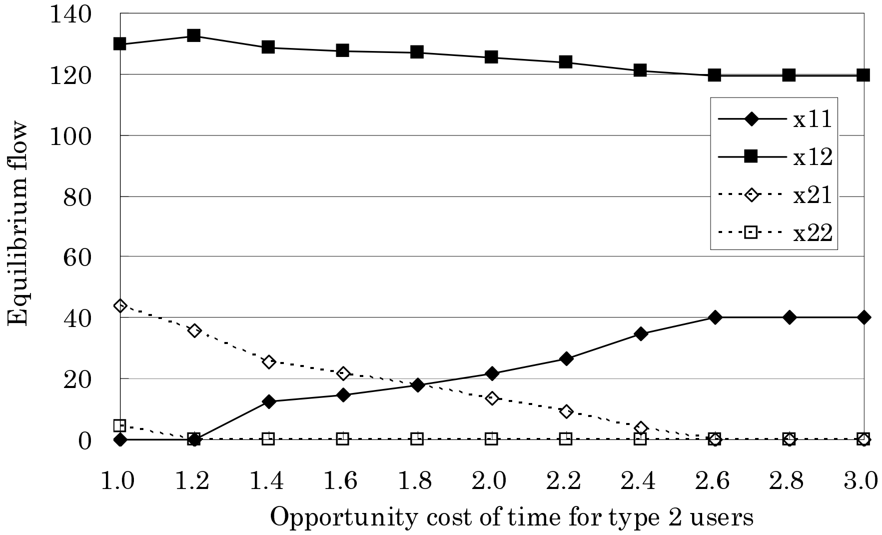

Figure 5: Second best flow — type-dependent pricing

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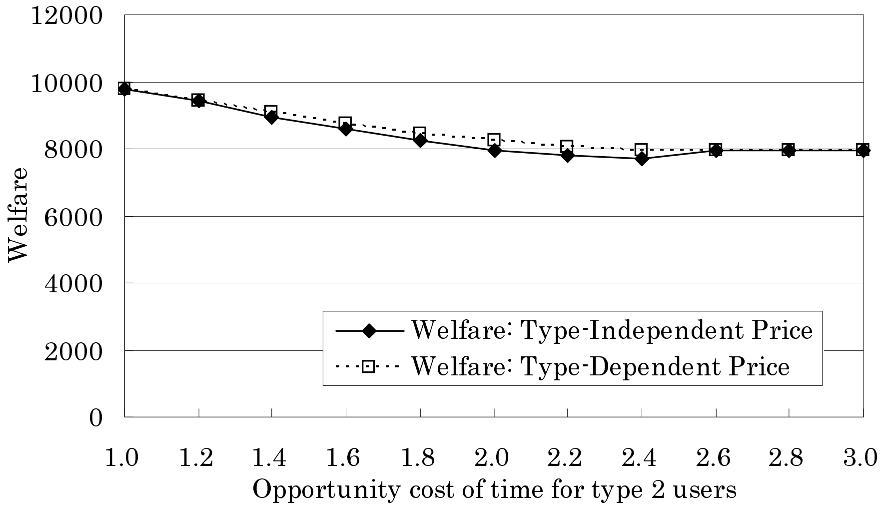

Figure 6: Second best flow — type-dependent pricing

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A. Derivation of (3.3)

Let

$$\begin{aligned} L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= F(\boldsymbol{y},\boldsymbol{z}) - \boldsymbol{\lambda}^T(\boldsymbol{H}^T M(\boldsymbol{z}) - c\boldsymbol{A}^T W(\boldsymbol{y}) - \boldsymbol{B}\boldsymbol{p}) \\ &\quad -\boldsymbol{\mu}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{\sigma}^T(\boldsymbol{H}\boldsymbol{x} - \boldsymbol{z}) \end{aligned}$$

where $F(\boldsymbol{y},\boldsymbol{z}) = \mathbf{1}^T V(\boldsymbol{z}) - c\boldsymbol{y}^T W(\boldsymbol{y})$. Taking the derivative of L with respect to $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{p}$, and setting them to be equal to $\mathbf{0}$, one has the first order condition to (3.2) as

