Journal of the
Operations Research
Society of Japan

VOLUME 2 | *January 1960* | NUMBER 3

A SYMMETRIC DUAL THEOREM FOR QUADRATIC PROGRAMS

W. S. DORN

New York University
(*Received June 20, 1959*)

Duality relationships have been established previously [1, 2] for classes of quadratic programming problems. The present note presents a pair of dual quadratic programs, symmetric in the sense that the dual of the dual is clearly the original program again, and proves the duality relationship. As in [2] the proof rests on the dual theorem of linear programming [3].

In what follows capital letters A, Ω, $\cdots$denote matrices and small letters x, λ, $\cdots$are column vectors. Prime denotes transpose and $x'y$ is the inner product of the vectors x and y. A vector inequality applies to each component of the vector, i. e., $x \geq 0$ implies that each component of x is non-negative.

The dual theorem for linear programs [3] states that if a solution to

$$\text{Minimize } p'x$$

subject to

$$Ax \geq b$$
$$x \geq 0$$

exists and is finite, then a solution also exists to its dual problem

$$\text{Maximize } b'v$$

subject to

$$A'v \leq p$$
$$v \geq 0$$

and moreover

$$\text{Minimum } p'x = \text{Maximum } b'v.$$

Consider now the following pair of quadratic programming problems

$$\left.\begin{array}{r}\text{Minimize } f(z) = z'\ \Omega z + \lambda' z \\ (\Omega + \Gamma) z \geq \mu \\ z \geq 0\end{array}\right\} \qquad (\mathrm{I})$$

$$\left.\begin{array}{r}\text{Maximize } g(w) = w'(-\Omega^{**}) w + \mu' w \\ (-\Omega^{*} + \Gamma') w \leq \lambda \\ w \geq 0\end{array}\right\} \qquad (\mathrm{II})$$

where the matrices and vectors are partitioned as follows

$$\Omega = \begin{Bmatrix} \frac{1}{2}C, & 0 \\ 0, & 0 \end{Bmatrix} \qquad \Gamma = \begin{Bmatrix} \frac{1}{2}C, & -C \\ 0, & A \end{Bmatrix}$$

$$\lambda = \begin{pmatrix} 0 \\ p \end{pmatrix} \qquad \mu = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$z = \begin{pmatrix} y \\ x \end{pmatrix} \qquad w = \begin{pmatrix} u \\ v \end{pmatrix}$$

and C is a symmetric, positive semi-definite, $n \times n$ matrix[1], A is an $m \times n$ matrix, y, x, u, p are n dimensional vectors, and v, b are m dimensional vectors.

Notice that Ω is $2n \times 2n$, Γ is $(m+n) \times 2n$, λ and z are $2n \times 1$, μ and w are $(m+n) \times 1$. The matrix Ω^* in Problem II is equal to Ω with an appropriate number of columns of zeroes added or subtracted so that Ω^* is $2n \times (m+n)$. Similarly Ω^{**} is equal to Ω^* with the same number of rows of zeroes added or subtracted so that Ω^{**} is $(m+n) \times (m+n)$.

Dual Theorem: If a solution exists and is finite for either Problem I or Problem II, then a solution also exists for the other problem and

$$\text{Minimum } f(z) = \text{Maximum } g(w)$$

Proof: Assume a solution exists to Problem I which can be rewritten

1) The Symmetry restriction results in no loss of gerality while the positive semi-definiteness assures that C and hence Ω is convex and thus that a local minimum of $f(z)$ is also a global one.

$$\text{Minimize } f^*(y,x)=\frac{1}{2}y'\ Cy+p'x$$
$$Cy-Cx\geq 0$$
$$Ax\geq b$$
$$y\geq 0$$
$$x\geq 0$$

Let $y=\bar{y}$, $x=\bar{x}$ be the minimizing vector, and consider the linear problem

$$\text{Minimize } F(y,\ x)=-\frac{1}{2}\bar{y}'\ C\bar{y}+\bar{y}'\ Cy+p'x \tag{1}$$
$$Cy-Cx\geq 0 \tag{2}$$
$$Ax\geq b \tag{3}$$
$$y\geq 0 \tag{4}$$
$$x\geq 0 \tag{5}$$

Denote this as Problem I′. Clearly $\bar{y}$, $\bar{x}$ is a feasible solution to this problem. Assume there exists another feasible solution y^*, x^* such that

$$F(y^*,\ x^*)<F(\bar{y},\ \bar{x}),$$

i. e.,

$$\bar{y}'\ C(y^*-\bar{y})+p'(x^*-\bar{x})<0 \tag{6}$$

Define y_0, x_0 as

$$y_0=ky^*+(1-k)\bar{y}$$
$$0<k<1$$
$$x_0=kx^*+(1-k)\bar{x}$$

Then y_0, x_0 is also a feasible solution and

$$f(y_0,\ x_0)-f(\bar{y},\ \bar{x})=k[\bar{y}'C(y^*-\bar{y})+p'(x^*-\bar{x})+\frac{1}{2}k(y^*-\bar{y})'\ C(y^*-\bar{y})]$$

