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MODELING INPUT ALLOCATIVE EFFICIENCY VIA DISTANCE AND QUASI-DISTANCE FUNCTIONS

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Abstract Data envelopment analysis (DEA) is an important analytical tool for estimating the technical efficiency of decision-making units. However, input allocative efficiency is less frequently investigated in the DEA framework because the measure of input allocative efficiency-derived as the ratio of cost efficiency to Farrell input technical efficiency-may be biased because of slack in the constraints that define the technology or because of differences in input prices paid by firms. We extend the DEA methodology by proposing a new measure of input allocative efficiency based on the direct and indirect output quasi-distance functions. An empirical illustration of the new and existing methods for measuring input allocative efficiency is offered for a sample of U.S. banks operating in 1994-95.

Keywords: DEA, quasi-distance functions, efficiency, productivity, U.S. banks

1. Introduction

Data envelopment analysis (DEA) is an important tool for estimating the technical efficiency of a decision-making unit. For example, in a review of depository financial institutions, Berger and Humphrey [3] identified sixty-two studies that used DEA to measure the technical efficiency of banks and other depository financial institutions. However, input allocative inefficiency is less frequently investigated in the DEA framework. Aly *et al.* [1] provide one such example for U.S. banks. One problem with existing measures of input allocative efficiency derived as the ratio of overall cost efficiency to Farrell input technical efficiency, is due to the possible existence of slack in the constraints that define the reference technology.

An alternative to the Farrell measure of input technical efficiency is the non radial Russell input technical efficiency measure introduced by Färe and Lovell [7] and extended by Färe *et al.* [8]. However, we show that if one attempts to define input allocative efficiency as the ratio of cost efficiency to Russell input technical efficiency, the measure does not have the standard interpretation of the Farrell input allocative efficiency.

In the DEA literature of efficiency measurement, there are several alternatives to the Farrell measure of input allocative efficiency. One alternative is that of Tone [20] who constructs measures of technical and allocative efficiency based on spending on inputs, rather than input quantities and input prices separately. Tone's technique accounts for the possibility that not only do firms sometimes use too much input to produce output, but also sometimes choose input prices that are too high relative to what could actually be obtained. A second alternative, introduced by Grosskopf *et al.* [12], uses Shephard's [17, 18] direct and indirect output distance functions-they use output as opposed to the input based efficiency measures-building on Shephard's duality theory. Since the direct output distance function is no less than the indirect output distance function, Grosskopf *et al.* use the ratio of the indirect to the direct output distance function to measure the gain in output due to an optimal reallocation of inputs. They use the gain measure to analyze the potential consequences of school district decentralization and deregulation in the United States. More recently, Fukuyama *et al.* [10] have built on Grosskopf *et al.*'s measure and derived an indirect Farrell-based input allocative efficiency change index as one of the components of a Malmquist productivity index with an application to Japanese credit cooperatives. Related to the indirect measures of efficiency is a study by Shaffer [16]. Shaffer estimated a cost function for U.S. banks taking revenues as the exogenous variable, rather than output as is common for cost functions. Using a "revenue indirect" cost function Shaffer found scale economies existing over a wider range of bank assets than previous researchers had found. Subsequent research validated Shaffer's findings and suggests that "indirect" measures of efficiency might yield useful information.

While the indirect Farrell input allocative efficiency measure is valid, it might also be biased when estimated via DEA because of slack in the constraints that define the reference technology. The use of direct and indirect dualities, though, is interesting itself, and has important extensions. Taking the limitations of Grosskopf *et al.*'s measure into account and building on the standard duality results established by Shephard [17, 18] and Färe and Primont [9] we introduce a new measure of input allocative efficiency which does not leave any output slack in linear programming problems. We accomplish this task by defining direct and indirect quasi-distance functions. We call our indicator the indirect Russell input allocative efficiency measure, since quasi-distance functions are reciprocals of the Russell efficiency measures.

Thus, the purposes of this paper are twofold. First, we develop a new methodology for computing input allocative efficiency and compare it to existing measures. Second, we offer an empirical illustration of the new and existing methods of measuring input allocative efficiency for a sample of U.S. banks. In the next two sections the new methodology for measuring input allocative efficiency using DEA is developed and compared with three existing DEA measures of input allocative efficiency. The third section provides an empirical example for U.S. banks operating in 1994 and 1995. For completeness, we also estimate Tone's [20] measures of overall cost efficiency and its decomposition into technical and allocative efficiency. A discussion of the empirical results of the four measures of input allocative efficiency is followed by a concluding section.

2. General Production Framework

The production theoretical framework of Shephard [17, 18] forms the basis for our model. Let $y \in R_+^M$ and $x \in R_+^N$ denote vectors of M outputs and N inputs. The production technology is represented by the input requirement set:

$$L(y) = \{x : y \text{ can be produced from } x\}. \tag{1}$$

The graph of $L(y)$ is defined as $G = \{(y, x) : x \in L(y)\}$. The technology is assumed to satisfy the following conditions:

(a) L-Regular
  (i) inaction is possible,
  (ii) there is no free lunch, and
  (iii) G is closed. (2)
(b) Technology Convexity; G is convex.
(c) Strong Input and Output Disposability:
  If $(y, x) \in G$ and $(-y, x) \leq (-y', x')$, then $(y', x') \in G$.

Relative to $L(y)$, the (direct) Farrell input technical efficiency measure[1], $F_i(y,x)$, is defined as:

$$F_i(y,x) = \min\{\theta : \theta x \in L(y)\}. \tag{3}$$

The reciprocal of the direct Farrell input technical efficiency measure is equal to the input distance function. The properties of $F_i(y,x)$ are:

$F_i.1$ $F_i(y,x)$ is non decreasing in $y > 0$,
$F_i.2$ $F_i(y,x)$ is non increasing in $x > 0$,
$F_i.3$ $F_i(y,x)$ is homogeneous of degree minus one in $x > 0$.

The direct Farrell input efficiency measure (3) has a radial efficiency interpretation because it is defined as the largest proportional contraction of inputs given outputs. As such, it gauges the amount of excess input usage relative to the input isoquant. For semi-positive outputs, the input isoquant is denoted by:

$$IsoqL(y) = \{x : x \in L(y) \text{ and } \theta x \notin L(y),\ \theta \in (0,1)\}. \tag{4}$$

The direct Farrell input efficiency measure and the input isoquant are related as:

$$F_i(y,x) = 1 \text{ if and only if } x \in L(y), \quad \text{or, } IsoqL(y) = \{x : F_i(y,x) = 1\}. \tag{5}$$

In addition, $F_i(y,x)$ is no greater than one if the input vector x is an element of $L(y)$.

If $r \otimes x$ is denoted by $(r_1x_1, \cdots, r_Nx_N)$, the (direct) Russell input technical efficiency measure, $R_i(y,x)$ is defined as:

$$R_i(y,x) = \min\left\{\frac{\sum_{n=1}^{N} r_n}{N} : r \otimes x \in L(y), r_n \geq 0,\ n = 1, \cdots, N\right\}. \tag{6}$$

The expression in (6) is presented by Färe *et al.* [8][2]. For $x \in L(y)$, $R_i(y,x) \leq 1$. In addition, if $r_1 = r_2 = \cdots = r_N > 0$ in equation (6), then $R_i(y,x)$ is equal to $F_i(y,x)$. Additional properties on $R_i(y,x)$ include:

$R_i.1$ $R_i(y,x)$ is non decreasing in $y > 0$,
$R_i.2$ $R_i(y,x)$ is decreasing[3] in $x > 0$,
$R_i.3$ $R_i(y,x)$ is homogeneous of degree minus one in $x > 0$.

