Journal of the O. R. Society of Japan,
Vol. 8, No. 3, June 1966

NOTES ON PARAMETRIC QUADRATIC PROGRAMMING

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(Received Oct. 6, 1965)

The primal-dual algorithm of parametric linear programming (e.g. [1]) can be extended in some sence to parametric quadratic programming problems as follows.

§ 1. Problem of the first type

Let us consider the following quadratic programming problem $P|\lambda$ with a parameter λ.

$$P|\lambda: \quad Min\{p'x+x'Cx|Ax\leqq b+\lambda d\} ,$$

where C is an $n\times n$ positive semi-definite matrix, A is an $m\times n$ matrix, p and x are n-vectors, and b and d are m-vectors.

We denote by M' the transpose of M.

Dorn[2] constructed the dual problem $D|\lambda$ of $P|\lambda$ with a vector variable y:

$$D|\lambda: \quad Max\{p'x+x'Cx+y'(Ax-b-\lambda d)|p+2Cx+A'y=0, \quad y\geqq 0\} .$$

The duality theorem holds between $P|\lambda$ and $D|\lambda$ as in linear programming. Now let (x, y) be the optimal solution of $P|\lambda$ and $D|\lambda$.

Put $w=b+\lambda d-Ax$ and $S=\{i|w_i=0,\ 0\leq i\leq m\}$.

The restricted primal and dual problems which we denote by RP and RD are constructed as follows.

$$RP: \quad Min\{\xi'C\xi|① \ A\xi+\sigma=d, \quad ② \ \sigma_i\geq 0 \text{ for } i\in S, \quad ③ \ \sigma'y=0\} ,$$

$$RD: \quad Max\{\xi'C\xi+\sum_{i\in S}\eta_i(A'_i\xi-d_i)|① \ 2C\xi+A'\eta=0, \quad ② \ \eta_i\geq 0 \text{ if } y_i=0, \quad ③ \ w'\eta=0\} ,$$

where A'_i means the i-th row vector of $A=\begin{pmatrix} A'_1 \\ A'_2 \\ \vdots \\ A'_m \end{pmatrix}$.

Theorem 1. If (ξ, η) is the optimal solution of RP and RD then $(x+\theta\xi, y+\theta\eta)$ is the optimal solution of $P|(\lambda+\theta)$ and $D|(\lambda+\theta)$ for any θ such that $0<\theta\leqq\theta_0$,

where θ_0 is defined as follows:

$$\theta_1=\begin{cases} min-w_i/\sigma_i(\text{where } \sigma_i<0) & \text{if there exists } i \text{ such that } \sigma_i>0 \\ \infty & \text{other wise,} \end{cases}$$

$$\theta_2=\begin{cases} min-y_i/\eta_i(\text{where } \eta_i<0) & \text{if there exists } i \text{ such that } \eta_i<0 \\ \infty & \text{other wise,} \end{cases}$$

and $\theta_0=min(\theta_1, \theta_2)$.

By the Kuhn-Tucker theorem, a necessary and sufficient condition that (x, y) (resp. (ξ, η)) be an optimal solution of $P|\lambda$, $D|\lambda$ (resp. RP, RD) is the following, from which the above theorem can easily be proved.

(I) Optimality of (x, y) for $P|\lambda$, $D|\lambda$:

$$Ax+w=b+\lambda d\ ,$$
$$2Cx+A'y=-p\ ,$$
$$w\geqq 0,\ y\geqq 0,\ \ w'y=0\ .$$

(II) Optimality of (ξ, η) for PR, RD:

$$A\xi+\sigma=d\ ,$$
$$2C\xi+A'\eta=0\ ,$$
$$\sigma_i\geqq 0 \text{ if } w_i=0,\ \ \eta_i\geqq 0 \text{ if } y_i=0\ ,$$
$$\sigma'y=0,\ \ w'\eta=0,\ \ \sigma'\eta=0\ .$$

§ 2. Problem of the second type

Next, let us consider the following problem.

$P'|\lambda$: $\quad Min\{(p'+\lambda_q')x+x'Cx|Ax\leqq b\}$.

The Kuhn-Tucker conditions for $P'|\lambda$ are as follows.

