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A NEW CHARACTERIZATION OF M$^\natural$-CONVEX SET FUNCTIONS BY SUBSTITUTABILITY

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Abstract The concepts of M-convex functions and M$^\natural$-convex functions play central roles in the theory of discrete convex analysis which has been applied to mathematical economics. On the other hand, substitutability, which is a key property guaranteeing the existence of a stable matching in generalized stable marriage models, is known as a nice property in mathematical economics. In this paper, we introduce new properties, which are extensions of substitutability, and present new characterizations of M$^\natural$-convex set functions by these properties.

Keywords: Discrete optimization, game theory

1. Introduction

Discrete convex analysis, proposed by Murota [11, 12], is a unified framework of discrete optimization (see [13, 14] for details). The concepts of M-convex functions due to Murota [11, 12] and M$^\natural$-convex functions due to Murota and Shioura [15] are considered as a backbone in discrete convex analysis, and have close relations to nice properties in mathematical economics, called the gross substitutability and the single improvement property. Under the gross substitutability, Kelso and Crawford [9] proposed a matching model with money and showed the nonemptiness of the core of their model. The single improvement property was first introduced by Gul and Stacchetti [8], and the equivalence between these two properties was verified for set functions by them. Furthermore, the equivalence between the single improvement property and M$^\natural$-concavity for set functions was shown by Fujishige and Yang [7]. Relations among these three properties are extended to the general case by Danilov, Koshevoy and Lang [2] and by Murota and Tamura [16]. This guarantees that an M$^\natural$-concave function has nice features as a utility function from the point of view of mathematical economics. In fact, several economic models utilizing M$^\natural$-concave utility functions have been proposed. Danilov, Koshevoy and Murota [3] showed the existence of a competitive equilibrium in an exchange economy with indivisible commodities when the utility function of each agent is quasilinear in money and its reservation value function is M$^\natural$-concave. Murota and Tamura [17] proposed an efficient algorithm for finding a competitive equilibrium of the Danilov-Koshevoy-Murota model. B. Lehmann, D. Lehmann and Nisan [10] discussed a combinatorial auction with M$^\natural$-concave utilities. Eguchi and Fujishige [4] extended the stable marriage model to the framework of discrete convex analysis. Fujishige and Tamura [6] proposed a common generalization of the stable marriage model and the

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assignment game model by utilizing M$^\natural$-concave utilities and verified the existence of a stable solution in their general model.

On the other hand, "substitutability" is a key property guaranteeing the existence of a stable matching in generalized stable marriage models in terms of choice functions by Roth [18], Sotomayor [19], Alkan and Gale [1] and Fleiner [5], who defined "substitutability" in distinct manners and showed the existence of stable matchings of their models.

In this paper, we define two properties as variations of "substitutability." Let V be a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. We denote by $\mathbf{Z}^V$ the set of integral vectors $x = (x(v) : v \in V)$ indexed by V, where $x(v)$ denotes the v-component of vector x. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $U \subseteq \mathbf{Z}^V$, the set of minimizers of f on U is defined by

$$\arg\min\{f(y) \mid y \in U\} = \{x \in U \mid \forall y \in U : \ f(x) \leq f(y)\}.$$

The two properties are described as follows, where $z \wedge x$ for two vectors z and x denotes the vector whose v-component $(z \wedge x)(v)$ is given by $\min\{z(v), x(v)\}$.

(SC^1) For any $z_1, z_2 \in \mathbf{Z}^V$ such that $z_1 \geq z_2$ and $\arg\min\{f(y) \mid y \leq z_2\} \neq \emptyset$, if $x_1 \in \arg\min\{f(y) \mid y \leq z_1\}$, then there exists x_2 such that

$$x_2 \in \arg\min\{f(y) \mid y \leq z_2\}, \quad z_2 \wedge x_1 \leq x_2.$$

(SC^2) For any $z_1, z_2 \in \mathbf{Z}^V$ such that $z_1 \geq z_2$ and $\arg\min\{f(y) \mid y \leq z_1\} \neq \emptyset$, if $x_2 \in \arg\min\{f(y) \mid y \leq z_2\}$, then there exists x_1 such that