For $x \in L(y)$, the Farrell and Russell measures of input technical efficiency satisfy:

$$1 \geq F_i(y,x) \geq R_i(y,x). \tag{7}$$

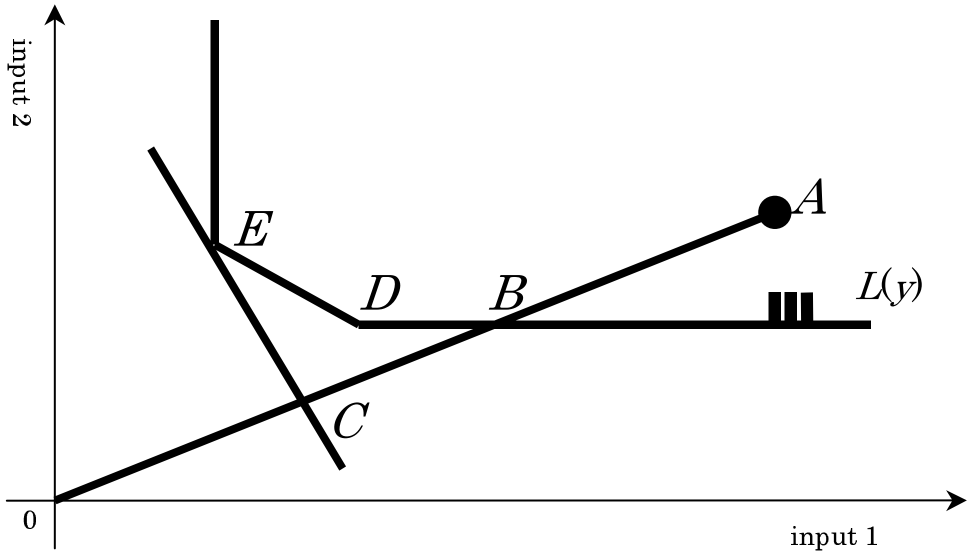
The cost function is defined for positive input prices, w, as:

$$C(y,w) = \min\{wx : x \in L(y)\}. \tag{8}$$

Based on the cost function defined by (8) and the direct Farrell input technical efficiency measure given by (3), the (direct) Farrell input allocative efficiency measure, AE^F, is derived as:

$$AE^F = \frac{C(y,w)}{wx} \times \frac{1}{F_i(y,x)}. \tag{9}$$

In Figure 1 the direct Farrell input technical efficiency measure for the firm represented by A is equal to $0B/0A$ and overall cost efficiency is $0C/0A$. If input allocative efficiency is computed by (9) then $AE^F = 0C/0B$. However, the magnitude of AE^F includes the horizontal segment of the isoquant DB, and thus we get a biased estimate.



Cost Efficiency=$\frac{0C}{0A}$, $F_i(y,x) = \frac{0B}{0A}$, $AE^F = \frac{0C}{0B}$

Figure 1: Components of cost efficiency

In order to avoid this kind of bias, Färe *et al.* [8] introduce the (direct) Russell input allocative efficiency measure, AE^R, defined as:

$$AE^R = \frac{C(y,w)}{wx} \times \frac{1}{R_i(y,x)}. \tag{10}$$

Unfortunately, AE^R does not have the standard interpretation of the Farrell input allocative efficiency measure given in (9). One can see this numerically in view of Table 1 and Figure 2 which depicts the observations from Table 1. For observation A_1, $R_i(y,x) = (1/2+4/5)/2 = 13/20 = 0.65$. Since $(w_1, w_2) = (0.1, 2)$, the minimum cost for A_1 is $C(y,w) = 0.1\times2+2\times4 = 8.2$ and overall cost efficiency equal $0.79 = (8.2/10.4)$. Clearly, cost efficiency is greater than Russell input technical efficiency for the first observation[4]. Therefore, AE^R can be greater than one, unlike the Farrell measure, AE^F, which is always no greater than one. This result indicates that the excess cost caused by input allocative inefficiency cannot be gauged by the value $(1 - AE^R)$.

Table 1: An example

Obs.	y_1	x_1	x_2	w_1	w_2	$wx = c$	$R_i(y,x)$	$C(y,w)/wx$
A_1	1	4	5	0.1	2	10.4	0.65	0.79(= 8.2/10.4)
A_2	1	2	4	0.1	2	8.2	1	1

Are there any other ways to construct reasonable input allocative efficiency indicators? Recently, Grosskopf *et al.* [12] constructed an indicator by exploiting Shephard's [17, 18] direct and indirect output distance functions. To show this, define the output set as:

$$P(x) = \{y : x \in L(y)\}, \tag{11}$$

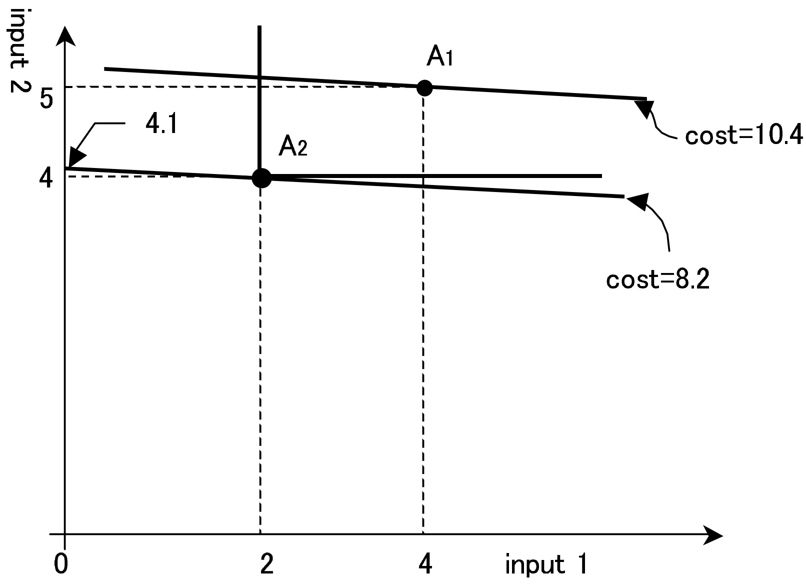


Figure 2: Example from Table 1

which is derived from the inverse relation between the input and the output set. Relative to $P(x)$, the (direct) output distance function[5], $D_o(x, y)$, can be defined by:

$$\frac{1}{D_o(x, y)} = \max\{\varphi : \varphi y \in P(x)\}. \tag{12}$$

The reciprocal of the output distance function gives the proportional (radial) output expansion for a given level of input. The properties of $D_o(x, y)$ include:

D_o.1 $D_o(x, y)$ is non increasing in $x > 0$,
D_o.2 $D_o(x, y)$ is non decreasing in $y > 0$,
D_o.3 $D_o(x, y)$ is homogeneous of degree one in $y > 0$,
D_o.4 $D_o(x, y)$ is convex in $y > 0$.

Denoting c as the budget available to hire inputs and following Shephard [18], the indirect output set is defined as:

$$IP(w/c) = \{y : y \in P(x), (w/c)x \leq 1\}. \tag{13}$$

The set $IP(w/c)$ gives the set of all outputs that can be produced from inputs at a cost less than or equal to some prescribed cost. Relative to $IP(w/c)$, the indirect output distance function is:

$$\frac{1}{ID_o(w/c, y)} = \max\{\delta : \delta y \in IP(w/c)\} = \max\{\delta : C(\delta y, w/c) \leq 1\}. \tag{14}$$

The properties on $ID_o(w/c, y)$ include:

ID_o.1 $ID_o(w/c, y)$ is non decreasing in $(w/c) > 0$,
ID_o.2 $ID_o(w/c, y)$ is non decreasing in $y > 0$,
ID_o.3 $ID_o(w/c, y)$ is homogeneous of degree one in $y > 0$,
ID_o.4 $ID_o(w/c, y)$ is convex in $y > 0$.

For proofs, see Shephard (1974, pp. 63–66). The indirect Farrell input allocative efficiency measure is:

$$IAE^F = \frac{ID_o(w/c, y)}{D_o(x, y)} \leq 1. \tag{15}$$

The measure, IAE^F, takes a value no greater than one and its reciprocal gives the proportional output expansion that would be feasible if the firm operated at the point associated with the cost-minimizing input vector. While Grosskopf *et al.* [12] consider (15) as a gain under budget constraint, Fukuyama *et al.* [10] interpret it as an indicator of input allocative efficiency as well.

Although values of AE^F and IAE^F are in general different, it is of great importance to examine when the two are the same. For this purpose, suppose a production technology exhibits constant returns to scale, i.e., the graph of the input requirement set can be characterized by $G = \mu G,\ \mu > 0$. In this case, the direct Farrell input technical efficiency is equal to the direct output distance function, i.e.

$$F_i(y, x) = D_o(x, y). \tag{16}$$

Appealing to this well-known result, (16), and substituting into the definition of the cost function, we can obtain the following characterization:

$$C(y, w) = \mu C(y/\mu, w), \tag{17}$$

for which the indirect output distance function becomes

$$ID_o(w/c, y) = \min\{\varphi : C(y/\varphi, w/c) \leq 1\} = C(y, w/c) \tag{18}$$

where the second equality is due to (17). For (18), see Färe and Primont [9]. By virtue of equations (16) and (18), the indirect input allocative efficiency becomes

$$IAE^F = \frac{ID_o(w/c, y)}{D_o(x, y)} = \frac{C(y, w)}{wx} \times \frac{1}{F_i(y, x)} \tag{19}$$

where we set $c = wx$. Since the right hand side of (19) is equal to AE^F, we have

$$AE^F = IAE^F. \tag{20}$$

One problem with (15) as a measure of input allocative efficiency though, is the potential for slack in the output constraints associated with $D_o(x, y)$. To alleviate the problem of slack associated with (15) we construct a new measure of input allocative efficiency. The measure relies on output quasi-distance functions which are reciprocals of two output-based Russell efficiency measures estimated for the $P(x)$ and $IP(w/c)$ technologies[6]. The first is the direct output quasi-distance function[7]:

$$\frac{1}{Q_o(x, y)} = \max\left\{\frac{\sum_{m=1}^{M} s_m}{M} : s \otimes y \in P(x), s_m \geq 0, m = 1, \cdots, M\right\} \tag{21}$$

for positive outputs. The measure defined in (21) is the reciprocal of the direct Russell output technical efficiency measure[8] defined by Färe *et al.* [8].

