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SUFFICIENT CONDITIONS FOR NONEMPTY CORE OF MINIMUM COST FOREST GAMES

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Abstract This paper studies problems of establishing a minimum cost network and of determining a fair cost allocation among customers. Each supplier offers a different type of service to the customers, and each customer wishes to be connected with the suppliers which he needs. The characteristic function game is deduced from minimum costs for constructing subnetworks. By introducing an equivalence relation on the set of customers, we provide sufficient conditions to have a nonempty core, which solves the above problems. It is shown that the game has a nonempty core as long as the optimal grand network becomes a forest which is composed of the collection of the minimum spanning trees on the above equivalence classes. It is further shown that, whenever the game consists of at most two equivalence classes, the core is nonempty.

Keywords: Game theory, core, minimum spanning tree, forest

1. Introduction

We are having increasingly more opportunities to get information or other services through networks such as an intranetwork, a cable television network, and internet. It will be reasonable to conjecture that many new kinds of networks would emerge in the future. Thus, if many people or organizations share a network, it is important to construct a network with the minimum cost and to allocate fairly the cost among the users.

Claus and Kleitman [2] consider a cost allocation in a minimum spanning tree, where there is exactly one supplier and all customers wish to be connected with the supplier. They discuss various schemes of cost allocation. Bird [1] formulates this problem as a cooperative game, called minimum cost spanning tree (MCST) game. His model assumes that a coalition S, a subset of customers, is allowed to use only the vertices of S and the supplier vertex. It is shown that a refinement of the core is stable. Granot and Huberman [5] prove that the above model has always a nonempty core even when each coalition S can use any external vertex. Megiddo [7] independently gives a proof of the nonemptiness of the core. For a model including more than one supplier, Rosenthal [9] deals with minimum cost spanning forest game. In his model, all suppliers provide identical services, and customers need to maintain a path to at least one supplier. Thus, the resulting network is not required to be a tree. He shows that the core of this game is nonempty. Granot and Granot [4] introduce fixed cost spanning forest (FCSF) game, where there is a fixed cost for establishing a supplier in addition to the link cost of edges. Each customer wishes to physically connect himself with a supplier. They give a sufficient condition under which FCSF game has a nonempty core, and propose a strongly polynomial algorithm to compute the nucleolus for a special case.

Kuipers [6] introduces a generalized model of the MCST game, called minimum cost forest (MCF) game. This model would enable us to deal with a variety of real situations for the cost allocation problem on a network. This paper explores his model further, and examines the existence of the core.

In an MCF game, there may be more than one supplier. Customers and suppliers are represented by vertices on a network. Suppliers are not identical in services they provide. Each customer has a subset of suppliers which he needs, and satisfies his demand only if he is physically connected with them. Every coalition S of customers may utilize the external vertices. Notice that it is not necessary for a customer to be connected with the suppliers by a direct link. Link cost is given for every pair of vertices. Some customers may have common suppliers which they need. Then, they might cooperatively construct a lower cost network by sharing links. We now set the following problems. What is a minimum cost network such that all customers' demands are satisfied? Under what condition is there a fair cost allocation among the customers to establish the optimal network? Here, this problem is studied by a cooperative game theoretic approach. The MCF game does not always have a nonempty core as illustrated later. Kuipers [6] shows that the core of MCF game is nonempty if there is at least one customer who needs all suppliers or if there are at most two suppliers. It is ingenious that he shows the nonemptiness of the core by using the properties of the minimal network. In this paper, we provide more general sufficient conditions by considering an equivalence relation of customers. These conditions include the sufficient conditions of [6] as special cases. Associated with the subset of suppliers each customer requires, an equivalence relation on the set of customers defines the connectivity requirement among customers. Then, based on the concept of the equivalence relation, a sufficient condition for the nonemptiness of the core of the MCF game is given. By using this result, it is verified that the core is nonempty if the number of equivalence classes is less than or equal to two.

The remainder of this paper is organized as follows. In Section 2, we formally describe the MCF game. In Section 3, we discuss the existence of nonempty core, and then provide the sufficient conditions under which the core is nonempty. Section 4 summarizes our findings and gives directions for future research.

