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SPECIAL CLASSES OF QUIET ACCUMULATION GAMES

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Abstract In an accumulation game, a hider places objects at locations, and a seeker examines these locations. If the seeker discovers an object then he/she confiscates it. The goal of the hider is to accumulate a certain number of objects before a given time, and the goal of the seeker is to prevent this. In this paper we discuss the quiet accumulation game in which the hider is informed of the location searched on a turn only if the seeker finds an object there. We solve the case where the number of steps is 3 and the goal of the hider is to accumulate 2 objects, and the case where the number of steps is equal to the hider's goal.

Keywords: Game theory, search, optimal strategy, inspection

1. Definitions and Notation

The type of search game that we analyze models the following types of situations: (1) An illicit organization, such as a terrorist gang, attempts to accumulate a certain minimum amount of material and a law enforcement agency attempts to prevent this by means of a limited number of inspections. (2) A polluter attempts to illegally conceal a quantity of waste, and an enforcement agency tries to uncover this attempt. This leads to a two-person zero sum game between the organization and the law enforcement agency for which the payoff to the organization is 1 (it wins), if it accumulates m units of the material by time K, and is 0 (it loses), otherwise.

In an accumulation game a hider (called **H**) tries to accumulate a certain number of objects within a certain time by hiding them at a fixed number of locations, and a seeker (called **S**) attempts to prevent this. In general, **H** can place the objects at locations $1, 2, \ldots, n$, and the game is played in discrete time.

The followings are parameters and assumptions. The letter n indicates the total number of locations at which objects can be hidden. $I = \{1, 2, \ldots, n\}$ is the set of all locations. At each turn **H** acquires an object and can place it at any empty location. **S** can examine only one location at each turn. **S** will find an object with certainty if it is at the location searched.

In this paper we investigate quiet search where **H** knows the location searched on a turn only if **S** finds an object there. **H** can use this information to choose a location in subsequent turns. **H** has complete knowledge of the empty locations at each turn. **H** can place an object at a location where : (1) **H** has not placed an object yet and **S** has not searched yet, (2) **H** has not placed an object yet and **S** has already searched, or (3) **H** has placed another object but **S** has already searched and found the object. Note that after **S** examines an empty location, the location will return to its initial state. This means that **H** can not distinguish whether or not that location has been examined.

N is the number of objects that **H** wants to accumulate. The maximum number of steps is k. If **H** can hide N objects at the end of the t-th turn for some $t \quad (N \le t \le k)$, the game terminates and **H** wins (payoff 1) . Otherwise, if **H** cannot hide N objects within k turns, **H** loses (payoff 0).

We express this game as (n, N, k) (i.e. a game with n locations, N objects, and k steps).

In Section 2 of this paper we analyze the case $N = 2$ and $k = 3$. Our results suggest that it is difficult to extend the analysis even to games in the special case where $k = N+1$. Some variations of the case $k = N + 1$ will be analyzed somewhere. We analyze the case where $N = k$ in Section 3. This case is closely related to what we call the very quiet accumulation game (See [5]).

In [2], we analyzed noisy search in which **H** knows the location searched on each turn. We presented the solution for that game for all but some marginal cases. In [3], we analyzed the noisy case where **H** can place a continuous material at discrete locations and the game is played in discrete time. We shall usually assume that the game begins in an initial state where **H** has no objects. However, most of our analysis also applies to situations where the game begins with a number of objects already hidden, but **S** knows only the number of objects hidden and has no information about their location. The reader can find a general theory of search games in [1], and information on geometric search games (with notes on open problems) in [4]. Although accumulation games form a new kind of search game, there are many related two-person tactical games including search games. References 1-23 in [2] describe a variety of two-person zero-sum search games. Table 1 in [2] describes some variations on accumulation games.

2. Quiet Accumulation Game for $(n, 2, 3)$

Since the total number of turns (the third coordinate) is greater than the number of objects by 1, **H** will definitely lose if all objects are found within two turns. We define the outcomes as follows.

$\mathcal{N}$: **S** fails to find an object on the indicated turn.
$\mathcal{F}$: **S** finds an object on the indicated turn.

We often use the following notation to indicate the outcome from **H**'s viewpoint, keeping in mind the information obtained by **H**.

$\mathcal{F}i$: On the second step, **S** finds an object hidden at the ith step, $i = 1, 2$.

Since the game ends when **H** succeeds in the first two steps, or when **H** fails in the first two steps, the following sequences of outcomes can occur.

$$\mathcal{NN}, \mathcal{NFN}, \mathcal{FNN}, \mathcal{NFF}, \mathcal{FNF}, \mathcal{FF}$$

H wins in the first three sequences out of six, i.e., $\mathcal{NN}, \mathcal{NFN}, \mathcal{FNN}$.

A pure strategy for **H**, denoted as $\bar{h} \equiv (h_1, \bar{h}_N, \bar{h}_F)$, is defined as follows, where $\bar{h}_N \equiv (h_2^N, h_3^{F1}, h_3^{F2})$ and $\bar{h}_F \equiv (h_2^F, h_3^N)$ are the choices at the second and the third steps, assuming the outcomes of the first step are $\mathcal{N}$ and $\mathcal{F}$ respectively. Each subscript indicates the step at which the choice is made.

h_1:the choice at the first step
$h_i^N, i = 2, 3$:the choice at the ith step, assuming the outcome of the $(i-1)-$th step is $\mathcal{N}$
h_2^F:the choice at the second step, assuming the outcome of the first step is $\mathcal{F}$
$h_3^{Fi}, i = 1, 2$:the choice at the third step, assuming the outcome of the second step is $\mathcal{F}i$

A pure strategy for **S**, denoted as $\bar{s} \equiv (s_1, \bar{s}_N, \bar{s}_F)$, is defined as follows, where $\bar{s}_N \equiv (s_2^N, s_3^F)$ and $\bar{s}_F \equiv (s_2^F, s_3^N)$. Each subscript indicates the step at which the choice is made.

s_1:the choice at the first step
$s_i^N, i = 2, 3$:the choice at the ith step, assuming the outcome of the $(i-1)$-th step is $\mathcal{N}$
$s_i^F, i = 2, 3$:the choice at the ith step, assuming the outcome of the $(i-1)$-th step is $\mathcal{F}$

Since the game is sequential, we use behavioral strategies rather than mixed strategies. So, at each decision point a player determines a probability distribution on all alternatives.

For **S**, a behavioral strategy is given by

$$\bar{q} \equiv (q(\cdot), q(\cdot|s_1, \mathcal{N}), q(\cdot|s_1, \mathcal{N}, s_2^N, \mathcal{F}), q(\cdot|s_1, \mathcal{F}), q(\cdot|s_1, \mathcal{F}, s_2^F, \mathcal{N})),$$

where each component is a probability distribution at each decision point. We let $\bar{q}_N \equiv (q(\cdot|s_1, \mathcal{N}), q(\cdot|s_1, \mathcal{N}, s_2^N, \mathcal{F}))$ and $\bar{q}_F \equiv (q(\cdot|s_1, \mathcal{F}), q(\cdot|s_1, \mathcal{F}, s_2^F, \mathcal{N}))$. For **H**, a behavioral strategy is given by

$$\bar{p} \equiv (p(\cdot), p(\cdot|h_1, \mathcal{N}), p(\cdot|h_1, \mathcal{N}, h_2^N, \mathcal{F}1), p(\cdot|h_1, \mathcal{N}, h_2^N, \mathcal{F}2), p(\cdot|h_1, \mathcal{F}), p(\cdot|h_1, \mathcal{F}, h_2^F, \mathcal{N})).$$

We let $\bar{p}_N \equiv (p(\cdot|h_1, \mathcal{N}), p(\cdot|h_1, \mathcal{N}, h_2^N, \mathcal{F}1), p(\cdot|h_1, \mathcal{N}, h_2^N, \mathcal{F}2))$ and $\bar{p}_F \equiv (p(\cdot|h_1, \mathcal{F}), p(\cdot|h_1, \mathcal{F}, h_2^F, \mathcal{N}))$. $f(\bar{p}, \bar{q})$ is the probability of **H**'s winning when **H** and **S** use behavioral strategies $\bar{p}$ and $\bar{q}$ respectively.