The second Russell output technical efficiency measure is defined as an indirect output quasi-distance function:

$$\begin{aligned}\frac{1}{IQ_o(w/c,y)} &= \max\left\{\frac{\sum_{m=1}^{M} q_m}{M} : q \otimes y \in IP(w/c), q_m \geq 0, m = 1, \cdots, M\right\} \\ &= \max\left\{\frac{\sum_{m=1}^{M} q_m}{M} : C(q \otimes y, w/c) \leq 1, q_m \geq 0, m = 1, \cdots, M\right\}\end{aligned} \tag{22}$$

for positive outputs. The indirect output quasi-distance function extends the indirect output distance function and the quasi-distance function defined by (21).

If $y \in P(x)$, the properties on $Q_o(x, y)$ include:

- Q_o.1 $Q_o(x, y)$ is non increasing in $x > 0$,
- Q_o.2 $Q_o(x, y)$ is increasing in $y > 0$,
- Q_o.3 $Q_o(x, y)$ is homogeneous of degree one in $y > 0$,
- Q_o.4 $Q_o(x, y)$ is convex in $y > 0$.

Similarly, if $y \in IP(w/c)$, the properties on $IQ_o(w/c, y)$ include:

- IQ_o.1 $IQ_o(w/c, y)$ is non decreasing in $(w/c) > 0$,
- IQ_o.2 $IQ_o(w/c, y)$ is increasing in $y > 0$,
- IQ_o.3 $IQ_o(w/c, y)$ is homogeneous of degree one in $y > 0$,
- IQ_o.4 $IQ_o(w/c, y)$ is convex in $y > 0$.

The proofs of the properties on $Q_o(x, y)$ and $IQ_o(w/c, y)$ are given in Appendix A-1 and Appendix A-2.

Taking the ratio of $Q_o(x, y)$ and $IQ_o(w/c, y)$, we obtain the indirect Russell input allocative efficiency measure:

$$IAE^R = \frac{IQ_o(w/c, y)}{Q_o(x, y)} \leq 1. \tag{23}$$

The reciprocal of the indirect Russell input allocative efficiency measure provides the asymmetric output expansion that would be feasible if the firm operated at the point consistent with cost minimization. The output quasi-distance function and the indirect output quasi-distance function diverge when a firm employs a non cost minimizing input vector. In order to show that the definition (23) is a reasonable indicator[9], we can show that

$$Q_o(x, y) = \max\{IQ_o(w/c, y) : (w/c)x \leq 1\} \tag{24}$$

for $x \in L(y)$. From the definition of $Q_o(x, y)$ and $IQ_o(w/c, y)$ and the relation between $P(x)$ and $IP(w/c)$, we have

$$Q_o(x, y) \geq IQ_o(w/c, y) \tag{25}$$

for $(w/c)x \leq 1$. Hence, the left-hand side is greater than or equal to the right-hand side in (24).

To show "$\leq$", start by assuming the contrary, i.e., $Q_o(x, y) > \max\{IQ_o(w/c, y) : (w/c)x \leq 1\}$. If the right-hand side is k^*, then we obtain

$$Q_o(x, y/k^*) > \max\{IQ_o(w/c, y/k^*) : (w/c)x \leq 1\} = 1 \tag{26}$$

due to the homogeneity of $Q_o(x, y)$ and $IQ_o(w/c, y)$ in outputs. Consequently, we have

$$IQ_o(w/c, y/k^*) \leq 1 \quad \text{and} \quad C(y/k^*, w/c) \leq 1. \tag{27}$$

The statement (26) also means $Q_o(x, y/k^*) > 1$, which in turn implies $x \notin L(y/k^*)$. The implication of the strong hyper-plane theorem is that $w^o x < C(y/k^*, w^o)$ for some w^o. Given the identity $1 = w^o x / w^o x$ and linear homogeneity of the cost function in input prices, we obtain $1 < C(y/k^*, w^o/c^o)$ where $c^o = w^o x$, contradicting (27). It follows that IAE^R is well defined.

To summarize the differences between the indirect Farrell and indirect Russell measures of input allocative efficiency we distinguish between the output isoquant and output efficient subset and show that the Farrell measure expands outputs to the output isoquant, while the output quasi-distance functions expand outputs to the direct and indirect Russell output efficient subsets of the output isoquant. The output isoquant is defined as:

$$ISOQ\ P(x) = \{y : y \in P(x), \lambda y \notin P(x), \text{ for } \lambda > 1\}. \tag{28}$$

Similarly, the output isoquant for the indirect output set is given as:

$$ISOQ\ IP(w/c) = \{y : y \in IP(w/c), \theta y \notin IP(w/c) \text{ for } \theta > 1\}. \tag{29}$$

The efficient subsets of $ISOQ\ P(x)$ and $ISOQ\ IP(w/c)$ are defined as:

$$Eff\, P(x) = \{y \in P(x) : y' \geq y, y' \neq y, y' \notin P(x)\} \tag{30}$$

and

$$Eff\, IP(w/c) = \{y \in IP(w/c) : y' \geq y, y' \neq y, y' \notin IP(w/c)\}. \tag{31}$$

We illustrate these two isoquants and the two efficient subsets in Figure 3. We observe five firms each producing different combinations of two outputs. We assume that all five firms face the same input prices, w, and have the same budget, c, with which to hire inputs. Firms A, B, and C, all use inputs x', while firm D uses inputs x'' and firm E uses inputs x''', but $wx' = wx'' = wx''' = c$. For a piecewise linear DEA technology (which we describe in greater detail in the next section) the direct output set, $P(x')$, for firms A, B, and C, is the set of outputs bounded by {0JBCH0}. The output isoquant, $ISOQ\ P(x')$, equals the line segments JB, BC, and CH. The output efficient subset, $Eff\, P(x')$, for firms A, B, and C is the line segment BC. Similarly, $IP(w/c)$ is the set of outputs bounded by {0KDBEI0} and $Eff\, IP(w/c)$ =DBE. For firm A, $D_o(x', y) = 0A/0F$ and $ID_o(w/c, y) = 0A/0G$. Clearly the direct and indirect output distance functions expand observed output to the respective output isoquants, but not to the efficient subsets of $P(x)$ and $IP(w/c)$, since both the direct and indirect output distance functions leave slack in the constraints that define $P(x)$ and $IP(w/c)$. Once firm A's outputs have been radially expanded to point F on $ISOQ\ P(x')$, output 1 could be further expanded (in a non radial manner) to point C. Similarly, once firm A's outputs have been radially expanded to point G on $ISOQ\ IP(w/c)$, output 1 could be expanded to point E at no opportunity cost.

Given the potential existence of slack in the constraints that define $ISOQ\ P(x)$ and $ISOQ\ IP(w/c)$ it is desirable to compare actual outputs to a subset of the output isoquants for the respective sets. Rather than compare outputs to the efficient subsets, our output quasi-distance functions compare observed outputs to the direct Russell efficient subset and the indirect Russell efficient subset. The direct Russell efficient subset is defined as

$$\begin{aligned} R - Eff\, P(x) = \ & \{y : y \in P(x), (s_1 y_1, \cdots, s_M y_M) \notin P(x), \\ & \text{for } (s_1 + \cdots + s_M) > M, \text{ all } s_m \geq 0\}. \end{aligned} \tag{32}$$