(I′) $$Ax+w=b\ ,$$
$$2Cx+A'y=-p-\lambda q\ ,$$
$$w\geqq 0,\quad y\geqq 0,\quad w'y=0\ .$$

The restricted problem then becomes as follows.

RP: $Min\{q'\xi+\xi'C\xi \mid ①\ A\xi+\sigma=0,\ \ ②\ \sigma_i\geqq 0 \text{ for } i\in S,\ \ ③\ \sigma'y=0\ ;$

where $S=\{i \mid w_i=0,\ \ 0\leqq i\leqq m\}$.

The Kuhn-Tucker conditions for RP are;

(II′) $$A\xi+\sigma=d\ ,$$
$$2C\xi+A'\eta=-q\ ,$$
$$\sigma_i\geqq 0 \text{ if } w_i=0,\quad \eta_i\geqq 0 \text{ if } y_i=0,$$
$$\sigma'y=0,\quad w'\eta=0,\quad \sigma'\eta=0\ .$$

Theorem 2. If (x, y) is the solution of (I′) (i.e. the optimal solution of $P'|\lambda$, $D'|\lambda$), and if (ξ, η) is the solution of II′ based on (x, y) (i.e. the optimal solution of RP', RD'), then $(x+\theta\xi,\ y+\theta\eta)$ is the optimal solution of $P'|(\lambda+\theta)$ and $D'|(\lambda+\theta)$ for any θ such that $0<\theta\leqq\theta_0$ where θ_0 is defined by the same relation as in § 1.

§ 3. Remarks

1. Problems of the form $Min\{p'x+x'Cx \mid Ax\leqq b\}$ can easily be solved by Wolfe's simplex method, if $p=0$ or if C is strictly positive-definite (cf., e.g., Die Kurze Form in **[3]**, p. 115).

RP and (II) in § 1 satisfy this condition.

2. Starting from an optimal solution (x, y, w) of $P|\lambda$ and $D|\lambda$, we can obtain the optimal solution of (x_1, y_1, w_1) of $P|(\lambda+\theta_0)$ and $D|(\lambda+\theta_0)$ by the method explained in § 1, where $x_1=x+\theta_0\xi$, $y_1=y+\theta_0\eta$ and $w_1=w+\theta_0\sigma$.

To solve $P|\lambda'$ and $D|\lambda'$ for $\lambda'>\lambda+\theta_0$, we must solve (II) for (x_1, y_1, w_1), that is to find a solution satisfying ① $A\xi+\sigma=d$, $2C\xi+A'\eta=0$, ② $\sigma_i\geqq 0$ if $w_{1i}=0$, $\eta_i\geqq 0$ if $y_{1i}=0$, ③ $\sigma'y_1=0$, $w'_1\eta=0$, $\sigma'\eta=0$.

But the solution (ξ, η, σ) of (II) for (x, y, w) satisfies automatically ① and ③ of (II) for (x_1, y_1, w_1).

To obtain the solution of (II) for (x_1, y_1, w_1) satisfying ② by using

(ξ, η, σ), it seems to be enough to solve by the simplex method the problem of minimizing $\sum_{w_{1i}=0} \sigma_i^{(-)} + \sum_{y_{1i}=0} \eta_i^{(-)}$ under the condition ① and ③, where $\sigma_i = \sigma_i^{(+)} - \sigma_i^{(-)}$ $(\sigma_i^{(+)}, \sigma_i^{(-)} \geqq 0)$ and $\eta_i = \eta_i^{(+)} - \eta_i^{(-)}$ $(\eta_i^{(+)}, \eta_i^{(-)} \geqq 0)$.

However, we have not yet obtained the rigorous proof of this fact.

3. Let us consider a solution of the problem $Min\{p'x + x'Cx | Ax \leqq b\}$, $b \geqq 0$.

Consider the following parametric problem:

$$Min\{\lambda p'x + x'Cx | Ax \leqq b\}, \quad b \geqq 0 .$$

$(x=0, y=0)$ is evidently the optimal solutions of this problem for $\lambda=0$. It suffices to solve this problem for increasing λ by the method in § 2 and then to stop at $\lambda=1$.

4. Markowitz [4] investigated a form of parametric quadratic programming, in connection with the problem of porto-folio selection.

References

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