2. Model

Let $N = \{1, ..., n\}$ be a set of customers and $M = \{n+1, ..., n+m\}$ be a set of suppliers. For each customer $i \in N$, $M(i) \subseteq M$ represents the set of suppliers required by i. We assume that $M(i) \neq \emptyset$ for all $i \in N$. For each $S \subseteq N$, $M(S) \equiv \bigcup_{i \in S} M(i)$. Let d be a nonnegative weight function on the edges in the complete graph with vertex set $N \cup M$. We denote by d_{ij} the cost of constructing a direct link between i and j. Assume that d_{ij} is symmetric, i.e., $d_{ij} = d_{ji}$. Given $E \subseteq \mathcal{P}_2(N \cup M)$, we define the graph $G_E = (N \cup M, E)$, where $\mathcal{P}_2(N \cup M)$ is the collection of all subsets of $(N \cup M)$ with cardinality 2. We call G_E *S-feasible* if for any $i \in S \subseteq N$ and any $j \in M(i)$ there exists a path from i to j in G_E. For any $i \in S$, it is allowed that i uses the vertices of $N \cup M$ outside S and $M(S)$. Notice that N-feasible graph is always S-feasible for any $S \subseteq N$. Our objects are to find an N-feasible graph G_E such that the cost $\sum_{(i,j) \in E} d_{ij}$ is minimized and to allocate the cost of the optimal graph among the customers.

If G_E has a cycle, we can remove at least one edge while keeping N- or S-feasibility, reducing the cost of the network. Hence, we may restrict our attention to the set of N-feasible forests. Note that a minimal N- or S-feasible graph may not be a tree. For example, if $M(S) \cap M(S^c) = \emptyset$ for some $S \subseteq N$, a forest composed of the two minimum spanning trees corresponding to $S \cup M(S)$ and $S^c \cup M(S^c)$ has a fewer number of edges compared with the minimum spanning tree on $N \cup M$. Thus, a forest may have a lower cost than a tree.

In this paper, the cost allocation problem is analyzed by means of a cooperative game theory. The characteristic function of each coalition $S \subseteq N$ is defined as follows:

$$C(S) \equiv \min\{ \sum_{(i,j)\in E} d_{ij} | G_E \text{ is an } S\text{-feasible forest}\}.$$

It is clear that this characteristic function satisfies the subadditivity: $C(S) + C(T) \geq C(S \cup T)$, for any $S, T \subseteq N$.

We define a *minimum cost forest game* (MCF game) as an ordered pair (N, C). Moreover, core allocation is one of the popular solution concepts in the game theory, and is defined as follows:

$$\text{Core}(C) \equiv \{\mathbf{x} \in \mathbf{R}^n | \sum_{i\in N} x_i = C(N) \text{ and } \sum_{i\in S} x_i \leq C(S) \text{ for any } S \subset N\}.$$

Core gives fair allocations of the cost for the optimal N-feasible forest, if any.

3. Nonemptiness of Core

Is there always a core allocation of MCF game? Kuipers [6] shows a simple example demonstrating that the core of MCF game may be empty. Let $N = \{1, 2, 3\}$ be the set of players,

