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DETERMINING AN EQUITABLE ALLOCATION OF NEW INPUT AND OUTPUT USING DATA ENVELOPMENT ANALYSIS

Gholamreza Jahanshahloo
Teacher Training University

Alireza Amirteimoori Sohrab Kordrostami
Islamic Azad University

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Abstract In this paper, we consider the problem of demanding fixed aggregated output from producers for an equitable allocating of new input. In many applications, there is often a fixed input and output which are imposed on all decision making units. In such situations, the choice of an allocation pattern often seems to be rather subjective. This paper, presents a method for obtaining an equitable allocation and demand of these new input and output which is based on two principals: invariance and Pareto-minimality.

Keywords: DEA, optimization, linear programming, efficiency

1. Introduction

Data Envelopment Analysis (DEA) has been used in a very wide range of applications to measure the relative efficiency of organizational decision making units (DMUs). In these applications, there is often two fixed factors (one input and one output) that are imposed on all DMUs. These two factors are added to DMUs as a new input and new output. It has been assumed that allocating more resources, proportionally increase more outputs. The problem arises as to how these factors can be allocated and demanded in an equitable way to the various DMUs. This is the main subject under discussion in this paper. Cook et al.(1999) [6] proposed a method which only considered an equitable allocation of shared input. Consider the example of a set of departments of a university. A department of university is a DMU with research and teaching activities. Each department receives research grants, equipment grants, fees and funding council income. These inputs generate some outputs. Let the outputs be research students, undergraduate students, research quality rating and etc. Suppose that the university is interested in assigning of a certain amount of research capital to all departments and needs a certain number of Ph.D. students. The question is : how the capital can be allocated and students be wanted in an equitable way to the various departments. Clearly, these two factors that are imposed on a DMU, constitute an additional input and output which may alter the relative efficiency of the DMU. The management's objective is to allocate and demand these factors in an equitable way to the various units. It should be emphasized that the DMU has no control on these new input and output. Its performance relies entirely on its existing inputs and outputs. We will assume that any allocation and demand that does not alter the value of relative efficiency measure, is equitable. We therefore take this as a necessary condition for such allocation and demand. This paper presents a procedure to obtaining an equitable allocation and demand of new input and output which is based on two principals: invariance and Pareto-minimality. It should be noted that consistent with the assumption the observed inputs and outputs define an empirical production function, the equitable allocation and demand should have no effect on this function. We call this requirement as *invariance* of relative efficiency. The paper is structured as follows: Section 2 discusses a prioritization method for frontier DMUs. The third section of this paper gives the method for single-input/single-output case. The proposed method for multi-input/multi-output case is given in Section 4. Some numerical examples are solved in Section 5. Conclusions appear in Section 6.

2. Prioritization Model for Frontier DMUs

Several methods for prioritizing efficient units are reported in literature (see for instance [1]). In this section, we address a method for prioritizing DMUs. In the proposed method, in order to ranking an efficient unit, first, this unit is omitted from production possibility set (PPS) and then the shortest distance(L_2 norm) from this unit to new frontier is determined. Unit with most distance has the best rank. Consider the following linear programming problem:

$$\begin{array}{l} D_p = \ Max \ -\alpha^T X_p + \ \beta^T Y_p \\ s.t. \\ -\alpha^T X_j + \ \beta^T Y_j \leq 0, \ j = 1, \ldots, n, j \neq p, \\ \alpha^T 1_m + \ \beta^T 1_s = 1 \ , \\ \alpha \geq \epsilon .1_m \ , \\ \beta \geq \epsilon .1_s . \end{array} \qquad (1)$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)^T$, $\beta = (\beta_1, \ldots, \beta_s)^T$, $\epsilon > 0$, $X_j = (x_{1j}, \ldots, x_{mj})^T$, $Y_j = (y_{1j}, \ldots, y_{sj})^T$ and $1_. = (1,\ 1,\ \ldots,\ 1)^T$. This problem is always feasible, because, if we choose ϵ sufficiently small, it is easy to show that

$$\beta_i = \epsilon;\ i = 1,\ \ldots,\ s,\ \alpha_i = \tfrac{1-s.\epsilon}{m};\ i = 1,\ \ldots,\ m,$$

is a feasible solution for (1). It can be seen that (1) is bounded. Suppose that (α^*, β^*) be the optimal solution of (1). It is easy to show that

$$F' = \{(X, Y) : -\ \alpha^{*T} X + \beta^{*T} Y = 0\} \bigcap T_c^{'}$$

is an efficient surface of $T_c^{'}$ ($T_c^{'}$ is the new PPS obtained from omission DMU_p). Applying this problem in first stage, the farthest efficient surface of $T_c^{'}$ from DMU_p is determined and in second stage, the shortest distance from DMU_p to $T_c^{'}$ is found. Toward this end, solve problem (1) for $DMU_p \in J = \{DMU_1, \ldots, DMU_n\}$ and suppose that (α^*, β^*) be the optimal solution. Consider the following two cases:

(i) (α^*, β^*) is unique. In this case, let

$$EF_p = \frac{D_p}{\sqrt{\alpha^{*T}\alpha^* + \ \beta^{*T}\beta^*}},$$

(ii) (α^*, β^*) is not unique. In this case, let $X_\lambda = \sum_{j\neq p} \lambda_j X_j$, $\ Y_\lambda = \sum_{j\neq p} \lambda_j Y_j$ in which $\lambda_j \geq 0$; for all j. Let

$$EF_p = Min_{\lambda\geq 0}\{ \text{ The distance between } (X_p, Y_p) \text{ and } (X_\lambda, Y_\lambda) \ \ (L_2 - norm)\}$$

It is to be noted that in a real data set, the second case is of seldom occurrence. Now, we define

$$\text{Efficiency index } \hat{EF}_p = \frac{EF_p}{Max_{1\leq j\leq n}\{EF_j\}} \leq 1.$$

The following theorem gives a necessary and sufficient condition for efficiency of DMU_p.

Theorem 1 *Consider problem (1). $D_p \geq 0$ if and only if DMU_p is efficient in CCR model. (i.e. $DMU_p \in RE = E \bigcup E' \bigcup F$, the set of all DEA-scale efficient)(see [5]).*

Proof:

We shall first prove that if $D_p \geq 0$, DMU_p is non-dominant in T_c. Suppose in contrary that DMU_p is dominant. Then, there exist some $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that: $\lambda_j \geq 0,\ j = 1, \ldots, n$ and

$$X_p \geq \sum_{j \neq p} \lambda_j X_j = X_\lambda$$
$$Y_p \leq \sum_{j \neq p} \lambda_j Y_j = Y_\lambda$$

and strict inequality is hold true for at least one component. Now,

$$0 \leq -\ {\alpha^*}^T X_p +\ {\beta^*}^T Y_p < -{\alpha^*}^T X_\lambda +\ {\beta^*}^T Y_\lambda, \tag{2}$$

where $(\alpha^*,\ \beta^*)$ is an optimal solution for (1). But, we note that $(X_\lambda,\ Y_\lambda) \in T_c$ and

$$H^- = \{(X, Y) :\ -\ {\alpha^*}^T X +\ {\beta^*}^T Y \leq 0\}$$

is a convex set and therefore we must have $-\ {\alpha^*}^T X_\lambda +\ {\beta^*}^T Y_\lambda \leq 0$, which contradicts (2). To show converse, suppose that DMU_p is CCR-efficient, this means that ${Z_p}^* \geq 1$ in which

$$\begin{array}{l} {Z_p}^* =\ Max\ \ U^T Y_p \\ s.t. \\ V^T X_p = 1\ , \\ U^T Y_j -\ V^T X_j \leq 0,\ j = 1, \ldots, n, j \neq p, \\ U \geq 0, \\ V \geq 0. \end{array} \tag{3}$$

Hence, if (U^*, V^*) be the optimal solution of this problem, we must have ${U^*}^T Y_p -\ {V^*}^T X_p \geq 0$. It is easy to show that (U^*, V^*) is the optimal solution of this problem iff $(\frac{U^*}{1U^*+1V^*}, \frac{V^*}{1U^*+1V^*}) = (\alpha^*, \beta^*)$ be the optimal solution of (1). Hence, we have

$$\sum_{r=1}^{s} \frac{{u_r}^*}{1U^*+1V^*} y_{rp} - \sum_{i=1}^{m} \frac{{v_i}^*}{1U^*+1V^*} x_{ip} \geq 0$$

$$\Rightarrow {\beta^*}^T Y_p - {\alpha^*}^T X_p = D_p \geq 0.$$

This completes the proof.□

As a result of the foregoing theorem, note that in (1):

$D_p > 0$ if and only if DMU_p is extreme efficient, (i.e. belongs to E),

$D_p < 0$ if and only if DMU_p is inefficient,

$D_p = 0$ if and only if DMU_p is non-extreme efficient, $(i.e. belongs to E' \bigcup F)$.