$$x_1 \in \arg\min\{f(y) \mid y \leq z_1\}, \quad z_2 \wedge x_1 \leq x_2.$$

These properties are interpreted as follows. Here, we assume that V denotes the set of indivisible commodities, $x \in \mathbf{Z}^V$ the numbers $x(v)$ of commodities v produced by a producer and f a cost function of the producer. (SC^1) says that when the producible quota of each commodity decreases or remains the same, the producer wants a production such that the numbers of the commodities whose quotas remain the same do not decrease. (SC^2) says that when each quota increases or remains the same, the producer wants a production such that the numbers of the commodities which fail to fill the original quotas do not increase. If f is a set function (i.e., is defined on the hypercube $\{0,1\}^V$) then (SC^1) and (SC^2) are equivalent to conditions of substitutability in Sotomayor [19, Definition 4], and if arg min always gives a singleton (in this case (SC^1) and (SC^2) coincide) then these are equivalent to persistence (substitutability) in Alkan and Gale [1]. In the model in [4] and the restricted version of that in [6], the concave-versions[1] of (SC^1) and (SC^2) are key properties certifying the existence of a stable matching, because M$^\natural$-convexity implies (SC^1) and (SC^2) (see Lemma 3.1 in Section 3). The model in [4] can be recognized as a concrete example (in terms of utility functions) of the models in [1, 5, 18, 19] (in terms of choice functions with "substitutability").

This work is motivated by the above recent results on generalized stable marriage models with M$^\natural$-concavity or substitutability. Our aim is to investigate relations between M$^\natural$-convexity and substitutability. This paper gives two examples showing the independence between (SC^1) and (SC^2), and furthermore, introduces two strengthened properties (SC$^1_{\rm G}$) and (SC$^2_{\rm G}$) of (SC^1) and (SC^2). Our main contribution is to show the equivalence among (SC$^1_{\rm G}$), (SC$^2_{\rm G}$) and M$^\natural$-convexity for set functions.

[1]Concave-versions are obtained by replacing "arg min" with "arg max."

2. M$^\natural$-Convexity

In this section, we review some definitions and known results about M$^\natural$-convex functions. For a vector $z \in \mathbf{Z}^V$, we define the *positive support* and the *negative support* of z by

$$\mathrm{supp}^+(z) = \{v \in V \mid z(v) > 0\}, \quad \mathrm{supp}^-(z) = \{v \in V \mid z(v) < 0\}.$$

The *characteristic vector* χ_S of $S \subseteq V$ is defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \in V \setminus S, \end{cases}$$

where χ_u is used instead of $\chi_{\{u\}}$ for each $u \in V$. For convenience, χ_0 is defined by the zero vector on V. We define the *effective domain* of a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\mathrm{dom} f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called *M$^\natural$-convex* [15] if it satisfies the following condition:

(M$^\natural$-EXC) For $x, y \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

The following property, which is a generalized version of the gross substitutability in [9], is introduced for a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ in [16].

(M$^\natural$-GS$_\mathrm{W}$) For $p, q \in \mathbf{R}^V$ and $x \in \arg\min f[p]$ with $p \leq q$ and $\arg\min f[q] \neq \emptyset$, there exists $y \in \arg\min f[q]$ such that

$$p(v) = q(v) \implies y(v) \geq x(v),$$

where $f[p]$ denotes the function in $x \in \mathbf{Z}^V$ defined by

$$f[p](x) = f(x) + \sum_{v \in V} p(v)x(v).$$

(M$^\natural$-GS$_\mathrm{W}$) is equivalent to M$^\natural$-convexity for set functions.

Theorem 2.1 ([7, 16]) *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\emptyset \neq \mathrm{dom} f \subseteq \{0,1\}^V$ is M$^\natural$-convex if and only if it satisfies* (M$^\natural$-GS$_\mathrm{W}$).

3. New Characterizations of M$^\natural$-Convexity

In this section, we give two strengthened properties (SC$^1_\mathrm{G}$) and (SC$^2_\mathrm{G}$) of (SC^1) and (SC^2) and present new characterizations of M$^\natural$-convexity for set functions by these new properties.

Before defining (SC$^1_\mathrm{G}$) and (SC$^2_\mathrm{G}$), we give examples showing the independence between (SC^1) and (SC^2).

Example 3.1 *Define $f : \{0,1\}^2 \to \mathbf{R}$ by*

$$f(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ 1 & \text{if } x = (1,0) \\ 1 & \text{if } x = (0,1) \\ 0 & \text{if } x = (1,1). \end{cases}$$

We can easily check that f satisfies (SC^1). *However, it does not satisfy* (SC^2) *for $z_1 = (1,1)$ and $z_2 = (0,1)$.*

Example 3.2 *Define* $f : \{0,1\}^2 \to \mathbf{R}$ *by*

$$f(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ 0 & \text{if } x = (1,0) \\ 2 & \text{if } x = (0,1) \\ 0 & \text{if } x = (1,1). \end{cases}$$

We can easily check that f *satisfies* (SC^2)*. However, it does not satisfy* (SC^1) *for* $z_1 = (1,1)$ *and* $z_2 = (0,1)$.