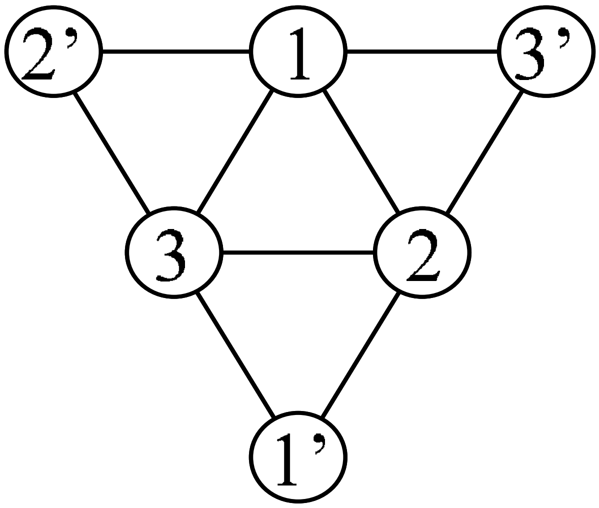

Figure 1: Example such that the core is empty

and $M = \{1', 2', 3'\}$ the set of suppliers. The sets of suppliers which each player requires are as follows: $M(1) = \{1'\}$, $M(2) = \{2'\}$, and $M(3) = \{3'\}$. The weight of all edges drawn in Figure 1 is 3, and the weight of all the other edges is 5. For any player $i \in N$, the optimal forest is obtained by connecting the edge (i, i'). Thus, for any $i \in N$, $C(\{i\}) = 5$. For every coalition of two players, the cost of the optimal forest is 9. For example, the optimal forest for $C(\{1, 2\})$ is established by the edges $(2', 1)$, $(1, 2)$, and $(2, 1')$. One of the optimal forests for the grand coalition N consists of the optimal forest for $\{1, 2\}$ together with the optimal forest for $\{3\}$. Therefore, $C(N) = 14$. If the core is nonempty, the allocation $(14/3, 14/3, 14/3)$ should be in the core by the symmetry. However, the allocation is blocked by the coalition $\{1, 2\}$, since $14/3 + 14/3 > C(\{1, 2\})$. Hence, the core is empty.

Now, we proceed to the discussion of the sufficient conditions under which the core of MCF game is nonempty. Through this paper, let us denote a *network* by an ordered pair (V, d) of the vertex set V and the weight function d of the edges of the complete graph with vertex set V. We start with describing some lemmas needed for our argument. The following is an important property of minimum spanning tree (see Ford and Fulkerson [3]).

Lemma 3.1 *Let (V, d) be a network and Γ be a spanning tree on V. For each pair $v, w \in V$, let $d^{\Gamma}_{v,w}$ denote the maximum over all weights among the edges along the unique path from v to w in Γ. Then the following statements are equivalent.*

1) *Γ is a minimum spanning tree in the network (V, d).*
2) *$d \geq d^{\Gamma}$.*

It has been shown that, given a network (V, d), the unique weight function d^{Γ} can be determined. For the proof, see Kuipers [6].

Lemma 3.2 ([6]) *Let (V, d) be a network, and Γ and Ω be two minimum spanning trees on V in this network. Let d^{Γ} and d^{Ω} be the weight functions defined in Lemma 3.1. Then $d^{\Gamma} = d^{\Omega}$.*

For a network (V, d), the weight function d^{Γ} defined in Lemma 3.1 is determined regardless of the particular minimum spanning tree Γ. In the following, we suppress Γ and denote the weight function by $\bar{d}$. We call $\bar{d}$ *minimal weight function* induced from the network (V, d), and the network $(V, \bar{d})$ *minimal network*. The minimal network has the following properties as verified in Kuipers [6].

Lemma 3.3 ([6]) *Let (V, d) be a network and Γ be a minimum spanning tree on V in this network. Then*

1*) the cost of a minimum spanning tree in the minimal network $(V, \bar{d})$ equals the cost of a minimum spanning tree in the original network (V, d), and*

2*) for each $U \subseteq V$, there exists a minimum spanning tree $\bar{\Gamma}_U$ in the minimal network $(V, \bar{d})$ such that U is a connected set in $\bar{\Gamma}_U$.*

Lemma 3.4 ([6]) *Let $(N, \bar{C})$ be an MCF game with set of suppliers M, defined on a minimal network $(N \cup M, \bar{d})$. Then the game is submodular.*

The above Lemma implies that the MCF game on a minimal network always has a nonempty core (for submodular game, see Shapley [10]).

If the customers $i_1, i_2 \in N$ require a common supplier, they must belong to the same tree on any N-feasible forest. Moreover, even if the customers i_1 and i_2 do not require any common supplier, they necessarily belong to the same tree on any N-feasible forest if another customer i_3 requires a common supplier not only with i_1, but also with i_2, i.e., $M(i_1) \cap M(i_3) \neq \emptyset$ and $M(i_2) \cap M(i_3) \neq \emptyset$.

For any $i, i' \in N$, we say that i is related to i' by the binary relation $\approx$ and write $i \approx i'$ if $M(i) \cap M(i') \neq \emptyset$. If there is a sequence $i_1, ..., i_k$ of the players such that $i_l \approx i_{l+1}$ for any $l = 1, ..., k-1$, we denote the extended relation by $i_1 \sim i_k$. It is clear that the relation $\sim$ defines an *equivalence relation*. Thus, N is always partitioned into the equivalence classes $N_1, ..., N_p$, where any pair of elements in each N_j $(j = 1, ..., p)$ has the equivalence relation

$\sim$. Accordingly, M is partitioned into $M_1, ..., M_p$. Let $V_j \equiv N_j \cup M_j$ for $j = 1, ..., p$. We call each V_j an *equivalence component* on a network.