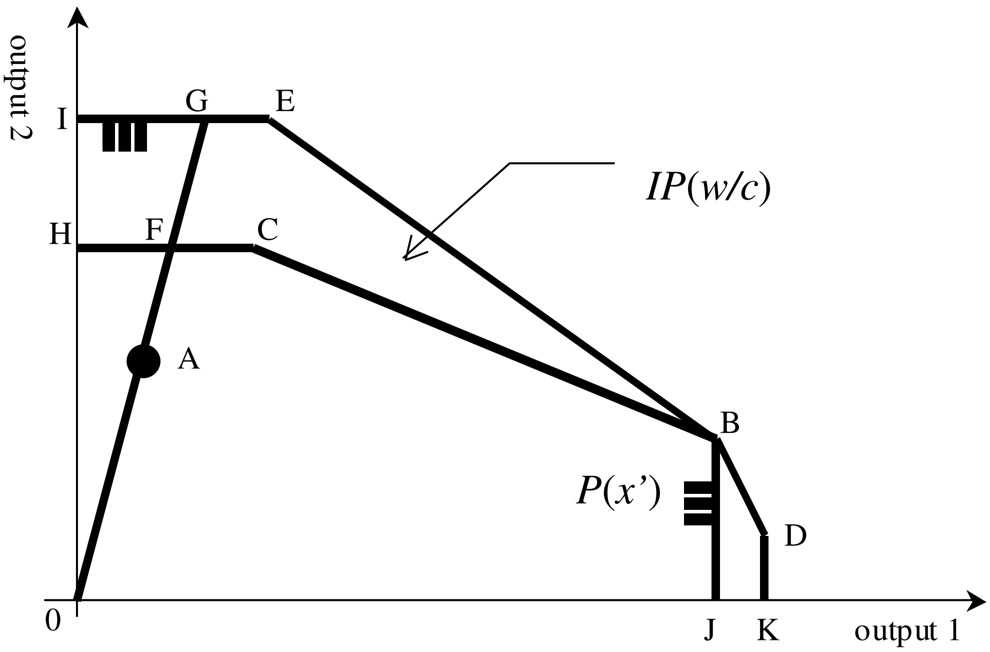


Figure 3: Illustration of output distance and quasi-distance functions

Similarly, the indirect Russell efficient subset is defined as

$$R - Eff\, IP(w/c) = \{y : y \in IP(w/c), (q_1 y_1, \cdots, q_M y_M) \notin IP(w/c), \text{ for } (q_1 + \cdots + q_M) > M, \text{ all } q_m \geq 0\}. \quad (33)$$

The direct and indirect output quasi-distance functions take values of one only when the actual output of a firm is part of the direct or indirect Russell efficient subsets. We can summarize these differences in the distance functions and the quasi-distance functions as follows:

$$D_o(x,y) = 1 \Leftrightarrow y \in ISOQ\ P(x) \text{ and } Q_o(x,y) = 1 \Leftrightarrow y \in R - Eff\, P(x). \quad (34)$$

Similarly,

$$ID_o(w/c,y) = 1 \Leftrightarrow y \in ISOQ\ IP(w/c) \text{ and } IQ_o(w/c,y) = 1 \Leftrightarrow y \in R - Eff\, IP(w/c). \quad (35)$$

In general, $R - Eff\, P(x) \subseteq Eff\, P(x) \subseteq ISOQ\, P(x)$ and $R - Eff\, IP(w/c) \subseteq Eff\, IP(w/c) \subseteq ISOQ\ IP(w/c)$. For example, in Figure 3, suppose that the output of firm B is $y^{\mathrm{B}} = (3,1)$ and the output of firm C is $y^{\mathrm{C}} = (1,2)$. Firms B and C both use inputs x' to produce their output and

$$ISOQ\ P(x) = \{y : y_1 = 3 \text{ for } 0 \leq y_2 \leq 1, y_2 = 2.5 - 0.5y_1 \text{ for } 1 \leq y_1 \leq 2, \text{ and } y_2 = 2 \text{ for } 0 \leq y_1 \leq 1\},$$

$$Eff\, P(x) = \{y : y_2 = 2.5 - 0.5y_1, \text{ and } 1 \leq y_1 \leq 3\}, \text{ and}$$

$$R - Eff\, P(x) = \{y : y_1 = 2.5, \text{ and } y_2 = 1.25\}.$$

The isoquant, efficient subset, and the Russell efficient subset for the indirect output set can be constructed similarly.

3. DEA Framework

The development and major contributions of DEA were made by Farrell [6], Charnes *et al.* [4], Färe *et al.* [8] and Cooper *et al.* [5]. In the empirical section we consider a variable returns to scale technology with a single fixed input. The fixed input helps account for the risk-return profile of equity holders. Let J represent the number of decision making units, let Y represent the $(M \times J)$ matrix of observed outputs, let X represent the $(N \times J)$ matrix of observed variable inputs, let Z represent the $(V \times J)$ matrix of observed fixed inputs, and let λ represent a J-dimensional vector of intensity variables. For a DEA-based production technology the input requirement set, $L(y)$, the direct production possibility set, $P(x, z)$, and the indirect production possibility set, $IP(w/c, z)$ are represented by:

$$L(y) = \{(x, z) : Y\lambda \geq y, X\lambda \leq x, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0\}, \tag{36a}$$

$$P(x, z) = \{y : Y\lambda \geq y, X\lambda \leq x, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0\}, \text{ and} \tag{36b}$$

$$IP(w/c, z) = \{y : Y\lambda \geq y, X\lambda \leq x, Z\lambda \leq z, (w/c)x \leq 1, e\lambda = 1, \lambda \geq 0\} \tag{36c}$$

where e is a J-dimensional vector of ones and thus $e\lambda$ is the sum of the intensity variables. The restriction, $e\lambda = 1$, constrains the technology to variable returns to scale. The system of inequalities, $Y\lambda \geq y, X\lambda \leq x$, and $Z\lambda \leq z$, allows strong output and strong input disposability. In addition, the DEA technology satisfies convexity. In the previous section efficiency indicators were described as distance and quasi-distance functions. In order to estimate the indicators we solve the following linear programming problems:

$$F_i(y, x, z) = \min\{\theta : Y\lambda \geq y, X\lambda \leq \theta x, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0\}, \tag{37a}$$

$$C(y, w, z) = \min\{wx : Y\lambda \geq y, X\lambda \leq x, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0\}. \tag{37b}$$

$$1/D_o(x, y, z) = \max\{\varphi : Y\lambda \geq \varphi y, X\lambda \leq x, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0\}, \tag{38a}$$

$$1/ID_o(w/c, y, z) = \max\{\delta : Y\lambda \geq \delta y, X\lambda \leq x, Z\lambda \leq z, (w/c)x \leq 1, e\lambda = 1, \lambda \geq 0\}, \tag{38b}$$

$$1/Q_o(x, y, z) = \max\left\{\sum_{m=1}^{M} \frac{s_m}{M} : Y\lambda \geq s \otimes y, X\lambda \leq x, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0, s \geq 0\right\}, \tag{38c}$$

$$1/IQ_o(w/c, y, z) = \max\left\{\sum_{m=1}^{M} \frac{q_m}{M} : Y\lambda \geq q \otimes y, X\lambda \leq x, Z\lambda \leq z, (w/c)x \leq 1, e\lambda = 1, \lambda \geq 0, q \geq 0\right\}. \tag{38d}$$

It is important to note that in problems (37a), (38a), and (38c) that x is taken as given in determining the technology while in problems (37b) x is chosen to minimize costs and in (38b) and (38d) x is chosen to maximize output as long as it satisfies the budget constraint. Based on the linear programming problems presented in (37) and (38) estimates of AE^F, IAE^F, and IAE^R are computed.

4. An Empirical Example

Data from the U.S. banking industry are used to estimate the measures of input allocative efficiency described in the section above. The data were derived from a random sample of 1000 U.S. banks operating in 1994 taken from the June 1994 Federal Financial Institutions Examination Council Report of Income and Condition (Call Reports) and 921 banks in 1995 that were still in existence and reported complete data on the June 1995 Call Reports. We follow the user cost approach of Hancock [13] and assume that banks produce five outputs using three variable inputs and one fixed input. The bank outputs included the assets of securities investments (y_1), real estate loans (y_2), commercial and industrial loans (y_3), personal loans (y_4), and the liability of transaction account deposits (y_5). The three variable inputs included labor (x_1), physical capital of premises and fixed assets (x_2), and non transaction account deposits (x_3).

The Call Reports for each year require banks to report expenses on wages and salaries, expenses on premises and fixed assets and interest expenses on deposits. We follow the approach of most bank efficiency studies and construct the various input prices as the ratio of expenses to the quantity of input employed. The price of labor (w_1) equals bank salaries and wages divided by the number of full-time employees. The price of capital (w_2) equals bank expenses on premises and fixed assets divided by the value of physical capital. The price of non transaction account deposits (w_3) equals interest expenses on deposits divided by non transaction account deposits. Following Hughes and Mester [14], the risk-return tradeoff in banking is accounted for by treating bank equity capital (z_1) as a fixed input when solving the sequence of linear programming problems described in equations (29) and (30). Descriptive statistics are provided in Table 2.