Fujishige and Tamura [6] have shown that $\mathrm{M}^\natural$-convex functions satisfy (SC^1) and (SC^2) (for self-containment, we give a proof of their assertion).

Lemma 3.1 ([6]) *An* $M^\natural$*-convex function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *whose effective domain is bounded satisfies* (SC^1) *and* (SC^2).

Proof. Here we show that f satisfies (SC^1). Similarly we can show that it satisfies (SC^2). Let $x_1 \in \arg\min\{f(y) \mid y \leq z_1\}$. We choose $x_2 \in \arg\min\{f(y) \mid y \leq z_2\}$ such that it minimizes $\sum\{x_1(w) - x_2(w) \mid w \in \mathrm{supp}^+((x_1 \wedge z_2) - x_2)\}$. Assume to the contrary that $\mathrm{supp}^+((x_1 \wedge z_2) - x_2) \neq \emptyset$. Let $u \in \mathrm{supp}^+((x_1 \wedge z_2) - x_2)$, that is, $u \in \mathrm{supp}^+(x_1 - x_2)$. By $(\mathrm{M}^\natural\text{-EXC})$, there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \geq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v). \tag{3.1}$$

If $v \neq 0$ then we have $x_1(v) < x_2(v) \leq z_2(v) \leq z_1(v)$, and hence, $x_1 - \chi_u + \chi_v \leq z_1$; otherwise, obviously $x_1 - \chi_u + \chi_v \leq z_1$. Thus $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$, and by (3.1) $f(x_2) \geq f(x_2 + \chi_u - \chi_v)$ must hold. It follows from $x_2(u) < z_2(u)$ that $x_2 + \chi_u - \chi_v \leq z_2$, and hence, $f(x_2 + \chi_u - \chi_v) \geq f(x_2)$. Therefore, $x_2 + \chi_u - \chi_v \in \arg\min\{f(y) \mid y \leq z_2\}$, which contradicts the choice of x_2. Hence we have $x_1 \wedge z_2 \leq x_2$. ■

Unfortunately, the converse of the assertion of Lemma 3.1 does not hold, that is, $\mathrm{M}^\natural$-convexity does not follow from (SC^1) and (SC^2) as shown in the following example.

Example 3.3 *Define* $f : \{0,1\}^3 \to \mathbf{R} \cup \{+\infty\}$ *by*

$$f(x) = \begin{cases} -\sum_{i=1}^{3} 2^i x(i) & \text{if } x \leq (1,1,0) \text{ or } x \leq (0,0,1) \\ +\infty & \text{otherwise.} \end{cases}$$

We can check that f *satisfies both* (SC^1) *and* (SC^2)*, but not* $(\mathrm{M}^\natural\text{-EXC})$ *for* $x = (1,1,0)$ *and* $y = (0,0,1)$.

From the above discussion, we must consider stronger properties than (SC^1) and (SC^2) to characterize $\mathrm{M}^\natural$-convexity. For this, we introduce the following two properties of a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$.

$(\mathrm{SC}^1_\mathrm{G})$ For any $p \in \mathbf{R}^V$, $f[p]$ satisfies (SC^1).
$(\mathrm{SC}^2_\mathrm{G})$ For any $p \in \mathbf{R}^V$, $f[p]$ satisfies (SC^2).

The next lemma is a direct consequence of Lemma 3.1.

Lemma 3.2 *An* $M^\natural$*-convex function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *with a bounded effective domain satisfies* $(\mathrm{SC}^1_\mathrm{G})$ *and* $(\mathrm{SC}^2_\mathrm{G})$.

Proof. If f is $\mathrm{M}^\natural$-convex then $f[p]$ is also $\mathrm{M}^\natural$-convex for any $p \in \mathbf{R}^V$. The assertion follows from Lemma 3.1. ■

The lemma below shows that $(\mathrm{SC}^1_\mathrm{G})$ and $(\mathrm{SC}^2_\mathrm{G})$ are equivalent to each other for set functions.