By utilizing the concept of the equivalence relation, we proceed to argue about the MCF game. For any $U \subseteq (N \cup M)$, let $K(U)$ be the cost of a minimum spanning tree in the network (U, d). For each $j = 1, ..., p$, by using a minimum spanning tree Γ_j in the network (V_j, d) we can determine a unique minimal weight function $\tilde{d}$ and a minimal network $(V_j, \tilde{d})$ from Lemma 3.2. Hereafter, $\tilde{d}$ always refers to the weight function associated with the minimal network $(V_j, \tilde{d})$. For any $U_j \subseteq V_j$, we denote by $\tilde{K}(U_j)$ the cost of a minimum spanning tree in the network $(U_j, \tilde{d})$.

Now, consider the game $(N, \hat{C})$ defined on the network $(N \cup M, \hat{d})$, where

$$\hat{d}_{vw} = \begin{cases} \tilde{d}_{vw} & \text{if } v, w \in V_j \text{ for some } j \\ d_{vw} & \text{otherwise.} \end{cases}$$

Note that for any $j = 1, ..., p$ and for any $U_j \subseteq V_j$, $\hat{K}(U_j) = \tilde{K}(U_j)$. Thus, from Lemma 3.3-(1),

$$K(V_j) = \tilde{K}(V_j) = \hat{K}(V_j), \quad j = 1, ..., p. \tag{1}$$

We denote by $(N_j, \tilde{C})$ a game with set of customers N_j and set of suppliers M_j, defined on the network $(V_j, \tilde{d})$. Let $S_j \equiv S \cap N_j$ for $j = 1, ..., p$. The game $(N, \hat{C})$ has the following relationship with the game (N, C).

Lemma 3.5 *Let (N, C) be an MCF game with set of customers N and set of suppliers M, defined on the network $(N \cup M, d)$. Let $V_j = N_j \cup M_j$, $j = 1, ..., p$, be equivalence components. If*

$$C(N) = \sum_{j=1}^{p} K(V_j) \ , \tag{2}$$

then $\hat{C}(S) = \sum_{j=1}^{p} \tilde{C}(S_j)$ for all $S \subseteq N$.

Proof. One can easily show that $\hat{C}(N) = \sum_{j=1}^{p} \tilde{C}(N_j)$ holds (see Appendix). Thus, using Lemma 3.3, with respect to the weight function $\hat{d}$, an optimal N-feasible forest exists in which each set $S_j \cup M(S_j)$ $(j = 1, ..., p)$ is connected. Let E_j $(j = 1, ..., p)$ denote the set of edges with both endpoints in V_j, and let $E_j^S \subseteq E_j$ denote the set of edges with both endpoints in $S_j \cup M(S_j)$. Now take an optimal S-feasible forest and add the edges of the sets $E_j \backslash E_j^S$ $(j = 1, ..., p)$ to it. One can show that the resulting graph is N-feasible. Therefore, $\hat{C}(N)$ is at most the sum of $\hat{C}(S)$ and the cost of all edges in the sets $E_j \backslash E_j^S$ $(j = 1, ..., p)$. It follows that $\hat{C}(S)$ is at least the cost of all edges in the sets E_j^S $(j = 1, ..., p)$. Since the edges in each E_j^S trivially constitute an $(N_j \cap S)$-feasible forest with respect to the weight function $\tilde{d}$, it now follows that $\hat{C}(S) \geq \sum_{j=1}^{p} \tilde{C}(S_j)$. The reversed inequality is trivial. Thus, $\hat{C}(S) = \sum_{j=1}^{p} \tilde{C}(S_j)$ holds. ■

From Lemma 3.4 and Lemma 3.5, we are ready to state our main theorem regarding an MCF game composed of p equivalence classes.