2.1. An optimal strategy for the seeker

In this subsection we give the minimax value of the game and a behavioral strategy for **S** corresponding to the minimax value. The following diagram in Figure 1 is not a game tree, but is helpful for calculating the expected payoff to **H** when **S** uses a specified behavioral strategy and **H** uses a pure strategy.

$$h_1 \begin{cases} \xrightarrow[s_1\neq h_1]{N} h_2^N \begin{cases} \xrightarrow[s_2^N\neq h_1,h_2^N]{N} 1 \\ \xrightarrow[s_2^N=h_2^N]{F} h_3^{F2} \xrightarrow[s_3^F\neq h_1,h_3^{F2}]{N} 1 \\ \xrightarrow[s_2^N=h_1]{F} h_3^{F1} \xrightarrow[s_3^F\neq h_2^N,h_3^{F1}]{N} 1 \end{cases} \\ \xrightarrow[s_1=h_1]{F} h_2^F \xrightarrow[s_2^F\neq h_2^F]{N} h_3^N \xrightarrow[s_3^N\neq h_2^F,h_3^N]{N} 1 \end{cases}$$

Figure 1: A diagram for calculating the payoff to the hider.

S's problem is to minimize the probability of **H**'s winning, i.e., to choose a behavioral strategy $\bar{q}$ so that

$$max\{f(\bar{h}, \bar{q}) : \bar{h} = (h_1, \bar{h}_N, \bar{h}_F)\}$$

is minimized. In the first turn, **S** cannot distinguish any locations, and so **S** will choose each location with equal probability under any optimal strategy. This means $q(s_1) = \frac{1}{n}$ for all $s_1 \in I$. So **S** will find an object with probability $\frac{1}{n}$. If **S** finds an object, **S** and **H** will use strategies $\bar{q}_F$ and $\bar{h}_F$ respectively: otherwise, **S** and **H** will use strategies $\bar{q}_N$ and $\bar{h}_N$ respectively. The expected payoff to **H** is calculated separately as $f(\bar{h}, \bar{q}) = \frac{n-1}{n} f(\bar{h_N}, \bar{q_N}) + \frac{1}{n} f(\bar{h_F}, \bar{q_F})$, where $\bar{q} = ((\frac{1}{n}, \dots, \frac{1}{n}), \bar{q}_N, \bar{q}_F)$ and $\bar{h} = (h_1, \bar{h}_N, \bar{h}_F)$. By separating in this way, we see that we can analyze the cases $\mathcal{NN}$ and $\mathcal{NFN}$ and the case $\mathcal{FNN}$ separately. So in the following subsections we give the minimax or maximin strategies of both players separately for the above cases. Note that the argument above applies when we consider **H**'s strategy. So, for example, in subsection 2.1.1, we give a minimax strategy for **S** in the cases $\mathcal{NN}$ and $\mathcal{NFN}$.

2.1.1. Seeker's strategy in the cases $\mathcal{NN}$ and $\mathcal{NFN}$

First we define **S**'s strategy in the case $\mathcal{NN}$ and $\mathcal{NFN}$. Assuming $s_1 \neq h_1$, let

$$q(s_2^N | s_1, \mathcal{N}) = \begin{cases} \frac{1}{n-1}, & \text{for } s_2^N \in I \setminus \{s_1\}; \\ 0, & \text{for } s_2^N = s_1. \end{cases} \tag{1}$$

Assuming $s_1 \neq h_1$ and $s_2^N \in \{h_2^N, h_1\}$, define

$$q(s_3^F | s_1, \mathcal{N}, s_2^N, \mathcal{F}) = \begin{cases} 0, & \text{for } s_3^F = s_1; \\ \frac{1}{n}, & \text{for } s_3^F = s_2^N; \\ \frac{n-1}{n(n-2)}, & \text{for } s_3^F \in I \setminus \{s_1, s_2^N\}. \end{cases} \tag{2}$$

Note that $s_2^N \neq s_1$ by (1). (1) and (2) mean that **S** examines each box except the location which **S** examined where no object was found. Then calculating the expected payoff to **H**, case by case, for $\mathcal{NN}$ or $\mathcal{NFN}$ and any pure strategy $\bar{h}$ of **H**, we see that the maximum of the expected payoff is $\frac{(n-2)^2}{{}_nP_2} + \frac{1}{{}_nP_2}\{n-2+\frac{(n-1)(n-3)^2}{n(n-2)}\}$ (See Part 1 of the Appendix below).

2.1.2. Seeker's strategy in the case $\mathcal{FNN}$

Next define **S**'s strategy in the case $\mathcal{FNN}$. Assuming $s_1 = h_1$, define

$$q(s_2^F | s_1, \mathcal{F}) = \frac{1}{n} \text{ for all } s_2^F \in I.$$

$$q(s_3^N | s_1, \mathcal{F}, s_2^F, \mathcal{N}) = \begin{cases} 0, & \text{for } s_3^N = s_2^F; \\ \frac{1}{n-1}, & \text{for } s_3^N \in I \setminus \{s_2^F\}. \end{cases} \tag{3}$$

Then calculating the expected payoff to **H**, case by case, for $\mathcal{FNN}$ and any pure strategy $\bar{h}$ of **H**, we see that the maximum of the expected payoff is $\frac{(n-2)^2}{n^2(n-1)}$ (See Part 2 of the Appendix below). Hence if **S** uses the above strategies, the expected payoff to **H** is at most

$$P_H^* \equiv \frac{1}{n}\frac{(n-2)^2}{{}_nP_2} + \frac{(n-2)^2}{{}_nP_2} + \frac{1}{{}_nP_2}\{n-2+\frac{(n-1)(n-3)^2}{n(n-2)}\}.$$

2.2. An optimal strategy for the hider

In this subsection we give the maximin value of the game and a behavioral strategy for **H** corresponding to the maximin value. The next diagram in Figure 2 is not a game tree, but is helpful for calculating the expected payoff of **S** when **H** uses a specified behavioral strategy and **S** uses a pure strategy.

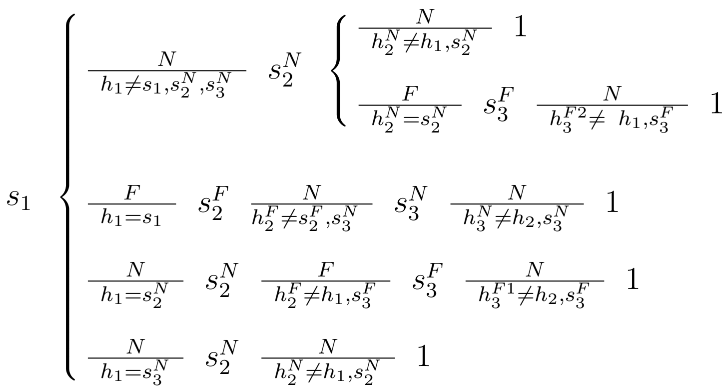

Figure 2 : A diagram for calculating the payoff to the seeker.

H's problem is to maximize the probability of winning, i.e., to choose a behavioral strategy $\bar{p}$ so that

$$min\{f(\bar{p},\bar{s}) : \bar{s} = (s_1, \bar{s}_N, \bar{s}_F)\}$$

is maximized.

2.2.1. Hider's strategy in the cases $\mathcal{NN}$ and $\mathcal{NFN}$

Next define **H**'s strategy in the cases $\mathcal{NN}$ and $\mathcal{NFN}$. Let

$$p(h_2^N|h_1,\mathcal{N}) = \begin{cases} \frac{1}{n-1}, & \text{for } h_2^N \in I \setminus \{h_1\}; \\ 0, & \text{for } h_2^N = h_1, \end{cases}$$

$$p(h_3^{F1}|h_1,\mathcal{N},h_2^N,\mathcal{F}1) = \begin{cases} 0, & \text{for } h_3^{F1} = h_2^N; \\ \frac{4n-9}{n(n-2)}, & \text{for } h_3^{F1} = h_1; \\ \frac{(n-3)^2}{n(n-2)^2}, & \text{for } h_3^{F1} \in I \setminus \{h_1, h_2^N\}, \end{cases} \tag{4}$$

$$p(h_3^{F2}|h_1,\mathcal{N},h_2^N,\mathcal{F}2) = \begin{cases} \frac{1}{n-2}, & \text{for } h_3^{F2} \in I \setminus \{h_1, h_2^N\}; \\ 0, & \text{for } h_3^{F2} \in \{h_1, h_2^N\}. \end{cases}$$

Then calculating the expected payoff to **H**, case by case, for $\mathcal{NN}$ or $\mathcal{NFN}$ and any pure strategy $\bar{s}$ of **S**, we see that the minimum expected payoff is $\frac{(n-2)^2}{{}_nP_2} + \frac{1}{{}_nP_2}\{n-2+\frac{(n-1)(n-3)^2}{n(n-2)}\}$.