By (6) the first term in square brackets is negative and from the positive semi-definiteness of C the second term is non-negative. If this second term vanishes, then the right hand member is negative. If the second term is positive then if k is sufficiently small, in fact, choosing k such that

$$k<\text{Min}\left\{1,\ -\frac{\bar{y}'C(y^*-\bar{y})+p'(x^*-\bar{x})}{\frac{1}{2}\ (y^*-\bar{y})'\ C(y^*-\bar{y})}\right\}$$

the right hand member is negative. Thus, it is always possible to choose k so that

$$f(y_0,\ x_0)-f(\bar{y},\ \bar{x})<0.$$

But $\bar{y}$, $\bar{x}$ minimizes $f(y, x)$ in contradiction to this last inequality. Therefore,

$$F(\bar{y},\ \bar{x})\leq F(y^*,\ x^*)$$

and $\bar{y}$, $\bar{x}$ minimizes Problem I′.

From the dual theorem of linear programming, a solution exists to the dual to Problem I′. The dual linear problem is

$$\text{Maximize } G(v)=-\frac{1}{2}\bar{y}'\ C\bar{y}+b'v$$
$$Cu\leq C\bar{y}$$
$$-Cu+A'v\leq p$$
$$u\geq 0$$
$$v\geq 0$$

which may be rephrased

$$\text{Maximize } G(v)=-\frac{1}{2}\bar{y}'\ C\bar{y}+b'v \tag{7}$$
$$A'v\leq p+C\bar{y} \tag{8}$$
$$v\geq 0 \tag{9}$$

Denote this as Problem II′. If $v=\bar{v}$ is the maximizing solution then by the dual theorem

$$b'\bar{v}=\bar{y}'\ C\bar{y}+p'\bar{x} \tag{10}$$

Now Problem II may be rewritten as

$$\text{Maximize } g^*(u,\ v)=-\frac{1}{2}u'\ Cu+b'v \tag{11}$$
$$-Cu+A'v\leq p \tag{12}$$
$$u\geq 0 \tag{13}$$
$$v\geq 0 \tag{14}$$

Clearly $u=\bar{y}$, $v=\bar{v}$ is a feasible solution to Problem II.

Now since C is positive semi-definite

$$\frac{1}{2}(u-\bar{y})'C(u-\bar{y})\geq 0$$

or

$$\frac{1}{2}u'Cu\geq\bar{y}'Cu-\frac{1}{2}\bar{y}'\ C\bar{y}$$

Subtracting $1/2\bar{y}'C\bar{y}$ from both sides

$$\frac{1}{2}u'Cu-\frac{1}{2}\bar{y}'C\bar{y}\geq\bar{y}\ C(u-\bar{y})$$

So

$$g^*(\bar{y},\ \bar{v})-g^*(u,\ v)=-\frac{1}{2}\bar{y}'C\bar{y}+\frac{1}{2}u'Cu+b'\bar{v}-b'v$$
$$\geq \bar{y}'C(u-\bar{y})+b'\bar{v}-b'v$$

From (10) then

$$g^*(\bar{y},\ \bar{v})-g^*(u,\ v)\geq \bar{y}'Cu+p'\bar{x}-b'v \tag{15}$$

By (3) and (14)

$$-b'v\geq -\bar{x}'Av$$

and by (5) and (12)

$$p'\bar{x}\geq v'A\bar{x}-u'C\bar{x}$$

Putting these in (15)

$$g^*(\bar{y},\ \bar{v})-g^*(u,\ v)\geq u'(C\bar{y}-C\bar{x})$$

and from (2) and (13)

$$g^*(\bar{y},\ \bar{v})-g^*(u,\ v)\geq 0$$

Thus $\bar{y}$, $\bar{v}$ is the maximizing solution to Problem II.

Finally from (10)

$$g^*(\bar{y},\ \bar{v})=-\frac{1}{2}\bar{y}'C\bar{y}+b'\bar{v}=\frac{1}{2}\bar{y}'\ C\bar{y}+p'\bar{x}=f^*(\bar{y},\ \bar{x})$$

which verifies the equality of the objective functions.

From the symmetry of Problems I and II, it becomes obvious that the existence of a solution to Problem II also implies that a solution to Problem I exists.

Finally it should be noted that since C is symmetric, positive semi-definite, it may be taken to be a matrix all of whose elements are zero. In this special case the dual quadratic programs reduce to the corresponding dual linear programs previously stated.

REFERENCES

1. Jack B. Dennis, *Mathematical Programming and Electrical Networks*, Interim Tech. Report 10, Fund. Inv. in Methods of OR. MIT, Cambridge, Mass. (Sept. 1958).
2. W. S. Dorn, *Duality in Quadratic Programming*, AEC Research and Development Report NYO-8676, New York University (Oct. 1958). To appear in Quart. App. Math.
3. G. B. Dantzig and A. Orden, *A Duality Theorem Based on the Simplex Method*, Sym. on Linear Inequalities and Programming, Project SCOOP. 51—55, (1951).