Hence, in order to ranking DMUs, DMU_j will have better rank as compared with DMU_i if and only if $\hat{EF_j} > \hat{EF_i}$. In what follows, we show that the suggested efficiency index $\hat{EF_j}$ is units invariant which should not be ignored. Suppose that $(X_j, Y_j),\ j = 1, \ldots, n$, is substituted by $(\lambda X_j, \lambda Y_j) = DMU_{\hat{j}}$, in which $\lambda > 0$. It suffices to show that $\hat{EF_{\hat{p}}} = \hat{EF_p}$. If (α^*, β^*) be the optimal solution of (1) for evaluation DMU_p, it also is the optimal solution of (1) for evaluation $DMU_{\hat{p}}$. Hence, $D_{\hat{p}} = \lambda D_p$. On the other hand,

$$EF_{\hat{p}} = \frac{D_{\hat{p}}}{\sqrt{{\alpha^*}^T \alpha^* +\ {\beta^*}^T \beta^*}} = \frac{\lambda D_p}{\sqrt{{\alpha^*}^T \alpha^* +\ {\beta^*}^T \beta^*}}$$

$$\hat{EF_{\hat{p}}} = \frac{EF_{\hat{p}}}{Max_{1 \leq j \leq n}\ \{EF_{\hat{j}}\}} = \hat{EF_p}.\square$$

Formally we define that an input allocation or an output demand is Pareto-minimal if no input or output can be transferred from one DMU to another without violating the invariance principal.

3. The Single-Input/Single-Output Case

The DEA formulation (1) for the single-input/single-output case is:

$$\begin{array}{l} D_p = Max \quad -\alpha x_p + \ \beta y_p \\ s.t. \\ -\alpha x_j + \ \beta y_j \leq 0, \ j = 1, \ldots, n, j \neq p, \\ \alpha + \ \beta = 1 \ , \\ \alpha, \beta \geq 0 \ , \end{array} \tag{4}$$

where DMU_p is each unit with rank ≥ 2. Without lose of generality, we assume that DMU_1 has rank 1. There is a new input R and new output Q that must be imposed on all DMUs. By imposing the new input δ_j and new output ϕ_j on DMU_j, we have the following problem:

$$\begin{array}{l} Max \quad -\alpha x_p + \ \beta y_p - \hat{\alpha}\delta_p + \hat{\beta}\phi_p \\ s.t. \\ -\alpha x_j + \ \beta y_j - \hat{\alpha}\delta_j + \hat{\beta}\phi_j \leq 0, \ j = 1, \ldots, n, j \neq p, \\ \alpha + \ \beta + \hat{\alpha} + \hat{\beta} = 1, \\ \alpha, \beta, \hat{\alpha}, \hat{\beta} \geq \epsilon. \end{array} \tag{5}$$

In order to have an equitable allocation and demand which does not alter the relative efficiency of DMUs, we must have $\hat{\alpha}$ and $\hat{\beta}$ as non-basic variable in (5). For each DMU_p with rank ≥ 2, $\hat{\alpha}$ and $\hat{\beta}$ are non-basic if and only if the reduced costs corresponding to $\hat{\alpha}$ and $\hat{\beta}$ be nonnegative and in order to preserve the condition that satisfies the Pareto-minimality condition, we must have:

$$\delta_p - \textstyle\sum_{j=1, j\neq p}^{n} \lambda_j^{(p)} \delta_j + \theta_p = 0, \tag{6}$$

$$-\phi_p + \textstyle\sum_{j=1, j\neq p}^{n} \lambda_j^{(p)} \phi_j + \theta_p = 0, \tag{7}$$

where θ_p and $\lambda_j^{(p)}$'s are the dual optimal variables of (5). The dual of (4) is as follows:

$$\begin{array}{l} Min \quad \theta_p \\ s.t. \\ \sum_{j=1, j\neq p}^{n} \lambda_j x_j - \theta_p \leq x_p, \\ \sum_{j=1, j\neq p}^{n} \lambda_j y_j + \theta_p \geq y_p, \\ \lambda_j \geq 0, \ for \ all \ j. \end{array} \tag{8}$$