2.2.2. Hider's strategy in the case $\mathcal{FNN}$

First we define **H**'s strategy in the case $\mathcal{FNN}$. Let $p(h_1) = \frac{1}{n}$ for all $h_1 \in I$. Let

$$p(h_2^F|h_1,\mathcal{F}) = \frac{1}{n} \text{ for all } h_2^F \in I.$$

$$p(h_3^N|h_1,\mathcal{F},h_2^F,\mathcal{N}) = \begin{cases} 0, & \text{for } h_3^N = h_2^F; \\ \frac{1}{n-1}, & \text{for } h_3^N \in I \setminus \{h_2^F\}. \end{cases} \tag{5}$$

Then calculating the expected payoff to **H**, case by case, for $\mathcal{FNN}$ and any pure strategy $\bar{s}$ of **S**, we see that the minimum expected payoff is $\frac{(n-2)^2}{n^2(n-1)}$ (See Part 3 of the Appendix below).

Hence if **H** uses the above strategies, the expected payoff to **H** is at least P_H^*. Consequently we have the following theorem. Interpretations of optimal strategies will be given in Section 4.

Theorem 1. The value of the quiet $(n, 2, 3)$ game is P_H^*. Optimal strategies for both players are given by (1)-(5) with uniform distributions in the first step.

Proof: By the analysis in the subsections 2.1.1 and 2.1.2 including Parts 1 and 2 in the Appendix, we have

$$max\{f(\bar{h},\bar{q}) : \bar{h} = (h_1, \bar{h}_N, \bar{h}_F)\} \leq P_H^*. \tag{6}$$

By the analysis in the subsections 2.2.1 and 2.2.2 including Parts 3 and 4 in the Appendix, we have

$$min\{f(\bar{p},\bar{s}) : \bar{s} = (s_1, \bar{s}_N, \bar{s}_F)\} \geq P_H^*. \tag{7}$$

By the basic theory of extensive games (noting that we are treating a game with perfect recall), there are mixed strategies $\bar{x}$ and $\bar{y}$ of **H** and **S** respectively such that $f(\bar{x},\bar{s}) = f(\bar{p},\bar{s})$ for all pure strategies $\bar{s}$ and $f(\bar{h},\bar{y}) = f(\bar{h},\bar{q})$ and for all pure strategies $\bar{h}$. From these and from (6) and (7), we have

$$f(\bar{x},\bar{s}) \geq P_H^* \geq f(\bar{h},\bar{y}) \text{ for all } \bar{h} \text{ and } \bar{s}. \tag{8}$$

Next suppose $\bar{p'}$ and $\bar{q'}$ are behavioral strategies of **H** and **S** respectively. There are mixed strategies $\bar{x'}$ and $\bar{y'}$ of **H** and **S** respectively such that $f(\bar{p},\bar{q'}) = f(\bar{x},\bar{y'})$ and $f(\bar{p'},\bar{q}) = f(\bar{x'},\bar{y})$. From these and (8), we have

$$f(\bar{p},\bar{q'}) \geq P_H^* \geq f(\bar{p'},\bar{q})$$

for all $\bar{p'}$ and $\bar{q'}$. Hence $(\bar{p},\bar{q})$ is an equilibrium point. Hence (1)-(5) are optimal (behavioral) strategies and P_H^* is the value of the game. ∎

Remark. The game tree for $(4,2,3)$, i.e., for $n=4$ is very complicated, but can be drawn. It shows that the roots of the subgames are at the points reached by the outcome of $\mathcal{F}$ in the first turn (By symmetry, there are 4 subgames). Indeed both players know they are at those points when the outcome of the first step is $\mathcal{F}$. We see (3) and (5) are optimal (behavioral) strategies for the subgames, by checking Parts 2 and 3 in Appendix, and by applying the same argument as in the proof of Theorem 1. The value of any subgame is $\frac{(n-2)^2}{n(n-1)}$. Since strategies given by (3) and (5) are parts of the whole strategies, we conclude that the pair of optimal (behavioral) strategies in Theorem 1 is a subgame perfect equilibrium.

3. Quiet Accumulation Game for (n,k,k)

In Section 2 we solved the quiet case where $N=2$ and $k=3$. In this section we solve the case where $N=k$, i.e., we solve the game for (n,k,k). Since the number of steps is equal to the number of locations, **S** will win (i.e., payoff 0) as soon as **S** finds an object at any step. Then, assuming **H** and **S** chose $h_1,\ldots,h_t$ and $s_1,\ldots,s_t$ at the previous t steps, we let $H_t \equiv (h_1,\ldots,h_t)$ and $S_t \equiv (s_1,\ldots,s_t)$ respectively. Without confusion and for convenience, we very often regard H_t and S_t as the sets whose elements are $h_1,\ldots,h_t$ and $s_1,\ldots,s_t$ respectively. $p(i|h_1,\ldots,h_t) \equiv p(i|H_t)$ is the probability that **H** chooses $i \in I\backslash H_t$. $q(i|s_1,\ldots,s_t) \equiv q(i|S_t)$ is the probability that **S** chooses $i \in I$.

Theorem 2. The value of the game for (n,k,k) is $v \equiv v(n,k) = \frac{(n-k)^k}{{}_nP_k}$. There are optimal strategies which are symmetric for both players such that $p(h_i|H_{i-1}) = \frac{1}{n-i+1}$ for $h_i \notin H_{i-1}$ and $q(s_i|S_{i-1}) = \frac{1}{n-i+1}$ for $s_i \notin S_{i-1}$.

The following identities in the lemmas are elementary but important in solving the game.

Lemma 1. For a fixed sequence $h_1,\ldots,h_k$ such that $h_i \neq h_j$ for all $i \neq j$,

$$\sum_{s_1 \notin H_1} \sum_{s_2 \notin H_2 \cup S_1} \cdots \sum_{s_k \notin H_k \cup S_{k-1}} 1 = (n-k)^k.$$

On the other hand, for a fixed sequence $s_1,\ldots,s_k$ such that $s_i \neq s_j$ for all $i \neq j$,

$$\sum_{h_1 \notin S_k} \sum_{h_2 \notin H_1 \cup \{s_2,\ldots,s_k\}} \cdots \sum_{h_k \notin H_{k-1} \cup \{s_k\}} 1 = (n-k)^k.$$

Proof of Lemma 1: Without loss of generality, we assume $h_i = i$ for $i=1,\ldots,k$. Then the left hand side becomes:

$$\begin{aligned}\sum_{s_1 \geq 2} \sum_{\substack{s_2 \geq 3 \\ s_2 \notin S_1}} \cdots \sum_{\substack{s_k \geq k+1 \\ s_k \notin S_{k-1}}} 1 &= \sum_{s_2 \geq 3} \sum_{\substack{s_1 \geq 2 \\ s_1 \notin \{s_2\}}} \sum_{\substack{s_3 \geq 4 \\ s_3 \notin S_2}} \cdots \sum_{\substack{s_k \geq k+1 \\ s_k \notin S_{k-1}}} 1 \\ &= \sum_{s_2 \geq 3} \sum_{\substack{s_3 \geq 4 \\ s_3 \notin \{s_2\}}} \cdots \sum_{\substack{s_k \geq k+1 \\ s_k \notin \{s_2,\ldots,s_{k-1}\}}} \sum_{\substack{s_1 \geq 2 \\ s_1 \notin \{s_2,\ldots,s_k\}}} 1\end{aligned}$$

$$\begin{aligned}
&= (n-k)\sum_{s_2\ge 3}\sum_{\substack{s_3\ge 4\\ s_3\notin\{s_2\}}}\cdots\sum_{\substack{s_k\ge k+1\\ s_k\notin\{s_2,\dots,s_{k-1}\}}} 1\\
&= (n-k)\sum_{s_3\ge 4}\sum_{\substack{s_4\ge 5\\ s_4\notin\{s_3\}}}\cdots\sum_{\substack{s_k\ge k+1\\ s_k\notin\{s_3,\dots,s_{k-1}\}}}\sum_{\substack{s_2\ge 3\\ s_2\notin\{s_3,\dots,s_k\}}} 1\\
&= \cdots = (n-k)^k.
\end{aligned}$$