Theorem 3.1 *Let (N, C) be an MCF game with set of customers N and set of suppliers M, defined on the network $(N \cup M, d)$. Let $V_j = N_j \cup M_j$, $j = 1, ..., p$, be equivalence components. If*

$$C(N) = \sum_{j=1}^{p} K(V_j) \ ,$$

then the game (N, C) *has a nonempty core.*

Proof. For each $j = 1, ..., p$, consider the game $(N_j, \tilde{C})$ defined on the network $(V_j, \tilde{d})$. From Lemma 3.4, we have a core allocation $\mathbf{x}_j$ such that $x(N_j) = \tilde{C}(N_j)$ and $x(S_j) \leq \tilde{C}(S_j)$ for any $S_j \subseteq N_j$, where $x(S) \equiv \sum_{i \in S} x_i$. Let $\mathbf{x} \equiv (\mathbf{x}_1, ..., \mathbf{x}_p) \equiv (x_1, ..., x_n)$. Then, for any $S \subset N$,

$$C(S) \geq \hat{C}(S) = \sum_{j=1}^{p} \tilde{C}(S_j) \geq \sum_{j=1}^{p} x(S_j) = x(S).$$

The first inequality follows from $d \geq \hat{d}$. The next equality is obtained from Lemma 3.5. Moreover, $x(N) = \sum_{j=1}^{p} x(N_j) = \sum_{j=1}^{p} \tilde{C}(N_j) = \sum_{j=1}^{p} K(V_j) = C(N)$, where the third equality follows from (1). Therefore, it is verified that $\mathbf{x}$ is a core allocation of the MCF game (N, C). ■

This theorem includes as a special case Lemma 7 in [6], which states that *if the optimal forest for the grand coalition consists of exactly* $|M|$ *components, then the MCF game has a nonempty core.*

Moreover, it is verified that an MCF game composed of at most two equivalence classes always has a nonempty core.

Theorem 3.2 *Let* (N, C) *be an MCF game with set of customers* N *and set of suppliers* M*, defined on the network* $(N \cup M, d)$*. Suppose that* N *is composed of at most two equivalence classes of the players. Then, the game has a nonempty core.*

Proof. First, the proof is presented for the case where N is composed of one equivalence class, i.e., $p = 1$. Consider the MCF game $(N, \bar{C})$ defined on the minimal network $(N \cup M, \bar{d})$ induced from the network $(N \cup M, d)$. The optimal forest for the grand coalition must be a tree in both games (N, C) and $(N, \bar{C})$ because of the equivalence relation of players. From Lemma 3.3-(1), $C(N) = \bar{C}(N)$. Since $d \geq \bar{d}$, $C(S) \geq \bar{C}(S)$ for any $S \subseteq N$. The MCF game $(N, \bar{C})$ has a nonempty core by Lemma 3.4. Since Core$(\bar{C}) \subseteq$ Core(C), the core of the MCF game (N, C) is nonempty.

Second, we discuss the case where the number p of the equivalence classes on MCF game is two. We may assume that every optimal forest for $C(N)$ is a tree, since otherwise by Theorem 3.1 the core of this game is nonempty. Moreover, it may be assumed that the network $(N \cup M, d)$ is not minimal, because otherwise by Lemma 3.4 the game has a nonempty core. Now, let us denote an optimal forest by Γ, which is a tree, and pick up an edge (v, w) such that $\bar{d}_{vw} < d_{vw}$. Lower the value of the weight d_{vw} until one of the following two cases happens. (i) The weight of the edge is equal to $\bar{d}_{vw}$, which is a minimal weight induced by Γ. (ii) There exists an optimal forest for grand coalition which is composed of two connected components. Let (N, C') be the game with the lowered weight. During this procedure, $C'(N) = C(N)$ by Lemma 3.3-(1), and $C'(S) \leq C(S)$ for any $S \subset N$. Therefore, to check the nonemptiness of the game (N, C), it is sufficient to examine if the game (N, C') has a nonempty core. If the case (ii) occurs, the game has a nonempty core from Theorem 3.1. If the case (i) happens, select another edge (v, w) which is not minimal and do the same thing. Then, if all edges are minimal, the game is submodular by Lemma 3.4, so has a nonempty core. In any cases, it follows that the core of the game (N, C') is nonempty. ■

Theorem 6 in [6] states that *if there is a player* $i \in N$ *with* $M(i) = M$*, then the game has a nonempty core.* Theorem 8 in [6] states that *if* $|M| = 2$*, then the MCF game has a nonempty core*. We note that Theorem 3.2 generalizes both of these statements.

4. Concluding Remarks

We study the nonemptiness of the core by considering the equivalence relation on the set of players. In view of constructing a network with the minimum cost, it is natural that an equivalence relation be introduced. Moreover, it is quite reasonable that a sufficient condition for a nonempty core involves this equivalence relation, as seen in Theorem 3.1. Note that this theorem includes Lemma 7 in [6] as a special case. Furthermore, it is verified that the core of MCF game is always nonempty if the game consists of at most two equivalence classes. In fact, as given in Section 3, if the game is composed three equivalence classes, we immediately have the possibility that the core is empty. Note also that this condition generalizes a conclusion given in [6] that the core is nonempty if there are at most two suppliers.

