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A NOTE ON COMPARISON OF "EXTRINSIC" AND "INTRINSIC" FORECASTING MODELS

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1. We may think of essentially two competing models to forcast future values of a time series Y. One is the Extrinsic model where the predictor is a function of some external variable. The Intrinsic approach, on the other hand, uses only information on the past values of Y. There are also a few models which combine in a way features of both the types.
2. The choice of one type or the other is influenced, among other factors, by i) the reliability of the estimates provided; ii) availability of the necessary data to a sufficient degree of accuracy and iii) computational ease in using the model in practice for forecasting. The main theoretical consideration is of course the first of these three and in what follows an attempt is made to compare the efficiency of the two methods under very simplifying conditions.
3. Consider the time series Y to be generated by the following process:

$$Y_{t+1}=\alpha+\beta_1 X_{t+1}+\beta_2 X_t+u_{t+1} \qquad \text{(I)}$$

The following assumptions are made:

α_1, β_1 and β_2 are known.

u_t follows a normal $N(0, \sigma_u^2)$ for all t.

$E(u_t\, u_{t+\tau})=0$ for all $\tau\neq 0$

X_t follows a normal $N(0, \sigma_x^2)$ and

$$E(X_t X_{t+s})=0 \text{ for } S \gtrless 0 \text{ and } E(X_t u_t)=0$$

Consider now the simple intrinsic model

$$Y_{t+1}=a+b\,Y_t+v_{t+1} \tag{II}$$

It will be shown in what follows that under the conditions imposed as above the second model, in general, will lead to poorer estimates and that the least squares estimation is invalid there because of interdependence of Y_t and v_{t+1}.

4. By the process assumed in I we have

$$Y_t=\alpha+\beta_1 X_t+\beta_2 X_{t-1}+u_t \tag{1}$$

or

$$\beta_2 Y_t=\beta_2\alpha+\beta_2\beta_1 X_t+\beta_2^2 X_{t-1}+\beta_2 u_t \tag{2}$$

$$(\beta_2 \neq 0)$$

similarly,

$$\beta_1 Y_{t+1}=\beta_1\alpha+\beta_1^2 X_{t+1}+\beta_1\beta_2 X_t+\beta_1 u_{t+1} \tag{3}$$

$$(\beta_1 \neq 0)$$

From (3) and (2) we further deduce

$$\beta_1 Y_{t+1}=\beta_1\alpha+\beta_1^2 X_{t+1}+\beta_2 Y_t-\beta_2\alpha-\beta_2^2 X_{t-1}-\beta_2 u_t+\beta_1 u_{t+1} \tag{4}$$

Comparison of (4) and II yields

$$a=\frac{\alpha(\beta_1-\beta_2)}{\beta_1}\,; \qquad b=\frac{\beta_2}{\beta_1}$$

and

$$v_{t+1}=\frac{\beta_1^2 X_{t+1}-\beta_2^2 X_{t-1}-u_t\beta_2+\beta_1 u_{t+1}}{\beta_1}\,;$$

hence

$$E(v_{t+1})=0 \text{ and } \sigma_v^2=\frac{(\beta_1^4+\beta_2^4)\sigma_x^2+(\beta_1^2+\beta_2^2)\sigma_u^2}{\beta_1^2}$$

v_{t+1} has a normal distribution with the above mean and variance, both the parameters being independent of t.

It is easily seen that:

$$E(v_t\, v_{t+1})=-\frac{\beta_2}{\beta_1}\sigma_u^2$$

$$E(v_t\, v_{t+2})=-\beta_2^2\,\sigma_x^2$$

$$E(Y_t\, v_{t+1})=-\frac{\beta_2}{\beta_1}(\sigma_u^2+\beta_2^2\,\sigma_x^2)=\sigma_{yv} \text{ say}$$

The covariance term between Y_t and v_{t+1} introduces bias in the least square estimates of parameters of a and b.

5. Keeping in mind the fact that I is used to forecast Y_{t+1} given X_t and II is used to forecast Y_{t+1} given Y_t, we have the following measures of their reliability.

$$\text{I} \quad E\{(Y_{t+1}-E(Y_{t+1}|X_t)\}^2=\beta_1^2\,\sigma_x^2+\sigma_u^2$$

$$\text{II} \quad E\{(Y_{t+1}-E(Y_{t+1}|Y_t)\}^2=\text{Var }\{b\,Y_t+v_{t+1})|Y_t\}$$

the calculation of which is done as below:

$$\text{Var }\{bY_t+v_{t+1})|Y_t\}=\text{Var }\{v_{t+1}|Y_t\}$$

Now from our assumptions we have

$$v_{t+1}\cap N(0,\sigma_v^2)$$

$$Y_t\cap N(0,\sigma_y^2) \text{ where it is assumed,}$$

without loss in generality, $\alpha=0$.