As seen in the table, there is significant variation in the input prices. A problem with our method of measuring efficiency arises if all banks purchase inputs in the same competitive input market[10]. For instance, consider two banks, A and B, that use the same inputs to produce the same outputs. That is, $x^{\mathrm{A}} = x^{\mathrm{B}}$ and $y^{\mathrm{A}} = y^{\mathrm{B}}$. Suppose bank A has costs that are twice as high as those of bank B because bank A is paying input prices twice as high as those needed to hire the inputs. That is $c^{\mathrm{A}} = 2c^{\mathrm{B}}$ because bank A has $w^{\mathrm{A}} = 2w^{\mathrm{B}}$. If we use the different input price vectors to estimate the cost function, then bank A will have the same cost efficiency and input allocative efficiency as bank B, even though it is less cost efficient. Tone [20] pointed out this problem as a "strange case" of the Farrell cost and input allocative efficiency measures. Instead of the Farrell technical and allocative efficiency measures that we describe in equations (3) and (9), Tone [20] proposes a new method of estimating technical, allocative, and overall cost efficiency, based on spending, rather than actual quantities.

Following Tone, let the spending on input n by firm k be denoted $\overline{x}_n^k = w_n^k x_n^k$. Once again, letting y represent output, z represent input, and e represent a vector of ones, Tone's cost-based output possibility set is[11]: $P^c(\overline{x}, z) = \{(\overline{x}, z, y) : \overline{X}\lambda \leq \overline{x}, Y\lambda \geq y, Z\lambda \leq z, e\lambda = 1, \lambda \geq 0\}$. From $P^c(\overline{x}, z)$ Tone formulates a technical (spending) efficiency measure for firm o as:

$$\overline{\theta}_o^* = \min\{\overline{\theta} : \overline{X}\lambda \leq \overline{\theta}\overline{x}_o, Y\lambda \geq y_o, Z\lambda \leq z_o, e\lambda = 1, \lambda \geq 0\}. \tag{39}$$

Tone also formulates an overall cost efficiency measure, $\overline{\gamma}^*$, as $\overline{\gamma}_o^* = e\overline{x}^*/e\overline{x}_o$, where $\overline{x}^*$ is found as the solution to

$$\overline{\gamma}_o^* = \min\{e\overline{x} : \overline{X}\lambda \leq \overline{x}, Y\lambda \geq y_o, Z\lambda \leq z_o, e\lambda = 1, \lambda \geq 0\}. \tag{40}$$

Finally, Tone's allocative efficiency measure, $\overline{\alpha}^*$, is estimated as $\overline{\alpha}_o^* = \overline{\gamma}_o^* \,/\, \overline{\theta}_o^*$.

To illustrate Tone's technique, let us return to our example in Table 1. We again assume that both firms have the input quantities as listed in the table and that firm A_2 still has input prices $w_1 = 0.1$ and $w_2 = 2$, but that firm A_1 faces prices $w_1 = 0.1$ and $w_2 = 1$. In this case, firm A_1 has overall costs of only 5.4 and is more overall cost efficient than firm A_2. For firm A_1, $\overline{\gamma}^* = 1 = 5.4/5.4$, $\overline{\theta}^* = 1$, and $\overline{\alpha}^* = 1$, but for firm A_2, $\overline{\gamma}^* = 0.66 = 5.4/8.2$, $\overline{\theta}^* = 1$, and $\overline{\alpha}^* = 0.66$. However, if the Farrell decomposition is used, then firm A_2 has overall cost efficiency of $C(y, w) \,/\, wx = 1$, input technical efficiency of $F_i(y, x) = 1$, and input

Table 2: Descriptive statistics (means and standard deviations)

Variable	1994	1995
Outputs		
y_1 = securities	110644.79	171327.67
	(791186.04)	(1617037.53)
y_2 = real estate loans	151379.99	71988.46
	(1464703.26)	(621434.95)
y_3 = commercial loans	61607.34	66167.66
	(539135.01)	(595892.33)
y_4 = personal loans	59880.37	105908.65
	(523499.50)	(872225.59)
y_5 = transaction deposits	128860.13	133033.57
	(12407222.22)	(1277517.63)
Variable Inputs		
x_1 = labor	231.06	219.52
	(2341.71)	(1971.10)
x_2 = capital	8660.31	9390.12
	(96955.30)	(102427.3)
x_3 = non-transaction deposits	236891.81	248296.70
	(1885866.53)	(1821344.42)
Variable input prices		
w_1	32.249	33.527
	(8.446)	(7.816)
w_2	0.367	0.358
	(0.365)	(0.347)
w_3	0.035	0.044
	(0.007)	(0.005)
Fixed input		
z_1 = equity	46242.80	49513.81
	(445723.65)	(466942.30)
wx = actual costs	18994.38	21646.66
	(177087.64)	(182243.56)

allocative efficiency of $IAE^F = 1$, while firm A_1 has $C(y,w)\,/\,wx = 0.78$, $F_i(y,x) = 0.8$, and $IAE^F = 0.97$. That firm A_1 has lower actual costs than firm A_2, but lower cost efficiency and input allocative efficiency are the "strange case" pointed out by Tone. Crucially then, the appeal of Tone's method over the traditional Farrell method of measuring cost and input allocative efficiencies depends on the extent to which firms have control over input prices. If local market conditions, government regulations, or union contracts dictate what input prices must be paid, then Tone's measures are likely to be biased. However, if all firms have the opportunity to choose the same input prices as would be consistent with national or global markets, but sometimes mistakenly or purposely pay too much, Tone's measures of efficiency would be preferred to the traditional Farrell measures, or the quasi-measures that we propose.

Table 3 provides the means and standard deviations of technical efficiency and input allocative efficiency and the solution values to the cost function, x_n^*s, the solution values

Table 3: Mean levels of efficiency (standard deviations)

Variable	1994	1995
Farrell Input Efficiency Measures		
$F_i(y,x,z)$	0.784	0.802
	(0.173)	(0.166)
$C(y,w,z)$	17465.15	21567.59
	(179975.2)	(215982.22)
${x_1}^*$	206.52	202.25
	(2340.40)	(1969.37)
${x_2}^*$	7574.42	8277.82
	(96861.54)	(102386.58)
${x_3}^*$	214674.7	222912.15
	(1884809.35)	(1815924.16)
$w_1{x_1}^*$	8401.32	9475.36
	(99430.08)	(112092.10)
$w_2{x_2}^*$	2784.26	2975.50
	(37659.70)	(39003.37)
$w_3{x_3}^*$	6293.35	9116.71
	(43619.05)	(66389.24)
AE^F	0.92	0.93
	(0.10)	(0.09)
Tone's Efficiency Measures		
Technical efficiency $\overline{\theta}^*$	0.78	0.82
	(0.19)	(0.17)
Allocative efficiency $\overline{\alpha}^*$	0.90	0.92
	(0.14)	(0.11)
Overall cost efficiency $\overline{\gamma}^*$	0.70	0.75
	(0.19)	(0.17)
$\overline{x_1}^*$	8371.55	9519.52
	(99462.10)	(112068.00)
$\overline{x_2}^*$	2859.29	3042.98
	(37679.67)	(38957.77)
$\overline{x_3}^*$	5196.46	9059.04
	(30258.03)	(66458.88)
Farrell Output Efficiency Measures		
$1/D_o(x,y,z)$	1.243	1.211
	(0.242)	(0.224)
$1/ID_o(w/c,y,z)$	1.342	1.283
	(0.236)	(0.953)
IAE^F	0.931	0.953
	(0.085)	(0.192)
Russell Output Efficiency Measures		
$1/Q_o(x,y,z)$	4.223	5.796
	(9.061)	(36.762)
${s_1}^*$	2.298	1.453
	(15.977)	(18.072)

Table 3 (continued)

Variable	1994	1995
${s_2}^*$	1.317	13.044
	(10.591)	(174.534)
${s_3}^*$	8.999	11.221
	(38.515)	(54.241)
${s_4}^*$	7.152	2.060
	(17.156)	(19.870)
${s_5}^*$	1.384	1.200
	(0.862)	(0.735)
$1/IQ_o(w/c, y, z)$	4.773	6.717
	(10.349)	(39.764)
${q_1}^*$	2.712	1.535
	(21.497)	(21.433)
${q_2}^*$	1.102	13.539
	(4.814)	(185.920)
${q_3}^*$	9.343	14.075
	(41.294)	(60.804)
${q_4}^*$	9.208	3.123
	(23.845)	(35.974)
${q_5}^*$	1.500	1.315
	(1.030)	(0.799)
IAE^R	0.919	0.919
	(0.110)	(0.124)

to the direct output quasi-distance function, s_m^*s, and the solution values to the indirect output quasi-distance function, q_m^*s. Direct Farrell input technical efficiency is 0.78 in 1994 and 0.80 in 1995. The solution values to the cost function indicate that in 1994 the average bank should employ 206 workers, invest \$7.574 million in physical capital in premises and fixed assets and use \$214.674 million in non-transaction account deposits in order to minimize costs. By 1995 the average bank would employ fewer workers, use more physical capital and more deposits to minimize variable costs of production, although the optimal input bundle was not significantly different from 1994. The direct Farrell input allocative efficiency measure, AE^F, increased from 0.92 to 0.93 during the period. Tone's efficiency measures indicate averages of technical efficiency and allocative efficiency in both years that are similar to the Farrell measures. A comparison of the optimal spending pattern for the Farrell measures ($w_k x_k^*$) and the Tone measures ($\overline{x}_k^*$) indicate no significant difference (based on a t-test) in spending on labor and deposits in both years, but significantly higher spending on physical capital for the Tone measure relative to the Farrell measure in both years.