The optimal extreme point of the feasible region of (8) is as $(\theta_p, 0, \ldots, 0, \lambda_1^{(p)}, 0, \ldots, 0)$, where

$$\begin{array}{c} \theta_p = -\alpha^* x_p + \beta^* y_p \\ \alpha^* = \frac{x_1}{x_1+y_1}, \quad \beta^* = \frac{y_1}{x_1+y_1}, \\ \lambda_t{}^{(p)} = \frac{y_p - \theta_p{}^*}{y_1} = \frac{x_p + \theta_p{}^*}{x_1}. \end{array}$$

From (6) and (7), we have

$$\begin{array}{l} \delta_p = \lambda_t^{(p)}\delta_t - \theta_p, \quad j = 1, \ldots, n, \\ \sum_{j=1}^{n} \delta_j = R, \end{array} \tag{9}$$

$$\begin{array}{l} \phi_p = \lambda_t^{(p)}\phi_t + \theta_p, \quad j = 1, \ldots, n, \\ \sum_{j=1}^{n} \phi_j = Q, \end{array} \tag{10}$$

which are two systems $n \times n$ of unknown $\delta_j,\ j = 1, \ldots, n$ and $\phi_j,\ j = 1, \ldots, n$. Thus we have provided the applicability of the proposed DEA approach for the single-input/single-output case.

4. Multi-input/Multi-output Case

Consider the general version of (1) as follows:

$$\begin{array}{l} Max \quad -\sum_{i=1}^{m} \alpha_i x_{ip} + \sum_{r=1}^{s} \beta_r y_{rp}, \\ s.t. \\ -\sum_{i=1}^{m} \alpha_i x_{ij} + \sum_{r=1}^{s} \beta_r y_{rj} \leq 0,\ j = 1, \ldots, n,\ j \neq p \\ \sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r = 1, \\ \alpha_i, \beta_r \geq 0,\ for\ all\ i, r. \end{array} \tag{11}$$

Suppose again that there is a new input R and new output Q that must be imposed on all DMUs. As before, by imposing the new input δ_j and new output ϕ_j on DMU_j, we have:

$$\begin{array}{l} Max \quad -\sum_{i=1}^{m} \alpha_i x_{ip} + \sum_{r=1}^{s} \beta_r y_{rp} - \hat{\alpha}\delta_p + \hat{\beta}\phi_p, \\ s.t. \\ -\sum_{i=1}^{m} \alpha_i x_{ij} + \sum_{r=1}^{s} \beta_r y_{rj} - \hat{\alpha}\delta_j + \hat{\beta}\phi_j \leq 0,\ j = 1, \ldots, n,\ j \neq p, \\ \sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r + \hat{\alpha} + \hat{\beta} = 1, \\ \alpha_i, \beta_r, \hat{\alpha}, \hat{\beta} \geq 0,\ for\ all\ i, r. \end{array} \tag{12}$$

In order to preserve the condition that satisfies the Pareto-minimality condition, we must have:

$$\begin{array}{ll} \delta_p - \sum_{j=1}^{n} \lambda_j^{(p)}\delta_j + \theta_p = 0 & p = 2, \ldots, n \\ -\phi_p + \sum_{j=1}^{n} \lambda_j^{(p)}\phi_j + \theta_p = 0 & p = 2, \ldots, n \end{array}$$

where $\lambda_j^{(p)}$'s and θ_p are the dual variables of (12).

N.B. It is assumed that by prioritizing the efficient DMUs, DMU_1, has rank one. The dual of (11) is as follows:

$$\begin{array}{l} Min \ \ \theta_p \\ s.t. \\ \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} - \theta_p \leq x_{ip}, \quad i = 1, \ldots, m, \\ \sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} + \theta_p \geq y_{rp}, \quad r = 1, \ldots, s, \\ \lambda_j \geq 0,\ for\ all\ j. \end{array} \tag{13}$$

Hence, we have the following two systems of equations:

$$\begin{array}{ll} \delta_p - \sum_{j=1}^{n} \lambda_j^{(p)}\delta_j = -\theta_p & p = 2, \ldots, n \\ \sum_{j=1}^{n} \delta_j = R & \end{array}$$

and

$$\begin{array}{ll} -\phi_p + \sum_{j=1}^{n} \lambda_j^{(p)}\phi_j = -\theta_p & p = 2, \ldots, n \\ \sum_{j=1}^{n} \phi_j = Q. & \end{array}$$

These two systems determine δ_js and ϕ_js.

5. Examples

To demonstrate the proposed method, we present two numerical examples.

5.1. Example 1

Suppose we have data on 6 DMUs with single input and single output as given in Table 1, and we are given a new resource allocation $R = 20$ and the output $Q = 25$ is in demand.