$$\boldsymbol{\mu}^T\boldsymbol{A} + \boldsymbol{\sigma}^T\boldsymbol{H} = \mathbf{0} \quad \text{(A.1)}$$

$$c\boldsymbol{y}^T W'(\boldsymbol{y}) + cW(\boldsymbol{y})^T - c\boldsymbol{\lambda}^T\boldsymbol{A}^T W'(\boldsymbol{y}) = \boldsymbol{\mu}^T \quad \text{(A.2)}$$

$$M(\boldsymbol{z})^T - \boldsymbol{\lambda}^T\boldsymbol{H}^T M'(\boldsymbol{z}) = \boldsymbol{\sigma}^T \quad \text{(A.3)}$$

$$\boldsymbol{\lambda}^T\boldsymbol{B} = \mathbf{0} \quad \text{(A.4)}$$

Substituting (A.2) and (A.3) into (A.1), one sees (3.3).

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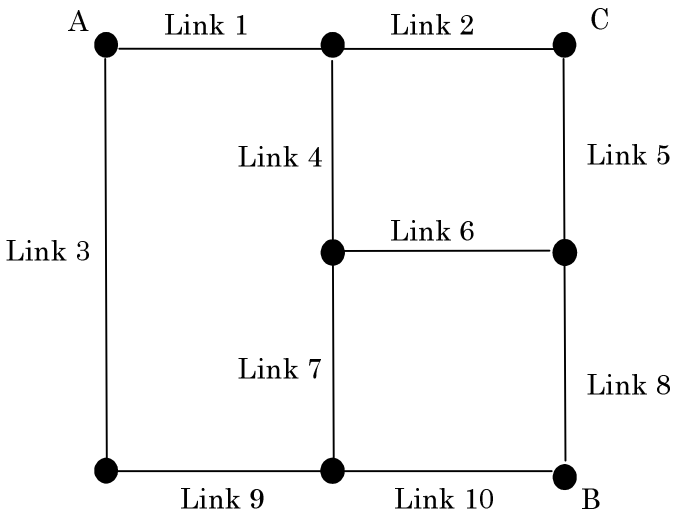

Figure 7: Network topology

Table 2: Set of Routes

ID	route
1	$1 \to 2 \to 5 \to 8$
2	$1 \to 2 \to 5 \to 6 \to 7 \to 10$
3	$1 \to 4 \to 6 \to 8$
4	$1 \to 4 \to 7 \to 10$
5	$3 \to 9 \to 10$
6	$3 \to 9 \to 7 \to 6 \to 8$
7	$3 \to 9 \to 7 \to 4 \to 2 \to 5 \to 8$

B. Networks satisfying the column full rank condition

In this appendix, we see a more complicated example of network (not) satisfying the rank condition (3.5). Consider the network as in Figure 7 with ten links, all of which are two way roads. We suppose that there is one OD flow from A to B. Then, there are 7 routes in the network as in Table 2. In this case, matrices $\boldsymbol{A}$ and $\boldsymbol{H}$ are respectively

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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Thus, the rank condition holds as

$$\text{rank}\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{H} \end{pmatrix} = R = 7$$

Table 3: Set of New Routes

ID	route
8	$5 \to 8$
9	$5 \to 6 \to 7 \to 10$
10	$2 \to 4 \to 6 \to 8$
11	$2 \to 4 \to 7 \to 10$
12	$2 \to 1 \to 3 \to 9 \to 10$
13	$2 \to 1 \to 3 \to 9 \to 7 \to 6 \to 8$

Suppose that there is a new OD flow from C to B. This new OD flow generates 6 additional routes, resulting in the total number of routes being 13, see Table 3. In this case, $\boldsymbol{A}$ and $\boldsymbol{H}$ respectively are $10 \times R$ and $2 \times R$ matrices with $R = 13$. Hence, the rank condition (3.5) cannot hold true.

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