In the left hand side of the second identity,

$$\sum_{h_i\notin H_{i-1}\cup\{s_i,s_{i+1},\dots,s_k\}} 1 = n-k$$

for $i = k, k-1, \dots, 1$. So we have the second half. ∎

Lemma 2.

$$\begin{aligned}
&\sum_{h_1=1}^{n}\sum_{s_1\notin H_1}\sum_{h_2\notin H_1}\sum_{s_2\notin H_2\cup S_1}\sum_{h_3\notin H_2}\sum_{s_3\notin H_3\cup S_2}\cdots\sum_{h_k\notin H_{k-1}}\sum_{s_k\notin H_k\cup S_{k-1}} 1\\
&=\sum_{h_1=1}^{n}\sum_{h_2\notin H_1}\cdots\sum_{h_k\notin H_{k-1}}\sum_{s_1\notin H_1}\sum_{s_2\notin H_2\cup S_1}\sum_{s_3\notin H_3\cup S_2}\sum_{s_4\notin H_4\cup S_3}\cdots\sum_{s_k\notin H_k\cup S_{k-1}} 1.
\end{aligned}$$

Lemma 3.

$$\begin{aligned}
&\sum_{s_1=1}^{n}\sum_{h_1\ne s_1}\sum_{s_2\notin H_1}\sum_{h_2\notin H_1\cup\{s_2\}}\cdots\sum_{s_k\notin H_{k-1}}\sum_{h_k\notin H_{k-1}\cup\{s_k\}} 1\\
&=\sum_{s_1=1}^{n}\sum_{s_2=1}^{n}\cdots\sum_{s_k=1}^{n}\sum_{h_1\notin S_k}\sum_{h_2\notin H_1\cup\{s_2,\dots,s_k\}}\sum_{h_3\notin H_2\cup\{s_3,\dots,S_k\}}\sum_{h_4\notin H_3\cup\{s_4,\dots,s_k\}}\cdots\sum_{h_k\notin H_{k-1}\cup\{s_k\}} 1.
\end{aligned}$$

Proof of Lemma 2 and Lemma 3: By considering the changes of orderings of summations carefully, we have Lemmas 2 and 3. ∎

Proof of Theorem 2: Denote the expected payoff to **H** by $f(p,q)$ when **H** and **S** use the mixed strategies p and q respectively. Denote p^* and q^* as follows, noting that the payoff is 0 once $s_i \in H_i$ occurs at any step i.

$$p^*(h_i|H_{i-1}) = \begin{cases}\frac{1}{n-i+1}, & \text{for } h_i \notin H_{i-1};\\ 0, & \text{for } h_i \in H_{i-1}.\end{cases}$$

$$q^*(s_i|S_{i-1}) = \begin{cases}\frac{1}{n-i+1}, & \text{for } s_i \notin S_{i-1};\\ 0, & \text{for } s_i \in S_{i-1}.\end{cases}$$

$$\begin{aligned}
f(p^*,q) &= \sum_{s_1=1}^{n} q(s_1)\sum_{h_1\ne s_1}\frac{1}{n}\cdots\sum_{s_k\notin H_{k-1}} q(s_k|H_{k-1})\sum_{h_k\notin H_{k-1}\cup\{s_k\}}\frac{1}{n-k+1}\\
&= \frac{1}{{}_nP_k}\sum_{s_1=1}^{n} q(s_1)\sum_{s_2=1}^{n} q(s_2|S_1)\cdots\sum_{s_k=1}^{n} q(s_k|S_{k-1})\sum_{h_1\notin S_k}\sum_{h_2\notin H_1\cup\{s_2,\dots,s_k\}}\cdots\sum_{h_k\notin H_{k-1}\cup\{s_k\}} 1\\
&\ge \frac{(n-k)^k}{{}_nP_k}\sum_{s_1=1}^{n} q(s_1)\sum_{s_2=1}^{n} q(s_2|S_1)\cdots\sum_{s_k=1}^{n} q(s_k|S_{k-1})\\
&= \frac{(n-k)^k}{{}_nP_k}
\end{aligned}$$

by Lemma 3 and since $|\{h_1|h_1 \notin S_k\}| \geq n-k$, $|\{h_2|h_2 \notin H_1 \cup \{s_2,\ldots,s_k\}\}| \geq n-k,\ldots,$ and $|\{h_k|h_k \notin H_{k-1} \cup \{s_k\}\}| \geq n-k$. Next,

$$\begin{aligned} f(p,q^*) &= \sum_{h_1=1}^{n} p(h_1) \sum_{s_1 \notin H_1} \frac{1}{n} \sum_{h_2 \notin H_1} p(h_2|h_1) \\ &\quad \sum_{s_2 \notin H_2 \cup S_1} \frac{1}{n-1} \cdots \sum_{h_k \notin H_{k-1}} p(h_k|H_{k-1}) \sum_{s_k \notin H_k \cup S_{k-1}} \frac{1}{n-k+1} \\ &= \frac{1}{{}_nP_k} \sum_{h_1=1}^{n} p(h_1) \sum_{h_2 \notin H_1} p(h_2|h_1) \cdots \sum_{h_k \notin H_{k-1}} p(h_k|H_{k-1}) \sum_{s_1 \notin H_1} \sum_{s_2 \notin H_2 \cup S_1} \cdots \sum_{s_k \notin H_k \cup S_{k-1}} 1 \\ &= \frac{(n-k)^k}{{}_nP_k}, \end{aligned}$$

by Lemmas 2 and 1. ∎

Remark. In the noisy case corresponding to (n,k,k), the game value becomes $\frac{{}_nP_{k+1}}{n^{k+1}}$ which is derived from Equation (10) of [2] and it is greater than or equal to $v(n,k) = \frac{(n-k)^k}{{}_nP_k}$. This is reasonable because **H** has more information in the noisy case than in the quiet case.

We conclude by sketching an alternative derivation of Theorem 2 that applies in any situation in which **H** loses if even one object is located. We leave the details to the reader. Consider an accumulation game with n locations in which at each turn **H** hides at h, **S** searches s, there are k turns and **H** must retain all objects hidden, $N = kh$. We assume **H** does not know the locations where **S** searches. We shall say that a location is *safe* at turn i if **S** will not examine that location at turn i or thereafter. If **H** places an object at a safe location then it will not be found. A location is called *open* if there is no object at that location. At turn i the largest number of locations that cannot be safe is $(k-i+1)s$. The maximal number is attained if **S** never searches a location more than once. This is possible if $ks < n$. Thus the smallest possible number of safe locations on turn i is $n-(k-i+1)s$. If **H** has previously placed objects only at safe locations on previous turns then the number of locations that are free and safe is $n-(k-i+1)s-(i-1)h$. The probability that **H** places h objects at locations that are safe during turn i is

$$\frac{n-(k-i+1)s-(i-1)h}{n-(i-1)h} \cdot \frac{n-(k-i+1)s-(i-1)h-1}{n-(i-1)h-1} \cdots \frac{n-(k-i+1)s-(i-1)h-h+1}{n-(i-1)h-h+1}.$$

The probability that **H** will win is the product of these numbers as i ranges from 1 to k. If $h = s$ then the probabilities become

$$\frac{n-ks}{n-(i-1)s} \cdot \frac{n-ks-1}{n-(i-1)s-1} \cdots \frac{n-(k+1)s+1}{n-is+1}.$$

If $h = s = 1$ then the probabilities become

$$\frac{n-k}{n-i+1}$$

so that the probability of **H** winning is

$$\prod_{i=1}^{k} \frac{n-k}{n-i+1} = \frac{(n-k)^k}{n \cdot (n-1) \cdots (n-k+1)}.$$

4. Conclusion and Comments

We have formulated a quiet accumulation game and solved two special cases: (1) Where the number of steps is 3 and the number of objects to be hidden is 2, $(n, 2, 3)$; (2) Where the number of steps equals the number of objects to be hidden.

In the game for $(n, 2, 3)$ each player should, in principle, choose locations with equal probability if he has the same information on each of those locations. Then, as play proceeds, both players must consider minimax and maximin as in the Appendix. In some cases, optimal strategies for both players are interpreted as follows.

