between C and $V - C$.) Recall that for each $k = 1, 2, \cdots, n$, v_k is a vertex $u \in V - U_{k-1}$ that maximizes $w_k(\{u\})$. We will write $w_k(u)$ instead of $w_k(\{u\})$ for $u \in V$ in the following.

Now, we show the following lemma. The validity of Nagamochi and Ibaraki's min-cut algorithm easily follows from this lemma.

Lemma 2.1: *For any $k = 1, 2, \cdots, n-1$ and any $u \in V - U_k$, $\{u\}$ is a u-v_k min-cut in G_k.*
(Proof) We show this lemma by induction. For $k = 1$ it trivially holds. So, suppose that it holds for $k = l$ with $1 \leq l < n - 1$. Consider any $u \in V - U_{l+1}$ and any u-v_{l+1} cut C. If C is a u-v_l cut, then by the induction hypothesis ($w_l(C) \geq w_l(u)$) and by the definitions of G_l and G_{l+1} we have

$$w_{l+1}(C) - w_{l+1}(u) \geq w_l(C) - w_l(u) \geq 0. \tag{2.1}$$

Also, if C is a v_l-v_{l+1} cut, then by the induction hypothesis ($w_l(C) \geq w_l(v_{l+1})$) and by the definition of v_{l+1} we have

$$w_{l+1}(C) \geq w_{l+1}(v_{l+1}) \geq w_{l+1}(u). \tag{2.2}$$

Hence, the lemma holds for $k = l + 1$. This completes the proof. □

This lemma also easily follows from the result of Nagamochi and Ibaraki [4] (but recall that our objective is to give a simple proof of the validity of their algorithm). It also follows from a lemma of Stoer and Wagner [7] and from a lemma of Frank [1]. Very recently, M. Queyranne [6] has found a (surprising) strongly polynomial combinatorial algorithm for minimizing symmetric submodular functions by generalizing the arguments in Stoer and Wagner [7]. The above lemma is almost the same as (but slightly different from) Lemma 1 of Queyranne [6] when specialized to cut functions of undirected graphs but the proofs are quite different.

The above lemma can also be slightly strengthened as follows.

Lemma 2.2: *For any $k = 1, 2, \cdots, n-1$ and any distinct $u, v \in V - U_{k-1}$, if $w_k(u) \leq w_k(v)$, then $\{u\}$ is a u-v min-cut in G_k.*
(Proof) Just replace v_{l+1} by v in the above proof of Lemma 1. □

These lemmas, especially Lemma 2.2, show more clearly the behavior of Nagamochi and Ibaraki's algorithm.

The above proofs also work for a symmetric submodular system (V, f) ([3]) by defining

$$w_k(C) = f(C) - (1/2)\{f(C \cap \overline{U_{k-1}}) + f(\overline{C} \cap \overline{U_{k-1}}) - f(\overline{U_{k-1}})\} \tag{2.3}$$

for each $k = 1, 2, \cdots, n - 1$ and $C \subseteq V$, where $\overline{X}$ denotes the complement of X in V and U_k ($k = 0, 1, \cdots, n$) are defined by the same procedure as described above by using this w_k for each $k = 1, 2, \cdots, n$, i.e., $v_k \in V - U_{k-1}$ attains the maximum value of $w_k(\{u\})$ ($u \in V - U_{k-1}$). The procedure is exactly the same as Queyranne's [6]. The present proof looks simpler than that in [6]. Note that when f is the cut function of a given weighted graph, the present w_k in (2.3) is exactly the same as that defined for the graph.

The above lemmas are useful for finding s-t min-cuts in G. Suppose that for some $k < n - 1$ there exists a vertex $u \in V - U_k$ such that u is not adjacent in G to any other vertices in $V - U_k$. Then we can conclude from the above lemmas that $\{u\}$ is a u-v_k min-cut in G and we can finish CAPFOREST. Also, instead of truncating the procedure CAPFOREST we may continue the procedure till the end to get more than one s-t pair and then merge each found s-t pair at the end of the procedure. This may be more effective in practical implementation of the algorithm. An efficient implementation technique was also given in [5].

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