Hence (v_{t+1}, Y_t) has a bivariate normal distribution with mean $(0,0)$ and variance-covariance matrix

$$\begin{pmatrix} \sigma_v^2 & \sigma_{vy} \\ \sigma_{vy} & \sigma_y^2 \end{pmatrix}$$

From this it is known that the conditional variance of v_{t+1} given

Y_t is $\dfrac{\sigma_v^2\sigma_y^2-\sigma_{vy}^2}{\sigma_y^2}$

Substituting the values

$$\sigma_v^2=\frac{(\beta_1^4+\beta_2^4)\,\sigma_x^2+(\beta_1^2+\beta_2^2)\,\sigma_u^2}{\beta_1^2}$$

$$\sigma_y^2=(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2$$

and

$$\sigma_{vy}=-\frac{\beta_2}{\beta_1}(\beta_2^2\,\sigma_x^2+\sigma_u^2)$$

in the expression for $V(v_{t+1}|Y_t)$ we have:

$$\frac{\beta_1^2\sigma_x^2\{\overline{\beta_1^4+\beta_2^4}\sigma_x^2+\overline{\beta_1^2+\beta_2^2}\sigma_u^2\}+(\beta_2^2\sigma_x^2+\sigma_u^2)(\beta_1^4\sigma_x^2+\beta_1^2\sigma_u^2)}{\beta_1^2\,(\overline{\beta_1^2+\beta_2^2}\,\sigma_x^2+\sigma_u^2)}$$

$$=\frac{\sigma_x^4\{\beta_1^4+\beta_2^4+\beta_1^2\,\beta_2^2\}+\sigma_u^4+2\beta_1^2\,\beta_2^2\,\sigma_x^2\,\sigma_u^2}{\overline{\beta_1^2+\beta_2^2}\,\sigma_x^2+\sigma_u^2}$$

$$=\frac{\overline{(\beta_1^2+\beta_2^2}\,\sigma_x^2+\sigma_u^2)^2-\beta_1^2\,\beta_2^2\,\sigma_x^4}{\overline{\beta_1^2+\beta_2^2}\,\sigma_x^2+\sigma_u^2}$$

$$V(u_{t+1}|Y_t)=(\beta_1^2\,\sigma_x^2+\sigma_u^2)+\frac{\beta_2^2\,\sigma_x^2\,[\beta_2^2\,\sigma_x^2+\sigma_u^2]}{(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2}$$

6. Nothing that the expression on R.H.S. is the sum of two terms, the first being $V(Y_{t+1}|X_t)$ as given by model I and the second an expression which is always positive, it is deduced that

$$V(Y_{t+1}|Y_t)\geq V(Y_{t+1}|X_t).$$

7. It may be also noted that if model II were to be estimated by the least squares method without assuming knowledge of b or a and

under the assumption $E(Y_t\, v_{t+1})=0$ we get an estimate of Y_{t+1} whose conditional variance given Y_t is the same as in the case of II. Thus under the Least Squares:

$$b=\frac{E(Y_t\, Y_{t+1})}{E(Y_t^2)}=\frac{\beta_1\,\beta_2\,\sigma_u^2}{(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2}$$

and

$$\begin{aligned}V(Y_{t+1}|Y_t)&=\sigma^{\,2}=\mathrm{Var}\{Y_{t+1}-a-bY_t\}\\&=V(Y_{t+1})+b^2\,\mathrm{Var}\,Y_t-2b\,\mathrm{cov}\,(Y_{t+1}, Y_t)\\&=\sigma_y^2\,(1+b^2)-2b\,\beta_1\,\beta_2\,\sigma_x^2\\&=\{(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2\}\left\{1+\frac{\beta_1^2\,\beta_2^2\,\sigma_x^4}{[(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2]^2}\right\}\\&\quad-\frac{2\,(\beta_1\,\beta_2\,\sigma_x^2)^2}{(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2}\end{aligned}$$

which reduces to

$$\beta_1^2\,\sigma_x^2+\sigma_u^2+\frac{\beta_2^2\,\sigma_x^2\,(\beta_2^2\,\sigma_x^2+\sigma_u^2)}{(\beta_1^2+\beta_2^2)\,\sigma_x^2+\sigma_u^2}$$

Which is the same as $V(Y_{t+1}|Y_t)$ in model II. This agreement of the two expressions which might seem to be rather strange at the first thought, is, in fact, a result of the special structure of the model under consideration. A general discussion on this point requires a further study.

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