Suppose the outcome of the first step is $\mathcal{F}$. Each player knows the choice of the oppsite player in the first step. Strategies (3) and (5) say that both players should choose strategies as if they were playing the game for $(n, 2, 2)$ in the following two steps. This is because both players must consider that the opposite player may make the same choice as in the first step. They calculate maximin and minimax respectively, and consequently must choose all locations at random.

Suppose the outcome of the first step is $\mathcal{N}$. In the second step, **H** should choose each location $h_2(\neq h_1)$ at random, since he does not know s_1 (The first expression in (4)). **S** should not choose s_1 since, with probability 1, **S** would fail to catch an object at h_1, if **S** chooses s_1. So **S** should choose $s_2(\neq s_1)$ at random (The expression (1)).

Suppose the outcomes of the two steps are $\mathcal{N}$ and $\mathcal{F}$. **S** would not be able to distinguish $\mathcal{NF}1$ and $\mathcal{NF}2$. In the third step **S** should not choose s_1 since, in the case where $s_2 = h_2$, and in the case where $s_2 = h_1$ and $s_1 \neq h_2$, **S** would fail to catch an object at h_1, with a probability of 1. Keeping in mind **H**'s strategy, the probability that $s_2 = h_1$ and $s_1 = h_2$ occur simultaneously is relatively small. On the other hand **S** may choose s_2^N in the third step since **H** may choose it, too (The expression (2)).

For **H**, the difference between $\mathcal{NF}1$ and $\mathcal{NF}2$ is that there is a possibility that $h_2 = s_1$ in $\mathcal{NF}1$ and there is no possibility that $h_2 = s_1$ or $s_1 = h_1$ in $\mathcal{NF}2$. This difference may affect **H**'s strategy in the third step (The second and third expressions in (4)).

In the game for (n, k, k), **H**'s strategy is clearly understood. In the ith $(i \geq 2)$ step **S** should not repeat his previous choices since, with probability of 1, **S** would fail to catch an object in $h_1, \ldots, h_{i-1}$ if **S** chooses one of $s_1, \ldots, s_{i-1}$. The reader can easily verify this in the case for $(n, 2, 2)$.

It seems to be difficult to solve the quiet accumulation game in general, i.e., in the case where the number of steps and the goal of **H** are not specified. On the other hand, it does not appear to be difficult to get a recursive relation of the value function on the number of steps, etc. But the relation will be very complicated, including many state variables.

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Appendix

The quantity in brackets on each edge in all of the figures in the Appendix is the conditional probability that the edge is reached under the behavioral strategy. Furthermore, the probability of each path is attached in brackets at each leaf.

Part 1. First we check **S**'s strategy in the cases $\mathcal{NN}$ and $\mathcal{NFN}$. Define $q(s_1) = \frac{1}{n}$ for all $s_1 \in I$. To simplify the notation, let $s_2 = s_2^N$ and $s_3 = s_3^F$. Assuming $s_1 \neq h_1$, define for $0 \leq a \leq \frac{1}{n-1}$,

$$q(s_2|s_1,\mathcal{N}) = \begin{cases} a, & \text{for } s_2 \in I \setminus \{s_1\}; \\ 1-(n-1)a, & \text{for } s_2 = s_1. \end{cases}$$

Assuming $s_1 \neq h_1$ and $s_2 \in \{h_2^N, h_1\}$ and $s_2 \neq s_1$, define

$$q(s_3|s_1,\mathcal{N},s_2,\mathcal{F}) = \begin{cases} x, & \text{for } s_3 = s_1; \\ y, & \text{for } s_3 = s_2; \\ z, & \text{for } s_3 \in I \setminus \{s_1, s_2\}, \end{cases}$$

where $x + y + (n-2)z = 1$ and $x, y, z \geq 0$. Assuming $s_1 = h_2^N$ and $s_2 = h_2^N$, define for $0 \leq d \leq \frac{1}{n-1}$,

$$q(s_3|s_1,\mathcal{N},s_1,\mathcal{F}) = \begin{cases} 1-(n-1)d, & \text{for } s_3 = s_1; \\ d, & \text{for } s_3 \in I \setminus \{s_1\}. \end{cases}$$

To simplify the notation, let $h_2 = h_2^N, h_{31} = h_3^{F1}$ and $h_{32} = h_3^{F2}$.

Case 1: h_1, h_2, h_{31} and h_{32} are all different.

$$h_1 \begin{cases}
\frac{N,[\frac{n-4}{n}]}{s_1\neq h_1,h_2,h_{31},h_{32}} \;\; h_2 \begin{cases} \frac{N,[1-2a]}{s_2\neq h_1,h_2} \;\; \left[\frac{(n-4)(1-2a)}{n}\right] \\ \frac{F,[a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-2z]}{s_3\neq h_{32},h_1} \;\; \left[\frac{(n-4)a(1-2z)}{n}\right] \\ \frac{F,[a]}{s_2=h_1} \;\; h_{31} \;\; \frac{N,[1-2z]}{s_3\neq h_{31},h_2} \;\; \left[\frac{(n-4)a(1-2z)}{n}\right] \end{cases} \\
\frac{N,[\frac{1}{n}]}{s_1=h_2} \;\; h_2 \begin{cases} \frac{N,[(n-2)a]}{s_2\neq h_1,h_2} \;\; \left[\frac{(n-2)a}{n}\right] \\ \frac{F,[1-(n-1)a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-2d]}{s_3\neq h_{32},h_1} \;\; \left[\frac{(1-(n-1)a)(1-2d)}{n}\right] \\ \frac{F,[a]}{s_2=h_1} \;\; h_{31} \;\; \frac{N,[1-x-z]}{s_3\neq h_{31},h_2} \;\; \left[\frac{a(1-x-z)}{n}\right] \end{cases} \\
\frac{N,[\frac{1}{n}]}{s_1=h_{32}} \;\; h_2 \begin{cases} \frac{N,[1-2a]}{s_2\neq h_1,h_2} \;\; \left[\frac{1-2a}{n}\right] \\ \frac{F,[a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-x-z]}{s_3\neq h_{32},h_1} \;\; \left[\frac{a(1-x-z)}{n}\right] \\ \frac{F,[a]}{s_2=h_1} \;\; h_{31} \;\; \frac{N,[1-2z]}{s_3\neq h_{31},h_2} \;\; \left[\frac{a(1-2z)}{n}\right] \end{cases} \\
\frac{N,[\frac{1}{n}]}{s_1=h_{31}} \;\; h_2 \begin{cases} \frac{N,[1-2a]}{s_2\neq h_1,h_2} \;\; \left[\frac{1-2a}{n}\right] \\ \frac{F,[a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-2z]}{s_3\neq h_{32},h_1} \;\; \left[\frac{a(1-2z)}{n}\right] \\ \frac{F,[a]}{s_2=h_1} \;\; h_{31} \;\; \frac{N,[1-x-z]}{s_3\neq h_{31},h_2} \;\; \left[\frac{a(1-x-z)}{n}\right] \end{cases}
\end{cases}$$

Figure 3 : Seeker's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 1.

We denote by $P_1^1(a,x,y,d)$ the expected payoff to $\mathbf{H}$: From Figure 3,

$$P_1^1(a,x,y,d) \equiv \frac{1}{n}\{(n-2)(1-a)+2(n-3)a(1-2z)+3a(1-x-z)+[1-(n-1)a](1-2d)\}.$$

Case 2: Assume $h_{31}=h_1$. h_1, h_2 and h_{32} are different.

$$
h_1 \begin{cases}
\frac{N,[\frac{n-3}{n}]}{s_1\neq h_1,h_2,h_{32}} \;\; h_2 \begin{cases}
\frac{N,[1-2a]}{s_2\neq h_1,h_2} \;\; [\frac{(n-3)(1-2a)}{n}] \\
\frac{F,[a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-2z]}{s_3\neq h_{32},h_1} \;\; [\frac{(n-3)a(1-2z)}{n}] \\
\frac{F,[a]}{s_2=h_1} \;\; h_1 \;\; \frac{N,[1-y-z]}{s_3\neq h_1,h_2} \;\; [\frac{(n-3)a(1-y-z)}{n}]
\end{cases} \\
\frac{N,[\frac{1}{n}]}{s_1=h_2} \;\; h_2 \begin{cases}
\frac{N,[(n-2)a]}{s_2\neq h_1,h_2} \;\; [\frac{(n-2)a}{n}] \\
\frac{F,[1-(n-1)a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-2d]}{s_3\neq h_{32},h_1} \;\; [\frac{(1-(n-1)a)(1-2d)}{n}] \\
\frac{F,[a]}{s_2=h_1} \;\; h_1 \;\; \frac{N,[(n-2)z]}{s_3\neq h_1,h_2} \;\; [\frac{a(n-2)z}{n}]
\end{cases} \\
\frac{N,[\frac{1}{n}]}{s_1=h_{32}} \;\; h_2 \begin{cases}
\frac{N,[1-2a]}{s_2\neq h_1,h_2} \;\; [\frac{1-2a}{n}] \\
\frac{F,[a]}{s_2=h_2} \;\; h_{32} \;\; \frac{N,[1-x-z]}{s_3\neq h_{32},h_1} \;\; [\frac{a(1-x-z)}{n}] \\
\frac{F,[a]}{s_2=h_1} \;\; h_1 \;\; \frac{N,[1-y-z]}{s_3\neq h_1,h_2} \;\; [\frac{a(1-y-z)}{n}]
\end{cases}
\end{cases}
$$

Figure 4 : Seeker's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 2.