Table 1. The data set for example 1

	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6
I	1	4	2	3	2	1
O	4	1	2	2	5	1

The systems of equations (9) and (10) for this set of data are as follows:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1.4 & 0 & 0 & 0 & -1 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{bmatrix} = \begin{bmatrix} -3 \\ -1.2 \\ -2 \\ -0.6 \\ -0.6 \\ 20 \end{bmatrix} \Rightarrow \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{bmatrix} = \begin{bmatrix} 2.2500 \\ 5.2500 \\ 3.0000 \\ 4.2500 \\ 3.7500 \\ 1.5000 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1.4 & 0 & 0 & 0 & -1 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.2 \\ 2 \\ 0.6 \\ 0.6 \\ 25 \end{bmatrix} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix} = \begin{bmatrix} 5.7857 \\ 2.7857 \\ 3.4286 \\ 3.7857 \\ 7.5000 \\ 1.7143 \end{bmatrix}$$

The equitable allocation and the efficiency rating of the model described in Section 3 are shown in Table 2.

Table 2. The Results for example 1

	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6
I	1	4	2	3	2	1
O	4	1	2	2	5	1
I_{new}	2.2500	5.2500	3.0000	4.2500	3.7500	1.5000
O_{new}	5.7857	2.7857	3.4286	3.7857	7.5000	1.7143
$Old\hat{E}F_j$	1	-6.50	-2.61	-4.34	-1.30	-1.30
$New\hat{E}F_j$	1	-6.50	-2.61	-4.34	-1.30	-1.30

5.2. Example 2

We provide another example which illustrates the procedure for multi-input/ multi-output case. The procedure is illustrated by means of 12 units data set from Cook et al.(1999)[6], which each unit consumes three inputs to produce two outputs. Raw data are listed in Table 3.

Table 3. The data set for example 2

DMU_j	Input1	Input2	Input3	Output1	Output2
1	350	39	9	67	751
2	298	26	8	73	611
3	422	31	7	75	584
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	17	83	1070
7	540	18	10	72	457
8	276	33	5	78	590
9	323	25	5	75	1074
10	444	64	6	74	1072
11	323	25	5	25	350
12	444	64	6	104	1199

The equitable allocation of δ_j's and the equitable demand of ϕ_j's using the proposed approach are presented in Table 4. The last column shows the equitable allocation of Cook et al.'s method. This method allocates only inputs and the new output can not be shared among DMUs. By the method proposed in Section 3, δ_js and ϕ_js are determined which are listed in Table 4.

Table 4. The results for example 2

DMU_j	δ_j	ϕ_j	$Old\hat{EF}_j$	$New\hat{EF}_j$	Cook's allocation
1	17.1955	77.6676	-0.14	-0.14	21.98
2	10.3651	83.5286	-0.09	-0.09	10.12
3	10.1645	86.7754	-0.06	-0.06	13.98
4	8.9830	79.9768	0.09	0.09	8.40
5	8.5746	85.5534	0.11	0.11	8.68
6	18.3650	95.8077	-0.11	-0.11	12.22
7	10.6564	82.4394	-0.07	-0.07	13.28
8	9.3167	88.0111	0.23	0.23	9.39
9	16.8416	86.6044	1	1	10.85
10	17.8102	85.5962	-0.03	-0.03	15.12
11	7.9294	29.1590	-0.10	-0.10	10.86
12	13.7980	118.8804	0.02	0.02	15.12
	R=150	Q=1000			150

6. Conclusion

The problem of finding an equitable allocation and equitable demand pattern is important in many managerial decision problems. In many applications, there is often a fixed or common input or output which are imposed on all decision making units. In these applications, the choice of an allocation and demand pattern is important. In this paper, we present a procedure to obtain an equitable allocation of the imposed input and an equitable demand of the imposed output which is based on two principals: invariance and Pareto-minimality. For the straightforward case of single -input/single-output to the multi-input/multi-output case, we have shown that the proposed DEA model can be used to find a characterization of an equitable allocation and demand. The proposed DEA model reflects the efficiency of each DMU and ranks the DMUs. It is shown that this prioritization model is feasible and bounded and hence the difficulties of the existing methods have been removed.

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Dr. Ali Reza Amirteimoori
Dept. of Mathematics
Islamic Azad University
Pol-e-Taleshan
Rasht, IRAN
Email: teimoori@kadous.gu.ac.ir