On the other hand, it is possible to make an example such that the core is nonempty while our condition in Theorem 3.1 is not satisfied. For example, consider the example in Section 3 where the value of d_{12} is modified to be 2. Then $C(N) = C(\{1,2\}) + C(\{3\}) = 8 + 5 = 13$ and an allocation $(4, 4, 5)$ is in the core. In addition, the example remains violating our condition. This implies our condition in Theorem 3.1 is not a necessary and sufficient condition. Hence, there may be a weaker sufficient condition such that the weights of the edges across the equivalence components are considered.

It is not easy to examine whether a problem satisfies the sufficient condition in Theorem 3.1, when the number n of the players is very large and the number p of the equivalence classes is as large as n. In many real world scenes, however, it is not so common that customers do not share a same type of service at all. Most of the servers are shared by lots of people, which increases the connectivity between the customers. Thus, p is supposed to be very small relative to n. Then, we can check the sufficient condition in Theorem 3.1 by the practical number of iteration of computations. Let $I \subseteq \{1, ..., p\}$. The cost $K(\bigcup_{j\in I} V_j)$ of minimum spanning tree on $\bigcup_{j\in I} V_j$ is compared with the sum of the cost of minimum spanning tree on each equivalence component V_j. If there exists at least one I such that $K(\bigcup_{j\in I} V_j) < \sum_{j\in I} K(V_j)$, it follows that $C(N) < \sum_{j=1}^{p} K(V_j)$. On the other hand, if $K(\bigcup_{j\in I} V_j) \geq \sum_{j\in I} K(V_j)$ for every I, we can see that the condition in Theorem 3.1 is satisfied, i.e., $C(N) = \sum_{j=1}^{p} K(V_j)$. Thus, the core of the game is nonempty. In addition, when n is large and p is small, Theorem 3.1 is effective, because we can dispense with examining the 2^n worths of the coalitions.

It appears that 2^p minimum spanning trees are needed in the worst case to solve the decision problem of whether the condition in Theorem 3.1 is satisfied or not. Thus, it is unlikely that this decision problem could be solved in a polynomial time on p. Developing a sufficient condition that can be verified within a polynomial time remains a topic of future research.

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Appendix

Proof of $\hat{C}(N) = C(N)$**:** Note that the condition (2) implies that for any $I \subseteq \{1, ..., p\}$, $C(\bigcup_{j\in I} N_j) = \sum_{j\in I} K(V_j)$.

Since $\hat{d} \leq d$, $\hat{C}(N) \leq C(N)$. Suppose that

$$\hat{C}(N) < C(N). \tag{3}$$

Let the graph $(V, \hat{E})$ be the optimal forest for $\hat{C}(N)$. Let $B = \{(v,w) \in \hat{E} \mid v \in V_j, w \in V_k, \text{ for } j \neq k\}$. We can observe that $B \neq \emptyset$ by the fact that $\hat{C}(N) < C(N) = \sum_{j=1}^{p} K(V_j) = \sum_{j=1}^{p} \hat{K}(V_j)$. Let $T_1, ..., T_t$ be the trees composing the optimal forest $(V, \hat{E})$. Since any two vertices in V_j can't belong to different trees by the equivalence relation, the forest can be represented by $T_k = (\bigcup_{j \in J_k} V_j, \hat{E}_k)$, $k = 1, ..., t$, where $\{J_k,\ k = 1, ..., t\}$ and $\{\hat{E}_k,\ k = 1, ..., t\}$ are partitions of $\{1, ..., p\}$ and $\hat{E}$, respectively. From (2) and (3), it can be seen that $\sum_{k=1}^{t} \hat{C}(\bigcup_{j \in J_k} N_j) = \hat{C}(N) < C(N) = \sum_{j=1}^{p} K(V_j) = \sum_{k=1}^{t} \sum_{j \in J_k} K(V_j)$. For any $k = 1, ..., t$, $\hat{C}(\bigcup_{j \in J_k} N_j) \leq \sum_{j \in J_k} \hat{K}(V_j) = \sum_{j \in J_k} K(V_j)$ holds by the $(\bigcup_{j \in J_k} N_j)$-feasibility. Therefore, there exists at least one tree T_i such that

$$\hat{C}(\bigcup_{j \in J_i} N_j) < \sum_{j \in J_i} K(V_j), \text{ where } |J_i| \geq 2.$$