We denote by $P_2^1(a,x,y,d)$ the expected payoff to $\mathbf{H}$: From Figure 4,

$$
\begin{aligned}
P_2^1(a,x,y,d) \;\; \equiv \;\; & \frac{1}{n}\{(n-2)(1-a)+(n-2)a(1-y-z) \\
& +a(1-x-z)+[1-(n-1)a](1-2d)+a[n-3-(n-4)z]\}.
\end{aligned}
$$

Case 3: Assume $h_{32}=h_2$. h_1, h_2 and h_{31} are different.

$$
h_1 \left\{ \begin{array}{l}
\frac{N,[\frac{n-3}{n}]}{s_1 \neq h_1, h_2, h_{31}} \quad h_2 \left\{ \begin{array}{l}
\frac{N,[1-2a]}{s_2 \neq h_1, h_2} \quad \left[\frac{(n-3)(1-2a)}{n}\right] \\
\frac{F,[a]}{s_2 = h_2} \quad h_2 \quad \frac{N,[1-y-z]}{s_3 \neq h_1, h_2} \quad \left[\frac{(n-3)a(1-y-z)}{n}\right] \\
\frac{F,[a]}{s_2 = h_1} \quad h_{31} \quad \frac{N,[1-2z]}{s_3 \neq h_{31}, h_2} \quad \left[\frac{(n-3)a(1-2z)}{n}\right]
\end{array} \right. \\
\frac{N,[\frac{1}{n}]}{s_1 = h_2} \quad h_2 \left\{ \begin{array}{l}
\frac{N,[(n-2)a]}{s_2 \neq h_1, h_2} \quad \left[\frac{(n-2)a}{n}\right] \\
\frac{F,[1-(n-1)a]}{s_2 = h_2} \quad h_2 \quad \frac{N,[(n-2)d]}{s_3 \neq h_1, h_2} \quad \left[\frac{(1-(n-1)a)(n-2)d}{n}\right] \\
\frac{F,[a]}{s_2 = h_1} \quad h_{31} \quad \frac{N,[1-x-z]}{s_3 \neq h_{31}, h_2} \quad \left[\frac{a(1-x-z)}{n}\right]
\end{array} \right. \\
\frac{N,[\frac{1}{n}]}{s_1 = h_{31}} \quad h_2 \left\{ \begin{array}{l}
\frac{N,[1-2a]}{s_2 \neq h_1, h_2} \quad \left[\frac{1-2a}{n}\right] \\
\frac{F,[a]}{s_2 = h_2} \quad h_2 \quad \frac{N,[1-y-z]}{s_3 \neq h_1, h_2} \quad \left[\frac{a(1-y-z)}{n}\right] \\
\frac{F,[a]}{s_2 = h_1} \quad h_{31} \quad \frac{N,[1-x-z]}{s_3 \neq h_{31}, h_2} \quad \left[\frac{a(1-x-z)}{n}\right]
\end{array} \right.
\end{array} \right.
$$

Figure 5 : Seeker's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 3.

We denote by $P_3^1(a,x,y,d)$ the expected payoff to **H** : From Figure 5,

$$
\begin{aligned}
P_3^1(a,x,y,d) \;\equiv\; & \frac{1}{n}\{(n-2)(1-a)+(n-2)a(1-y-z) \\
& +2a(1-x-z)+[1-(n-1)a](1-2d)+a(n-3)(1-2z)\}.
\end{aligned}
$$

Case 4: Assume $h_{32} = h_2$ and $h_{31} = h_1$ and $h_1 \neq h_2$.

$$
h_1 \left\{ \begin{array}{l}
\frac{N,[\frac{n-2}{n}]}{s_1 \neq h_1, h_2} \quad h_2 \left\{ \begin{array}{l}
\frac{N,[1-2a]}{s_2 \neq h_1, h_2} \quad \left[\frac{(n-2)(1-2a)}{n}\right] \\
\frac{F,[a]}{s_2 = h_2} \quad h_2 \quad \frac{N,[1-y-z]}{s_3 \neq h_1, h_2} \quad \left[\frac{(n-2)a(1-y-z)}{n}\right] \\
\frac{F,[a]}{s_2 = h_1} \quad h_1 \quad \frac{N,[1-y-z]}{s_3 \neq h_1, h_2} \quad \left[\frac{(n-2)a(1-y-z)}{n}\right]
\end{array} \right. \\
\frac{N,[\frac{1}{n}]}{s_1 = h_2} \quad h_2 \left\{ \begin{array}{l}
\frac{N,[(n-2)a]}{s_2 \neq h_1, h_2} \quad \left[\frac{(n-2)a}{n}\right] \\
\frac{F,[1-(n-1)a]}{s_2 = h_2} \quad h_2 \quad \frac{N,[(n-2)d]}{s_3 \neq h_1, h_2} \quad \left[\frac{(1-(n-1)a)(n-2)d}{n}\right] \\
\frac{F,[a]}{s_2 = h_1} \quad h_1 \quad \frac{N,[1-x-y]}{s_3 \neq h_1, h_2} \quad \left[\frac{a(1-x-y)}{n}\right]
\end{array} \right.
\end{array} \right.
$$

Figure 6 : Seeker's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 4.

We denote by $P_4^1(a,x,y,d)$ the expected payoff to **H** : From Figure 6,

$$
P_4^1(a,x,y,d) \equiv \frac{1}{n}\{(n-2)(1-a)+(n-2)a(1-y-z)+a(n-2)z+[1-(n-1)a](1-2d)\}.
$$

Let $a = \frac{1}{n-1}$, $x = 0$, $y = \frac{1}{n}$, $z = \frac{n-1}{n(n-2)}$ and let d be number. Then we have

$$P_2^1(\frac{1}{n-1},0,\frac{1}{n},d) \le P_1^1(\frac{1}{n-1},0,\frac{1}{n},d) = \frac{(n-2)^2}{{}_nP_2} + \frac{1}{{}_nP_2}\{n-2+\frac{(n-1)(n-3)^2}{n(n-2)}\},$$

$$P_4^1(\frac{1}{n-1},0,\frac{1}{n},d) \le P_3^1(\frac{1}{n-1},0,\frac{1}{n},d) < \frac{(n-2)^2}{{}_nP_2} + \frac{1}{{}_nP_2}\{n-2+\frac{(n-1)(n-3)^2}{n(n-2)}\}.$$

Part 2. Next we check **S**'s strategy in the case $\mathcal{FNN}$. Define $q(s_1) = \frac{1}{n}$ for all $s_1 \in I$. To simplify the notation, let $s_2 = s_2^F$ and $s_3 = s_3^N$. Define

$$q(s_2|s_1,\mathcal{F}) = \begin{cases} 1-(n-1)b, & \text{for } s_2 = s_1; \\ b, & \text{for } s_2 \in I \setminus \{s_1\}, \end{cases}$$

and

$$q(s_3|s_1,\mathcal{F},s_2,\mathcal{N}) = \begin{cases} e, & \text{for } s_3 = s_1; \\ 1-(n-2)f-e, & \text{for } s_3 = s_2; \\ f, & \text{for } s_3 \in I \setminus \{s_1,s_2\}, \end{cases}$$

where $0 \le b \le \frac{1}{n-1}$ and $0 \le (n-2)f+e \le 1$, $f,e \ge 0$. To simplify the notation, let $h_2 = h_2^F$ and $h_3 = h_3^{F1}$.
Case 1: h_1, h_2 and h_3 are different.

$$h_1 \xrightarrow[s_1=h_1]{F,[\frac{1}{n}]} h_2 \begin{cases} \xrightarrow[s_2\neq h_2,h_3]{N,[1-2b]} h_3 \xrightarrow[s_3\neq h_3,h_2]{N,[1-2f]} [\frac{(1-2b)(1-2f)}{n}] \\ \xrightarrow[s_2=h_3]{N,[b]} h_3 \xrightarrow[s_3\neq h_3,h_2]{N,[(n-3)f+e]} [\frac{b((n-3)f+e)}{n}] \end{cases}$$

Figure 7 : Seeker's strategy in $\mathcal{FNN}$: Case 1.