Lemma 3.3 *For a function* $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$, $(\mathrm{SC}^1_{\mathrm{G}})$ *holds if and only if* $(\mathrm{SC}^2_{\mathrm{G}})$ *holds.*

Proof. $(\Longrightarrow)$ Suppose that $(\mathrm{SC}^1_{\mathrm{G}})$ holds. It is enough to show that (SC^2) holds for f. Take any $x_2 \in \arg\min\{f(y) \mid y \leq z_2\}$. Define $q = \varepsilon p \in \mathbf{R}^V$ where

$$p = \chi_V - 2x_2,$$

and

$$0 < \varepsilon < \frac{\min\{|f(x) - f(y)| \mid f(x) \neq f(y), x, y \in \mathrm{dom} f\}}{|V|}.$$

Suppose that $f(x) < f(y)$ for $x, y \in \mathrm{dom} f$. Then we have

$$\begin{aligned} f[q](x) &= f(x) + \varepsilon\langle p, x - y\rangle + \langle q, y\rangle \\ &< f(x) + (f(y) - f(x))\frac{\langle p, x - y\rangle}{|V|} + \langle q, y\rangle \\ &\leq f[q](y), \end{aligned}$$

since $\langle p, x - y\rangle/|V| \leq 1$, where $\langle q, y\rangle = \sum_{v \in V} q(v)y(v)$. Hence we have

$$f(x) < f(y) \implies f[q](x) < f[q](y) \tag{3.2}$$

for $x, y \in \mathrm{dom} f$. By the definition of p, we can show that

$$\arg\min\{f[q](y) \mid y \leq z_2\} = \{x_2\}. \tag{3.3}$$

Implication (3.2) implies $\arg\min\{f[q](y) \mid y \leq z_1\} \subseteq \arg\min\{f(y) \mid y \leq z_1\}$. By $(\mathrm{SC}^1_{\mathrm{G}})$ for f, for all $x_1 \in \arg\min\{f[q](y) \mid y \leq z_1\}$, we have $z_2 \wedge x_1 \leq x_2$ by (3.3). Since $\arg\min\{f[q](y) \mid y \leq z_1\}$ is non-empty, (SC^2) holds for f, and thus $(\mathrm{SC}^2_{\mathrm{G}})$ holds.

$(\Longleftarrow)$ Suppose that $(\mathrm{SC}^2_{\mathrm{G}})$ holds. It is enough to show that (SC^1) holds for f. Take any $x_1 \in \arg\min\{f(y) \mid y \leq z_1\}$. In the same way as above, we can define $q \in \mathbf{R}^V$ satisfying (3.2) and

$$\arg\min\{f[q](y) \mid y \leq z_1\} = \{x_1\}. \tag{3.4}$$

Similarly, we have $\arg\min\{f[q](y) \mid y \leq z_2\} \subseteq \arg\min\{f(y) \mid y \leq z_2\}$ from (3.2). By $(\mathrm{SC}^2_{\mathrm{G}})$ for f, for all $x_2 \in \arg\min\{f[q](y) \mid y \leq z_2\}$, we have $z_2 \wedge x_1 \leq x_2$ by (3.4). Since $\arg\min\{f[q](y) \mid y \leq z_2\}$ is non-empty, (SC^1) holds for f, and thus $(\mathrm{SC}^1_{\mathrm{G}})$ holds. ■

We finally show that the converse of Lemma 3.2 holds for set functions.

Lemma 3.4 *If a function* $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* $(\mathrm{SC}^1_{\mathrm{G}})$, *then it is M*$^\natural$*-convex.*

Proof. By Theorem 2.1, we only need to prove that $(\mathrm{M}^\natural\text{-GS}_{\mathrm{W}})$ is satisfied. In addition, it is enough to consider the case when $q = p + \lambda\chi_u$ for some $u \in V$ and $\lambda > 0$. Let $x \in \arg\min f[p]$. If $x \in \arg\min f[q]$ then there is nothing to prove. Assume that $x \notin \arg\min f[q]$. Then, we have $x(u) = 1$ and $y(u) = 0$ for all $y \in \arg\min f[q]$. By taking $z_1 = \chi_V$ and $z_2 = z_1 - \chi_u$, we have

$$\begin{aligned} \arg\min f[p] &= \arg\min\{f[p](y) \mid y \leq z_1\}, \\ \arg\min f[q] &= \arg\min\{f[p](y) \mid y \leq z_2\}. \end{aligned}$$

Now since $(\mathrm{SC}^1_{\mathrm{G}})$ is satisfied, for any $x \in \arg\min f[p]$, there exists $y \in \arg\min f[q]$ such that $x \wedge z_2 \leq y$. It follows from $x \wedge z_2 \leq y$ that $y(v) \geq x(v)$ for all $v \in V$ with $p(v) = q(v)$ (i.e., for all $v \neq u$). ■

By Lemmas 3.2, 3.3 and 3.4, we obtain our main result.

Theorem 3.1 *For any function* $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$, *the following conditions are equivalent:*

(a) f *is* $M^\natural$*-convex,*

(b) f *satisfies* (SC^1_G),

(c) f *satisfies* (SC^2_G).

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