The reciprocal of the direct output distance function indicates that output could expand by approximately 24% in 1994 and 21% in 1995 if the average bank achieved the output technical efficiency of the best practice bank. The reciprocal of the indirect output distance function indicates that the average bank could expand output approximately 34% in 1994 and 28% in 1995 if it were to reallocate input, holding the overall input budget constant, and achieve the efficiency of the best practice bank. The indirect Farrell input allocative efficiency measure, IAE^F, averaged 0.93 in 1994 and increased to 0.95 in 1995. These indirect measures of input allocative efficiency suggest that an optimal reallocation of inputs, holding the budget with which to hire input constant, could result in a 7% expansion of output in 1994 and a 5% expansion of output in 1995.

The direct and indirect quasi-distance functions show a higher level of output expansion in asymmetric directions. For example, in 1994 greater technical efficiency would have allowed securities to be expanded by a factor of 2.30, real estate loans to be expanded by a factor of 1.32, commercial loans to be expanded by a factor of 9.00, personal loans to be expanded by a factor of 7.16, and transaction deposits to be expanded by a factor of 1.38 if the bank achieved greater technical efficiency. The scaling factors for the indirect quasi-distance function all indicate a greater asymmetric expansion for each of the five outputs that could be achieved by an optimal reallocation of inputs. Indirect Russell input allocative efficiency, IAE^R, averaged 0.92 in both 1994 and 1995.

Table 4: Correlation coefficients for the input allocative efficiency measures

	1994	1995	1994	1995	1994	1995	1994	1995
Pearson	$\overline{\alpha}^*$		AE^F		IAE^F		IAE^R	
$\overline{\alpha}^*$	1	1						
AE^F	0.28	0.37	1	1				
	(0.01)	(0.01)						
IAE^F	0.29	0.15	0.36	0.22	1	1		
	(0.01)	(0.01)	(0.01)	(0.01)				
IAE^R	0.17	0.16	0.14	0.13	0.15	0.12	1	1
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)		
Spearman	$\overline{\alpha}^*$		AE^F		IAE^F		IAE^R	
$\overline{\alpha}^*$	1	1						
AE^F	0.31	0.42	1	1				
	(0.01)	(0.01)						
IAE^F	0.33	0.35	0.45	0.50	1	1		
	(0.01)	(0.01)	(0.01)	(0.01)				
IAE^R	0.11	0.10	0.08	0.07	0.10	0.12	1	1
	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)		

In Table 4 we list the Spearman and Pearson (rank) correlation coefficients for the four allocative efficiency measures. The four measures of input allocative efficiency have a positive and significant correlation with one another. We further test for equality among the four measures of input allocative efficiency using an Anova F-test and the non-parametric Wilcoxon and Median tests. The non-parametric tests are rank tests and do not require the efficiency measures to be normally distributed. The test results, presented in Table 5, reveal a higher level of input allocative efficiency when measured by IAE^F than when measured by AE^F for the Median test in 1994 and the F-test and Median test in 1995. Similarly, the Russell measures of input allocative efficiency, IAE^R, are also greater than the Farrell measures, AE^F, in both years based on the Median tests in 1994 and the F-test and Median test in 1995. The tests results comparing IAE^F with IAE^R indicate a significant difference in the two distributions in both years for all but the Median test in 1994. Tone's allocative efficiency measure, $\overline{\alpha}^*$, is significantly different from the other three measures of input allocative efficiency in 1994. In 1995, there is weaker evidence of difference between Tone's allocative efficiency and the other three measures. The Tone measure and the Farrell measure give an equal ranking based on the Wilcoxon test, and the Russell measure and the Tone measure are not significantly different based on the F-test and the Median test.

Table 5: Tests of equality for input allocative efficiency measures (significance level)

Variable	Year	Anova-F (Prob> F)	Wilcoxon-X^2 (Prob > X^2)	Median-X^2 (Prob> X^2)
AE^F vs. IAE^F	1994	0.01	0.38	7.68
		(0.91)	(0.54)	(0.01)
	1995	5.60	0.51	10.33
		(0.02)	(0.48)	(0.01)
AE^F vs. IAE^R	1994	0.01	0.38	7.68
		(0.91)	(0.54)	(0.01)
	1995	5.60	0.51	10.33
		(0.02)	(0.48)	(0.01)
IAE^F vs. IAE^R	1994	6.77	7.88	0.03
		(0.01)	(0.01)	(0.86)
	1995	19.50	35.30	13.21
		(0.01)	(0.01)	(0.01)
AE^F vs. $\overline{\alpha}^*$	1994	20.63	15.96	3.96
		(0.01)	(0.01)	(0.05)
	1995	7.64	0.12	2.36
		(0.01)	(0.73)	(0.12)
IAE^F vs. $\overline{\alpha}^*$	1994	47.82	32.34	24.42
		(0.01)	(0.01)	(0.01)
	1995	22.00	15.86	10.94
		(0.01)	(0.01)	(0.01)
IAE^R vs. $\overline{\alpha}^*$	1994	18.51	10.26	21.01
		(0.01)	(0.01)	(0.01)
	1995	0.03	2.21	0.91
		(0.86)	(0.14)	(0.34)

The test results presented in Table 5 suggest that choice among the four measures of input allocative efficiency makes a difference in gauging the efficiency with which inputs are allocated. The preferred method will, of course, depend on the problem at hand. If researchers are studying agencies within the public sector, such as school districts which face a fixed budget but have discretion over the hiring of inputs, the Russell indirect or Grosskopf *et al.*'s indirect measures of input allocative efficiency would be preferred. However, if researchers are studying firms that are interested in minimizing costs, either the traditional Farrell method or the Tone method of measuring cost efficiency would be preferred. Comparing the Farrell and Tone methods can offer further insight. As pointed out by Tone, input quantity reduction is needed when $F_i(y, x)$ is low, but $\overline{\theta}^*$ is high. However, cost reduction is suggested when $F_i(y, x)$ is high, but $\overline{\theta}^*$ is low.

Recently, there has been renewed interest in the allocation of credit to small businesses. While large corporations have ready access to national and global capital markets, commercial banks tend to provide more than half the credit allocated to small businesses (Sullivan *et al.* [19]). Because of information asymmetries, banks which allocate a large share of assets to small business lending usually incur higher monitoring costs to offset the higher risk of lending to small businesses. For our sample, small business lending as a percent of bank assets averaged 4.9% (s =6.1%) in 1994 and 5.6% (s = 6.7%) in 1995. We are interested in finding out what effect the proportion of small business lending has on the three measures of input allocative efficiency. If banks with a higher proportion of small business loans have higher costs, or use larger amounts of labor to monitor small business loans, then it should be reflected in a lower level of input allocative efficiency. Conversely, if a higher proportion of small business lending is part of an overall strategy of risk minimization consistent with portfolio theory, then it might be reflected in greater input allocative efficiency. The effects of small business lending on the efficiency with which banks allocate inputs is therefore an empirical question of some policy importance.

To examine the effects of small business lending (SBL) on allocative efficiency several sub-samples were created and two thresholds of SBL/Assets were examined: banks with SBL/Assets > 5% and banks with SBL/Assets>10%. The Anova F-test and the non-parametric Wilcoxon test and the Median test were used to determine whether banks with SBL/Assets > threshold had the same input allocative efficiency as banks below the threshold. Table 6 presents the results using 5% significance levels. Overall we found very few patterns between SBL/Assets and the various input allocative efficiency measures. In 1994 there was some indication that banks with SBL/Asset ratios greater than 5% had a higher degree of input allocative efficiency for the indirect Farrell measure, IAE^F (all three tests), for the direct Farrell measure, AE^F (Median test), and for the Tone measure (F-test and Wilcoxon test). However, in 1995, banks with SBL/Asset ratios greater than 5% had lower input allocative efficiency for the direct Farrell measure, AE^F. When the threshold of SBL/Assets was 10%, banks with a higher proportion of small business loans had a higher indirect Farrell input allocative efficiency, IAE^F in 1994 for the F-test. However, the indirect Russell measure, IAE^R, indicated a lower level of allocative efficiency for banks above the threshold in 1994.