We denote by $P_1^2(b,e)$ the expected payoff to **H** : From Figure 7,

$$P_1^2(b,e) \equiv \frac{1}{n}\{b[(n-3)f+e]+(1-2b)(1-2f)\}.$$

Case 2: Assume $h_3 = h_1$ and $h_1 \neq h_2$.

$$h_1 \xrightarrow[s_1=h_1]{F,[\frac{1}{n}]} h_2 \xrightarrow[s_2\neq h_2]{N,[1-b]} h_1 \xrightarrow[s_3\neq h_1,h_2]{N,[1-f-e]} [\frac{(1-b)(1-f-e)}{n}]$$

Figure 8 : Seeker's strategy in $\mathcal{FNN}$: Case 2.

We denote by $P_2^2(b,e)$ the expected payoff to **H** : From Figure 8, $P_2^2(b,e) \equiv \frac{1}{n}(1-b)(1-f-e)$.
Case 3: Assume $h_2 = h_1$ and $h_1 \neq h_3$.

$$h_1 \xrightarrow[s_1=h_1]{F,[\frac{1}{n}]} h_1 \begin{cases} \xrightarrow[s_2\neq h_1,h_3]{N,[(n-2)b]} h_3 \xrightarrow[s_3\neq h_1,h_3]{N,[1-e-f]} [\frac{(n-2)b(1-e-f)}{n}] \\ \xrightarrow[s_2=h_3]{N,[b]} h_3 \xrightarrow[s_3\neq h_1,h_3]{N,[(n-2)f]} [\frac{b(n-2)f}{n}] \end{cases}$$

Figure 9 : Seeker's strategy in $\mathcal{FNN}$: Case 3.

We denote by $P_3^2(b,e)$ the expected payoff to **H** : From Figure 9, $P_3^2(b,e) \equiv \frac{1}{n}(n-2)b(1-e)$.Let $b = \frac{1}{n}$ and $e = f = \frac{1}{n-1}$. Then $P_1^2(\frac{1}{n},\frac{1}{n-1}) = P_3^2(\frac{1}{n},\frac{1}{n-1}) = \frac{(n-2)^2}{n^2(n-1)}$ is the maximum. Note that $P_2^2(\frac{1}{n},\frac{1}{n-1}) = \frac{(n-1)(n-3)}{n^2(n-1)} \le P_3^2(\frac{1}{n},\frac{1}{n-1})$.

Part 3. Next we check **H**'s strategy in the case $\mathcal{FNN}$. To simplify the notation, we let $h_2 = h_2^F$ and $h_3 = h_3^N$. Define $p(h_1) = \frac{1}{n}$ for all $h_1 \in I$. And for $0 \le a \le \frac{1}{n-1}$ and $0 \le b \le \frac{1}{n-2}$,

$$p(h_2|h_1,\mathcal{F}) = \begin{cases} 1-(n-1)a, & \text{for } h_2 = h_1 (= s_1); \\ a, & \text{for } h_2 \in I \setminus \{h_1\}, \end{cases}$$

$$p(h_3|h_1,\mathcal{F},h_2,\mathcal{N}) = \begin{cases} 1-(n-2)b, & \text{for } h_3 = h_1 (= s_1); \\ b, & \text{for } h_3 \in I \setminus \{h_1,h_2\}. \end{cases}$$

To simplify the notation, let $s_2 = s_2^F$ and $s_3 = s_3^N$.
Case 1: $s_1 \neq s_2, s_2 \neq s_3$ and $s_3 \neq s_1$.

$$s_1 \ \frac{F,[\frac{1}{n}]}{h_1 = s_1} \ s_2 \begin{cases} \frac{N,[1-(n-1)a]}{h_2 = s_1} \ s_3 \ \frac{N,[1-b]}{h_3 \neq h_2, s_3} \ [\frac{(1-(n-1)a)(1-b)}{n}] \\ \frac{N,[(n-3)a]}{h_2 \neq s_1,s_2,s_3} \ s_3 \ \frac{N,[1-b]}{h_3 \neq s_3, h_2} \ [\frac{(n-3)a(1-b)}{n}] \end{cases}$$

Figure 10 : Hider's strategy in $\mathcal{FNN}$: Case 1.

We denote by $P_1^3(a,b)$ the expected payoff to **H**: From Figure 10, $P_1^3(a,b) \equiv \frac{1}{n}\{(n-3)a(1-b) + [1-(n-1)a]\frac{n-2}{n-1}\}$.
Case 2: Assume $s_3 = s_2$ and $s_1 \neq s_2$.

$$s_1 \ \frac{F,[\frac{1}{n}]}{h_1 = s_1} \ s_2 \begin{cases} \frac{N,[1-(n-1)a]}{h_2 = s_1} \ s_2 \ \frac{N,[1-b]}{h_3 \neq h_2, s_2} \ [\frac{(1-(n-1)a)(1-b)}{n}] \\ \frac{N,[(n-2)a]}{h_2 \neq s_1,s_2} \ s_2 \ \frac{N,[1-b]}{h_3 \neq h_2, s_2} \ [\frac{(n-2)a(1-b)}{n}] \end{cases}$$

Figure 11 : Hider's strategy in $\mathcal{FNN}$: Case 2.

We denote by $P_2^3(a,b)$ the expected payoff to **H**: From Figure 11, $P_2^3(a,b) \equiv \frac{1}{n}\{(n-2)a(1-b) + [1-(n-1)a]\frac{n-2}{n-1}\}$.
Case 3: Assume $s_3 = s_1$ and $s_1 \neq s_2$.

$$s_1 \ \frac{F,[\frac{1}{n}]}{h_1 = s_1} \ s_2 \ \frac{N,[(n-2)a]}{h_2 \neq s_2, s_1} \ s_1 \ \frac{N,[(n-2)b]}{h_3 \neq h_2, s_1} \ [\frac{(n-2)^2 ab}{n}]$$

Figure 12 : Hider's strategy in $\mathcal{FNN}$: Case 3.

We denote by $P_3^3(a,b)$ the expected payoff to **H**: From Figure 12, $P_3^3(a,b) \equiv \frac{1}{n}(n-2)^2 ab$.
Case 4: Assume $s_2 = s_1$ and $s_3 \neq s_2$.

$$s_1 \ \frac{F,[\frac{1}{n}]}{h_1 = s_1} \ s_1 \ \frac{N,[(n-2)a]}{h_2 \neq s_3, s_1} \ s_3 \ \frac{N,[1-b]}{h_3 \neq h_2, s_3} \ [\frac{(n-2)a(1-b)}{n}]$$

Figure 13 : Hider's strategy in $\mathcal{FNN}$: Case 4.

We denote by $P_4^3(a,b)$ the expected payoff to **H**: From Figure 13, $P_4^3(a,b) \equiv \frac{1}{n}(n-2)a(1-b)$.
Case 5: Assume $s_2 = s_1$ and $s_3 = s_1$.

$$s_1 \ \frac{F,[\frac{1}{n}]}{h_1 = s_1} \ s_2 \ \frac{N,[(n-1)a]}{h_2 \neq s_1} \ s_1 \ \frac{N,[(n-2)b]}{h_3 \neq s_1, h_2} \ [\frac{(n-1)(n-2)ab}{n}]$$

Figure 14 : Hider's strategy in $\mathcal{FNN}$: Case 5.

We denote by $P_5^3(a,b)$ the expected payoff to $\mathbf{H}$: From Figure 14, $P_5^3(a,b) \equiv \frac{1}{n}(n-2)(n-1)ab$. Let $a = \frac{1}{n}$ and $b = \frac{1}{n-1}$. Then the minimum value is $\frac{(n-2)^2}{n^2(n-1)}$.

