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NOTE ON THE CRITICAL PATH ANALYSIS FOR A PROJECT WITH A DIVISIBLE ACTIVITY

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In a paper published in JORSA Vol. 13 (1965), W.S. Jewell [2] presented an algorithm for minimizing the total duration of a project with a divisible activity. The mathematical formulation of the problem is as follows:

$$
\begin{aligned}
&\text{Minimize} \quad t,\\
&\text{subject to} \quad v_j - v_i \geqq T_{ij}, && \text{for } (i,j)\in \bar{D},\\
&\left.\begin{aligned} & v_j - v_i - t_{ij} \geqq 0,\\ & t_{ij} \geqq 0, \end{aligned}\right\} && \text{for } (i,j)\in D,\\
&\sum_{(i,j)\in D} t_{ij} \geqq U,\\
&v_N - v_1 - t = 0,
\end{aligned}
$$

where $\bar{D}$, D, U, T's are given.

He considered the following parametric programming problem $P|Q$ and its dual one $D|Q$.

$P|Q$: Minimize $(Qt - \sum_{D} t_{ij})$,

$$
\begin{aligned}
&\text{subject to} \quad v_j - v_i \geqq T_{ij}, && \text{for } (i,j)\in \bar{D},\\
&\left.\begin{aligned} & v_j - v_i - t_{ij} \geqq 0,\\ & t_{ij} \geqq 0, \end{aligned}\right\} && \text{for } (i,j)\in D,\\
&v_N - v_1 - t = 0.
\end{aligned}
$$

$D|Q$: Maximize $\sum_{D} T_{ij} x_{ij}$,

subject to

$$\sum_{\substack{j \\ (i,j)\in D\cup\bar{D}}} x_{ij} - \sum_{\substack{j \\ (j,i)\in D\cup\bar{D}}} x_{ji} = \begin{cases} Q & (i=1), \\ 0 & (i=2,3,\cdots,N-1), \\ -Q & (i=N), \end{cases}$$

$$x_{ij} \geqq 0, \qquad \text{for } (i,j)\in\bar{D},$$

$$x_{ij} \geqq 1, \qquad \text{for } (i,j)\in D.$$

His algorithm consists of two parts, that is, the starting procedure for finding an initial solution of $P|Q$ for a sufficiently large positive Q, and the minimal-flow subroutine for decreasing Q so as to allocate more time to the divisible activity. The latter is one of the primal-dual algorithms. The procedure is terminated when $\sum_{D} t_{ij}$ reaches U, since it is proved in [2] that if an optimal solution of $P|Q$ for some Q, (v_i, t_{ij}, t), satisfies that $\sum_{D} t_{ij}=U$, then it is also optimal to the original problem.

Here, the author suggests that the usual CPM(critical path method) is directly applicable to the problem without introducing a modified algorithm. We consider the following parametric programming problem $D^*|T$ and its dual one $P^*|T$.

$D^*|T$: Maximize $\sum_{D} t_{ij}$,

$$\text{subject to} \quad v_j - v_i \geqq T_{ij}, \qquad \text{for } (i,j)\in\bar{D},$$

$$\left.\begin{array}{l} v_j - v_i - t_{ij} \geqq 0, \\ M \geqq t_{ij} \geqq 0, \end{array}\right\} \quad \text{for } (i,j)\in D,$$

$$v_N - v_1 = T,$$

where M is a sufficiently large positive number.

$P^*|T$: Minimize $[Tq + M\sum_{D} y_{ij} - \sum_{\bar{D}} T_{ij} x_{ij}]$,

$$\text{subject to} \quad x_{ij} \geqq 0, \qquad \text{for } (i,j) \in DU\bar{D},$$

$$y_{ij} \geqq 0, \qquad \text{for } (i,j) \in D,$$

$$\sum_{(i,j)\in DU\bar{D}} x_{1j} - q = 0,$$

$$\sum_{\substack{j \\ (i,j)\in DU\bar{D}}} x_{ij} - \sum_{\substack{j \\ (j,i)\in DU\bar{D}}} x_{ji} = 0 \quad (i=2,3,\cdots,N-1),$$

$$\sum_{\substack{j \\ (j,N)\in DU\bar{D}}} x_{jN} - q = 0,$$

$$x_{ij} + y_{ij} \geqq 1, \qquad \text{for } (i,j) \in D.$$

If $T < M$, the condition that $M \geqq t_{ij} \geqq 0$ for $(i,j) \in D$, may be replaced by the condition that $t_{ij} \geqq 0$ for $(i,j) \in D$, in $D^*|T$. and y's may be neglected in $P^*|T$. Hence, from Proposition 2.1. of Kurata [4], it is proved that if (v_i, t_{ij}) resp. (x_{ij}, y_{ij}, q) is the optimal solution of $D^*|T$ resp. $P^*|T$ for $T < M, (v_i, t_{ij}, t = T)$ resp. (x_{ij}) is the optimal solution of $P|Q=q$ resp. $D|Q=q$. Furthermore, $D^*|T$ is equivalent to one of the usual CPM problems:

Maximize $\sum_P c_{ij} t_{ij}$,

$$\text{subject to} \quad \left.\begin{array}{l} v_j - v_i - t_{ij} \geqq 0, \\ D_{ij} \geqq t_{ij} \geqq d_{ij}, \end{array}\right\} \quad \text{for } (i,j) \in P,$$

$$v_N - v_1 = T,$$

when $P = DU\bar{D}$,

$$D_{ij} = \begin{cases} M, & \text{for } (i,j) \in D, \\ T_{ij}, & \text{for } (i,j) \in \bar{D}, \end{cases}$$

$$d_{ij} = \begin{cases} 0, & \text{for } (i,j) \in D, \\ T_{ij}, & \text{for } (i,j) \in \bar{D}, \end{cases}$$

$$c_{ij} = \begin{cases} 1, & \text{for } (i,j) \in D, \\ 0, & \text{for } (i,j) \in \bar{D}. \end{cases}$$

So, we can obtain an optimal solution of the original problem by applying the usual CPM to $D^*|T$. In this case we can easily find a solution of $P^*|T=\infty$, as an initial solution, which is as follows:

$$x_{ij}=0 \qquad \text{for } (i,j)\in D\cup\bar{D},$$

$$y_{ij}=1 \qquad \text{for } (i,j)\in D,$$

$$q=0.$$

And by decreasing T, less time will be allocated to the divisible activity in the reverse order of Jewell's, and the procedure is terminated when $\sum_D t_{ij}$ reaches U.

References

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[4] Kurata, R. " Primal Dual Method of Parametric Programming and Iri's Theory on Network Flow Problems," JORSA 7, (1965), 104—144.