To further examine the information content of the four allocative efficiency measures we examined the effects of bank loan write-offs as a percent of equity capital on input allocative efficiency. Some banks may find it cost effective to incur higher monitoring costs for a lower level of loan write-offs. For other banks, the decline in monitoring costs might be more than enough to offset higher loan write-offs. Because of the possibility for these kinds of tradeoffs to occur we test whether banks with a higher level of loan write-offs as a percent of equity exhibit greater or less input allocative efficiency. Bank charges for loan write-offs as a percent of equity averaged 1.1% (s.d.= 2.5%) in both 1994 and 1995. Two thresholds of charges for Loan write-offs/Equity were considered: banks with Loan write-offs/Equity > 1% and banks with Loan write-offs/Equity> 2%. The null hypothesis is that banks with a ratio of charges for loan write-offs to equity greater than the threshold have the same input allocative efficiency as banks below the threshold. The results of the statistical tests (Anova, Kruskal-Wallis, and Median) are presented in Table 7. In 1994 and 1995, banks with charges for loan write-offs> 1% of equity exhibited greater input allocative efficiency when measured by IAE^F and $\overline{\alpha}^*$ (Wilcoxon test) than banks with loan write-offs as a percent of equity below 1%. When the threshold is changed to loan write-offs in excess of 2% of equity, the pattern is the same for IAE^F and AE^F in 1994 for the Wilcoxon test and in 1995 for the Wilcoxon and Median tests.

Table 6: Tests of the effects of small business lending on input allocative efficiency

Variable	Year	Anova-F (Prob>F)	Wilcoxon-X^2 (Prob> X^2)	Median-X^2 (Prob> X^2)
Banks with SBL/Assets> 0.05 vs. other banks				
AE^F	1994	0.01	1.81	5.39**
		(0.93)	(0.28)	(0.02)
IAE^F	1994	11.67**	6.72**	3.94**
		(0.01)	(0.01)	(0.05)
IAE^R	1994	0.50	1.62	0.60
		(0.48)	(0.20)	(0.44)
$\overline{\alpha}^*$	1994	4.30**	4.27	2.09
		(0.04)	(0.04)	(0.15)
AE^F	1995	2.38	7.24*	7.09*
		(0.12)	(0.01)	(0.01)
IAE^F	1995	0.21	0.04	0.14
		(0.65)	(0.85)	(0.71)
IAE^R	1995	0.15	1.10	0.14
		(0.70)	(0.29)	(0.71)
$\overline{\alpha}^*$	1995	0.75	0.13	0.49
		(0.39)	(0.72)	(0.48)
Banks with SBL/Assets>0.10 vs. other banks				
AE^F	1994	0.04	0.39	2.32
		(0.85)	(0.53)	(0.13)
IAE^F	1994	5.00**	2.69	2.96
		(0.03)	(0.10)	(0.09)
IAE^R	1994	0.26	7.00*	4.12*
		(0.61)	(0.01)	(0.04)
$\overline{\alpha}^*$	1994	0.52	0.06	0.11
		(0.47)	(0.81)	(0.74)
AE^F	1995	0.56	1.44	0.41
		(0.45)	(0.23)	(0.52)
IAE^F	1995	0.02	0.48	0.03
		(0.88)	(0.50)	(0.86)
IAE^R	1995	0.00	0.14	0.71
		(0.99)	(0.71)	(0.40)
$\overline{\alpha}^*$	1995	0.00	0.26	0.41
		(0.99)	(0.61)	(0.52)

*indicates allocative efficiency for banks with SBL/Assets above the threshold is less than allocative efficiency for banks with SBL/Assets below the threshold.

**indicates allocative efficiency for banks with SBL/Assets above the threshold is greater than the allocative efficiency for banks with SBL/Assets below the threshold.

Table 7: Tests of the effects of loan write-offs on input allocative efficiency

Variable	Year	Anova-F (Prob>F)	Wilcoxon-X^2 (Prob> X^2)	Median-X^2 (Prob> X^2)
Banks with Loan Write-offs/Equity > 0.01 vs. other banks				
AE^F	1994	0.04	1.20	0.00
		(0.83)	(0.27)	(0.99)
IAE^F	1994	8.87**	23.82**	11.41**
		(0.01)	(0.01)	(0.01)
IAE^R	1994	0.47	0.32	0.08
		(0.49)	(0.57)	(0.78)
$\overline{\alpha}^*$	1994	0.79	5.14**	1.88
		(0.37)	(0.02)	(0.17)
AE^F	1995	0.92	4.90**	1.26
		(0.34)	(0.03)	(0.26)
IAE^F	1995	4.67**	10.67**	11.11**
		(0.03)	(0.01)	(0.01)
IAE^R	1995	2.85	3.05	0.96
		(0.09)	(0.08)	(0.33)
$\overline{\alpha}^*$	1995	1.24	2.94	3.31
		(0.27)	(0.09)	(0.07)
Banks with Loan Write-offs/Equity > 0.02 vs. other banks				
AE^F	1994	0.79	3.70**	1.09
		(0.37)	(0.05)	(0.30)
IAE^F	1994	1.77	10.65**	6.08**
		(0.18)	(0.01)	(0.01)
IAE^R	1994	0	0.04	0.03
		(0.99)	(0.84)	(0.86)
$\overline{\alpha}^*$	1994	0.14	1.97	0.13
		(0.71)	(0.16)	(0.72)
AE^F	1995	2.63	9.34**	7.00**
		(0.11)	(0.01)	(0.01)
IAE^F	1995	10.98**	10.71**	8.07**
		(0.01)	(0.01)	(0.01)
IAE^R	1995	0.10	0.24	0.48
		(0.75)	(0.62)	(0.49)
$\overline{\alpha}^*$	1995	0.00	1.26	1.18
		(0.99)	(0.26)	(0.28)

*indicates allocative efficiency for banks with Loan Write-offs/Equity above the threshold is less than allocative efficiency for banks with Loan Write-offs/Equity below the threshold.

**indicates allocative efficiency for banks with Loan Write-offs/Equity above the threshold is greater than the allocative efficiency for banks with Loan Write-offs/Equity below the threshold.

5. Conclusions

The traditional Farrell measure of input allocative efficiency in a DEA framework is derived as the ratio of overall cost efficiency to input technical efficiency. Since the solution to the linear programming problem which estimates radial input technical efficiency can be subject to slack in the constraints which define the reference technology, the Farrell measure of input allocative efficiency might be biased. In this paper we derive a new measure of input allocative efficiency, based in Russell measures of output technical efficiency. Our new measure builds on Grosskopf *et al.*'s [12] measure of input allocative efficiency which was derived by comparing the radial distance between the direct and indirect production possibility sets. Compared to Grosskopf *et al.*'s radial measure of efficiency our measure is non radial and leaves no slack in the constraints which define the reference technology. In an interesting example, Tone [20] illustrates that the Farrell measure of cost efficiency can also be biased because a firm that chooses input prices that are higher, can have the same cost efficiency as another firm that correctly chose lower input prices.

An empirical examination of the four measures of input allocative efficiency was derived from a random sample of U.S. banks operating in 1994 and 1995. All four measures of input allocative efficiency average above 90% in both years. We found no conclusive evidence that the indirect Farrell measure (Grosskopf *et al.*'s) and our indirect Russell measure differ. In further empirical tests we found that banks might be accepting higher charges for loan write-offs for reduced monitoring costs.

Endnotes

1. We restrict the domains of the functions to be discussed in this paper so that these functions are strictly positive and can be denoted by 'max' or 'min'. In addition, properties on the production functions and efficiency measures are derived for positive outputs and inputs.
2. This definition is based on the assumption that all data are positive, since obtained data are usually positive. If some inputs are zero, then (6) can be modified as was done by Russell [15].
3. If we relax technology convexity, then $R_i(y,x)$ may not be decreasing in $x > 0$.
4. Our example also shows that Färe and Lovell's [7] original version of Russell input technical efficiency is less than overall cost efficiency for observation A_1.
5. In Farrell [6], the direct Farrell output technical efficiency measure is denoted as $D_o(x,y)$. Yet we use the reciprocal as the direct Farrell output technical efficiency measure because the output-oriented efficiency is usually computed as $1/D_o(x,y)$ in DEA literature (Charnes *et al.* [4]; Cooper *et al.* [5]) and in economics literature (Färe *et al.* [8]).
6. Fukuyama and Weber [11] introduced input quasi-distance functions and used them to construct a measure of output allocative efficiency.
7. It should be noted that production frontiers or reference plans based on Farrell measures are possibly different from those based on Russell measures.
8. Other types of Russell measures can be developed that account for the problem of slacks. See for example, Bardhan *et al.* [2].
9. The distance function version of (24) and its proof are provided in Färe and Primont [9].
10. We thank an anonymous referee for pointing out the potential for this kind of problem and providing the example.
11. While Tone's [20] example uses constant returns to scale the measures can be estimated under variable returns to scale. For comparison with our measures of technical and allocative efficiencies we assume variable returns to scale.