Part 4. Next we check $\mathbf{H}$'s strategy in the cases $\mathcal{NFN}$ and $\mathcal{NN}$. To simplify the notation, we let $h_2 = h_2^N$ and $h_{32} = h_3^{F2}$ and $h_{31} = h_3^{F1}$. Define

$$p(h_2|h_1,\mathcal{N}) = \begin{cases} \frac{1}{n-1}, & \text{for } h_2 \in I \setminus \{h_1\}; \\ 0, & \text{for } h_2 = h_1, \end{cases}$$

$$p(h_{31}|h_1,\mathcal{N},h_2,\mathcal{F}1) = \begin{cases} 0, & \text{for } h_{31} = h_2; \\ 1-(n-2)d, & \text{for } h_{31} = h_1; \\ d, & \text{for } h_{31} \in I \setminus \{h_1,h_2\}, \end{cases}$$

$$p(h_{32}|h_1,\mathcal{N},h_2,\mathcal{F}2) = \begin{cases} 0, & \text{for } h_{32} = h_1; \\ 1-(n-2)c, & \text{for } h_{32} = h_2; \\ c, & \text{for } h_{32} \in I \setminus \{h_1,h_2\}, \end{cases}$$

where $0 \le c,d \le \frac{1}{n-2}$. To simplify the notation, let $s_2 = s_2^N$ and $s_3 = s_3^F$.

Case 1: Assume s_1, s_2 and s_3 are different.

$$s_1 \begin{cases} \frac{N,[\frac{n-3}{n}]}{h_1\neq s_1,s_2,s_3} \quad s_2 \begin{cases} \frac{N,[\frac{n-2}{n-1}]}{h_2\neq h_1,s_2} \quad \left[\frac{(n-3)(n-2)}{n(n-1)}\right] \\ \frac{F,[\frac{1}{n-1}]}{h_2=s_2} \quad s_3 \quad \frac{N,[1-c]}{h_{32}\neq h_1,s_3} \quad \left[\frac{(n-3)(1-c)}{n(n-1)}\right] \end{cases} \\ \frac{N,[\frac{1}{n}]}{h_1=s_2} \quad s_2 \quad \frac{F,[\frac{n-2}{n-1}]}{h_2\neq h_1,s_3} \quad s_3 \quad \frac{N,[1-d]}{h_{31}\neq h_2,s_3} \quad \left[\frac{(n-2)(1-d)}{n(n-1)}\right] \\ \frac{N,[\frac{1}{n}]}{h_1=s_3} \quad s_2 \quad \frac{N,[\frac{n-2}{n-1}]}{h_2\neq h_1,s_2} \quad \left[\frac{n-2}{n(n-1)}\right] \end{cases}$$

Figure 15 : Hider's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 1.

We denote by $P_1^4(c,d)$ the expected payoff to $\mathbf{H}$: From Figure 15, $P_1^4(c,d) \equiv \frac{1}{n(n-1)}\{(n-3)(1-c)+(n-2)(1-d)+(n-2)^2\}$.

Case 2: Assume $s_3 = s_2$ and $s_2 \neq s_1$.

$$s_1 \begin{cases} \frac{N,[\frac{n-2}{n}]}{h_1\neq s_1,s_2} \quad s_2 \begin{cases} \frac{N,[\frac{n-2}{n-1}]}{h_2\neq h_1,s_2} \quad \left[\frac{(n-2)^2}{n(n-1)}\right] \\ \frac{F,[\frac{1}{n-1}]}{h_2=s_2} \quad s_2 \quad \frac{N,[(n-2)c]}{h_{32}\neq h_1,s_2} \quad \left[\frac{(n-2)^2c}{n(n-1)}\right] \end{cases} \\ \frac{N,[\frac{1}{n}]}{h_1=s_2} \quad s_2 \quad \frac{F,[1]}{h_2\neq h_1} \quad s_2 \quad \frac{N,[(n-2)d]}{h_{31}\neq h_2,s_2} \quad \left[\frac{(n-2)d}{n}\right] \end{cases}$$

Figure 16 : Hider's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 2.

We denote by $P_2^4(c,d)$ the expected payoff to $\mathbf{H}$: From Figure 16, $P_2^4(c,d) \equiv \frac{1}{n(n-1)}\{(n-2)^2c+(n-1)(n-2)d+(n-2)^2\}$.

Case 3: Assume $s_1 = s_2$ and $s_3 \neq s_1$.

$$s_1 \begin{cases} \dfrac{N,[\frac{n-2}{n}]}{h_1 \neq s_1, s_3} \quad s_1 \begin{cases} \dfrac{N,[\frac{n-2}{n-1}]}{h_2 \neq s_1, h_1} \quad [\frac{(n-2)^2}{n(n-1)}] \\ \dfrac{F,[\frac{1}{n-1}]}{h_2 = s_1} \quad s_3 \quad \dfrac{N,[1-c]}{h_{32} \neq s_3, h_1} \quad [\frac{(n-2)(1-c)}{n(n-1)}] \end{cases} \\ \\ \dfrac{N,[\frac{1}{n}]}{h_1 = s_3} \quad s_1 \quad \dfrac{N,[\frac{n-2}{n-1}]}{h_2 \neq s_1, h_1} \quad [\frac{n-2}{n(n-1)}] \end{cases}$$

Figure 17 : Hider's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 3.

We denote by $P_3^4(c,d)$ the expected payoff to **H**: From Figure 17, $P_3^4(c,d) \equiv \frac{1}{n(n-1)}\{(n-2)(1-c) + (n-1)(n-2)\}$.

Case 4: Assume $s_1 = s_3$ and $s_2 \neq s_1$.

$$s_1 \begin{cases} \dfrac{N,[\frac{n-2}{n}]}{h_1 \neq s_1, s_2} \quad s_2 \begin{cases} \dfrac{N,[\frac{n-2}{n-1}]}{h_2 \neq h_1, s_2} \quad [\frac{(n-2)^2}{n(n-1)}] \\ \dfrac{F,[\frac{1}{n-1}]}{h_2 = s_2} \quad s_1 \quad \dfrac{N,[1-c]}{h_{32} \neq h_1, s_1} \quad [\frac{(n-2)(1-c)}{n(n-1)}] \end{cases} \\ \\ \dfrac{N,[\frac{1}{n}]}{h_1 = s_2} \quad s_2 \quad \dfrac{F,[\frac{n-2}{n-1}]}{h_2 \neq s_2, s_1} \quad s_1 \quad \dfrac{N,[1-d]}{h_{31} \neq h_2, s_1} \quad [\frac{(n-2)(1-d)}{n(n-1)}] \end{cases}$$

Figure 18 : Hider's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 4.

We denote by $P_4^4(c,d)$ the expected payoff to **H**: From Figure 18, $P_4^4(c,d) \equiv \frac{1}{n(n-1)}\{(n-2)(1-c) + (n-2)(1-d) + (n-2)^2\}$.

Case 5: Assume $s_1 = s_2 = s_3$.

$$s_1 \quad \dfrac{N, [\frac{n-1}{n}]}{h_1 \neq s_1} \quad s_1 \begin{cases} \dfrac{N,[\frac{n-2}{n-1}]}{h_2 \neq s_1, h_1} \quad [\frac{(n-1)(n-2)}{n(n-1)}] \\ \dfrac{F,[\frac{1}{n-1}]}{h_2 = s_1} \quad s_1 \quad \dfrac{N,[(n-2)c]}{h_{32} \neq h_1, s_1} \quad [\frac{(n-1)(n-2)c}{n(n-1)}] \end{cases}$$

Figure 19 : Hider's strategy in $\mathcal{NN}$ and $\mathcal{NFN}$: Case 5.

We denote by $P_5^4(c,d)$ the expected payoff to **H**: From Figure 19, $P_5^4(c,d) \equiv \frac{1}{n(n-1)}\{(n-2)(n-1)c + (n-2)(n-1)\}$. Let $c = \frac{1}{n-2}$ and $d = \frac{(n-3)^2}{n(n-2)^2}$. The minimum value is $\frac{(n-2)^2}{{}_nP_2} + \frac{1}{{}_nP_2}\{n - 2 + \frac{(n-1)(n-3)^2}{n(n-2)}\}$.

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