Moreover, there exists at least one edge $(v,w) \in \hat{E}_i$ such that $\hat{d}_{vw} < d_{vw}$. Otherwise, we can construct a $(\bigcup_{j \in J_i} N_j)$-feasible tree with the cost less than $\sum_{j \in J_i} K(V_j)$ by using only the weight function d. Note that v and w belong to the same equivalence component because of $\hat{d}_{vw} < d_{vw}$. Let $\mathcal{T} = \{T^k\}$ be the set of minimum spanning trees in the network $(\bigcup_{j \in J_i} V_j, \hat{d})$. Choose $T^* = \arg\min_{T^k \in \mathcal{T}} |\{(v,w) \in \hat{E}^k \mid \hat{d}_{vw} < d_{vw}\}|$, where $\hat{E}^k$ is the edge set of tree T^k. In T^*, remove an edge $(v,w) \in \hat{E}^*$ such that $\hat{d}_{vw} < d_{vw}$. Then, T^* is divided into two trees T_1^*, T_2^*, and v, w are in different trees, say $v \in T_1^*$ and $w \in T_2^*$. Since $\hat{d}_{vw} < d_{vw}$, $v, w \in V_j$ for some j. Now, consider a minimum spanning tree Γ_j in the network (V_j, d). There exists a unique path $\mu_{vw} = [v = v_0, v_1, ..., v_l, w = v_{l+1}]$ $(l \geq 1)$ in Γ_j. There exists at least one edge (v_k, v_{k+1}) such that $v_k \in T_1^*$ and $v_{k+1} \in T_2^*$ since all vertices in μ_{vw} belong to either T_1^* or T_2^*. $d_{v_k v_{k+1}} \leq \hat{d}_{vw}$ holds, because $\hat{d}_{vw} = \max\{d_{vv_1}, d_{v_1 v_2}, ..., d_{v_{l-1} v_l}, d_{v_l w}\}$. By connecting T_1^* and T_2^* with the edge (v_k, v_{k+1}) of the weight $d_{v_k v_{k+1}}$, we can construct a $(\bigcup_{j \in J_i} N_j)$-feasible tree T^{**} with the edge set $\hat{E}^{**}$ different from T^*. The cost of this new tree can not be less than $\hat{C}(\bigcup_{j \in J_i} N_j)$ by the optimality and can not be larger by $d_{v_k v_{k+1}} \leq \hat{d}_{vw}$. Thus, the cost is equal to $\hat{C}(\bigcup_{j \in J_i} N_j)$. We constructed tree T^{**} such that $|\{(v,w) \in \hat{E}^{**} \mid \hat{d}_{vw} < d_{vw}\}|$ is one less than in T^*. This is a contradiction. Thus, $\hat{C}(N) = C(N) = \sum_{j=1}^{p} \tilde{C}(N_j)$. ■

References

[1] C. G. Bird: On cost allocation for spanning tree: A game theoretic approach. *Networks*, **6** (1976) 335-350.

[2] A. Claus and D. J. Kleitman: Cost allocation for a spanning tree. *Networks*, **3** (1973) 289-304.

[3] L. R. Ford and D. R. Fulkerson: *Flows in Networks* (Princeton University Press, Princeton, 1962).

[4] D. Granot and F. Granot: On the computational complexity of a cost allocation approach to a fixed cost spanning forest problem. *Mathematics of Operations Research*, **17** (1993) 765-780.

[5] D. Granot and G. Huberman: Minimum cost spanning tree games. *Mathematical Programming*, **21** (1981) 1-18.

[6] J. Kuipers: Minimum cost forest games. *International Journal of Game Theory*, **13** (1997) 367-377.

[7] N. Megiddo: Cost allocation for steiner trees. *Networks*, **8** (1978) 1-6.

[8] G. Owen: *Game Theory, Third Edition* (Academic Press, San Diego, California, 1995).

[9] E. C. Rosenthal: The minimum cost spanning forest game. *Economics Letters*, **23** (1987) 355-357.

[10] L. S. Shapley: Core of convex games. *International Journal of Game Theory*, **1** (1971) 11-26.

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