References

[1] H. Y. Aly, R. Grabowski, C. Pasurka and N. Rangan: Technical, scale, and allocative efficiencies in US banking: An empirical investigation. *Review of Economics and Statistics*, **72** (1990) 211–218.

[2] I. Bardhan, W. F. Bowlin, W. W. Cooper and T. Sueyoshi: Models and measures for efficiency dominance in DEA — Part II: Free Disposal Hull (FDH) and Russell Measures (RM) approach. *Journal of the Operations Research Society of Japan*, **39** (1996) 333–344.

[3] A. Berger and D. Humphrey: Efficiency of financial institutions: International survey and directions for future research. *European Journal of Operational Research*, **98** (1997) 175–212.

[4] A. Charnes, W. W. Cooper and E. Rhodes: Measuring the efficiency of decision-making units. *European Journal of Operational Research*, **2** (1978) 429–444.

[5] W. W. Cooper, L. M. Seiford and K. Tone: *Data Envelopment Analysis* (Kluwer Academic Publishers, Boston/Dordrecht/London, 2000).

[6] M. J. Farrell: The measurement of productive efficiency. *Journal of the Royal Statistical Society*, Series A, General, **120**, part 3 (1957) 253–281.

[7] R. Färe and C. A. K. Lovell: Measuring the technical efficiency of production. *Journal of Economic Theory*, **19** (1978) 150–162.

[8] R. Färe, S. Grosskopf and C. A. K. Lovell: *Production Frontiers* (Cambridge University Press, Cambridge, 1994).

[9] R. Färe and D. Primont: *Multi-Output Production and Duality: Theory and Applications* (Kluwer Academic Publisher, Boston/London/Dordrecht, 1995).

[10] H. Fukuyama, R. Guerra and W. L. Weber: Efficiency and ownership: Evidence from Japanese Credit Cooperatives. *Journal of Economics and Business*, **51** (1999) 473–487.

[11] H. Fukuyama and W. L. Weber: Estimating output allocative efficiency and productivity change: Application to Japanese banks. *European Journal of Operational Research*, **137** (2002) 177–190.

[12] S. Grosskopf, K. Hayes, L. Taylor and W. L. Weber: Anticipating the consequences of school reform: A new use of DEA. *Management Science*, **45** (1999) 608–619.

[13] D. Hancock: The financial firm: Production with monetary and nonmonetary goods. *Journal of Political Economy*, **93** (1985) 859–880.

[14] J. Hughes and L. Mester: Bank capitalization and cost: evidence of scale economies in risk management and signaling. *Review of Economics and Statistics*, **80** (1998) 314–325.

[15] R. R. Russell: Measures of technical efficiency. *Journal of Economic Theory*, **35** (1985) 109–126.

[16] S. Shaffer: A revenue-restricted cost study of 100 large banks. *Applied Financial Economics*, **4** (1994) 193–205.

[17] R. W. Shephard: *Theory of Cost and Production Functions* (Princeton University Press, Princeton, 1970).

[18] R. W. Shephard: *Indirect Production Functions Mathematical Systems in Economics*, No. 10 (Meisenheim Am Glan, Verlag Anton Hain, 1974).

[19] R. Sullivan, R. Johnson, R. Phillips and K. Spong: Small business lending by commercial banks in Colorado: 1994 to 1996. *Financial Industry Perspectives*, Federal Reserve Bank of Kansas City, (1998) 17–33.

[20] K. Tone: A strange case of the cost and allocative efficiencies in DEA. *Journal of the Operational Research Society*, **53** (2002) 1225–1231.

Appendix A-1

If $y \in P(x)$, the properties on $Q_o(x,y)$ are:

Q_o.1 $Q_o(x,y)$ is non increasing in $x > 0$,
Q_o.2 $Q_o(x,y)$ is increasing in $y > 0$,
Q_o.3 $Q_o(x,y)$ is homogeneous of degree one in $y > 0$,
Q_o.4 $Q_o(x,y)$ is convex in $y > 0$.

Proof of Q_o.1. Strong input disposability implies $x' \geq x \Rightarrow P(x) \subseteq P(x')$. From the definition of $Q_o(x,y)$ we have $1/Q_o(x',y) \geq 1/Q_o(x,y)$ or $Q_o(x,y) \geq Q_o(x',y)$.

Proof of Q_o.2. $1/Q_o(x,y') = \max\{\sum_m s_m/M : s \otimes y' \in P(x)\} < \max\{\sum_m s_m/M : s \otimes y \in P(x)\} = 1/Q_o(x,y)$ or $y' \geq y$ but $y' \neq y$. Hence we have $Q_o(x,y) < Q_o(x,y')$.

Proof of Q_o.3. $1/Q_o(x,\mu y) = \max\{\sum_m \mu s_m/(\mu M) : s \otimes \mu y \in P(x)\} = (1/\mu)[1/Q_o(x,y)]$.

Proof of Q_o.4. Using $Q_o(x,u) = Q_o(x,u)$ we obtain $Q_o(x,u/Q_o(x,u)) = 1$ by homogeneity. Since G is convex, $P(x)$ is convex. If $u^1/Q_o(x,u^1) \in P(x)$ and $u^2/Q_o(x,u^2) \in P(x)$ then $\alpha u^1/Q_o(x,u^1/Q_o(u^1,x)) + (1-\alpha)u^2/Q_o(x,u^2/Q_o(u^2,x)) \in P(x)$, for $\alpha \in [0,1]$. By taking $\alpha = Q_o(x,u^1)/[Q_o(x,u^1)+Q_o(x,u^2)]$, we obtain $(u^1+u^2)/[Q_o(x,u^1)+Q_o(x,u^2)] \in P(x)$. It follows that $Q_o(x,(u^1+u^2)) \leq Q_o(x,u^1,x) + Q_o(x,u^2)$ by the homogeneity of $Q_o(x,y)$ in outputs. If $u^1 = \alpha y^1$ and $u^2 = (1-\alpha)y^2$, then we obtain $Q_o(x,y^\alpha) \leq \alpha Q_o(x,y^1) + (1-\alpha)Q_o(x,y^2)$ where $y^\alpha = \alpha y^1 + (1-\alpha)y^2$. This implies that $Q_o(x,y)$ is convex in y. We adapt the distance function based proof of Shephard [17].

Appendix A-2

If $y \in P(x)$, the properties on $IQ_o(w/c,y)$ are:

IQ_o.1 $IQ_o(w/c,y)$ is non increasing in $(w/c) > 0$.
IQ_o.2 $IQ_o(w/c,y)$ is increasing in $y > 0$.
IQ_o.3 $IQ_o(w/c,y)$ is homogeneous of degree one in $y > 0$.
IQ_o.4 $IQ_o(w/c,y)$ is convex in $y > 0$.

Proof of IQ_o.1. $(w/c)' \geq (w/c)$ implies $C(y,w/c) \leq C(y,(w/c)') \leq 1$ from the definition of the cost function. Since $\{y : C(y,w/c) \leq 1\} = \{y : IQ_o(w/c,y) \leq 1\}$, we have $IQ_o(w/c,y)$ is non decreasing in $(w/c) > 0$.

Proof of IQ_o.2. $1/IQ_o(w/c,y) = \max\{\sum_m q_m/M : q \otimes y \in P(x)\} < \max\{\sum_m q_m/M : q \otimes y' \in P(x)\} = 1/IQ_o(w/c,y)$ if $y' \geq y$ but $y' \neq y$. Therefore we have $IQ_o(w/c,y') > IQ_o(w/c,y)$.

Proof of IQ_o.3. For $\mu > 0$ we have $1/IQ_o(w/c,\mu y) = \max\{\sum_m \mu q_m/(\mu M) : q \otimes \mu y \in IP(w/c)\} = (1/\mu)[1/IQ_o(w/c,y)]$.

Proof of IQ_o.4. The proof is similar to that of Q_o.4.

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