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EVALUATING ALL BERTRAND-NASH EQUILIBRIA IN A DISCRETE SPATIAL DUOPOLY MODEL

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Abstract This paper studies a spatial duopoly model where customers are located at nodes and the demand functions are given for each node. For any fixed location of two firms, we analyze Bertrand-Nash equilibrium and derive a necessary and sufficient condition for the existence of equilibrium. We present an algorithm to compute all equilibria, provided profit functions have a finite number of peaks. The algorithm terminates within polynomial time if the number of peaks is polynomial in the number of nodes.

Keywords: Bertrand-Nash equilibrium, discrete model, game theory, Hotelling's duopoly model, polynomial time algorithm, spatial competition

1. Introduction

While the pricing is a key factor for a firm's success in general, it renders an interesting form of competition when the customers are spatially non-homogeneous. Examples include access points for an Internet service provider, depots for a parcel delivery firm, branches for a bank, etc., see Dewan *et al.* [3]. In this paper, we explore a price equilibrium problem in a spatial competition model.

Since Hotelling, in his celebrated 1929 paper [9], presented the duopoly model, the spatial competition model has been studied successfully in economics. Hotelling examines the following duopoly model. The firms choose locations and compete in prices. The customers are distributed uniformly on a line. They pay the linear transportation cost and necessarily buy one unit of the product from the firm with the smallest sum of price and transportation cost. He analyzes a Nash equilibrium in prices (Bertrand-Nash equilibrium) given the fixed location of firms, and concludes that the equilibrium exists when firms locate close to each other, known as the "Principle of Minimum Differentiation". D'Aspremont *et al.* [2] however show no Bertrand-Nash equilibrium exists in Hotelling's original model unless the locations of the firms are relatively far apart, and point out that his conclusion is incorrect. Many extensions of Hotelling's original model have arisen thereafter. Economides [5] extends the duopoly model to a situation where customers are distributed uniformly on a plane. He proves the existence of Bertrand-Nash equilibrium for all *symmetric* locations of firms even if they are close to each other. Under more general distribution of customers' location and demand, Thisse *et al.* [12] and Zhang and Teraoka [13] examine a spatial duopoly model where the price discrimination is allowed. Other variations are explored by Anderson [1] and Rath [11]. For the review of the literature, the readers are referred to Gabszewicz *et al.* [6].

From the operational viewpoint, it is more important to obtain numerical solutions in actual settings faced by firms while the economists analyze the basic mechanism of equilibrium in relatively simple settings. A simple continuous model may yield various qualitative results, however, a discrete model is often more appropriate to formulate a complex situation. Dobson and Karmarkar [4] study the problem of locating facilities on a network in the presence of competition where customers are distributed at a finite number of nodes and the demands of customers are constant. Marianov *et al.* [10] formulate a hub location problem on a network in a competitive environment. Their works contribute to actual decision-makings through the numerical analysis in discrete models.

In this paper, we examine the Hotelling duopoly model from the operational view. We assume that customers are distributed discretely at n nodes on a plane and the demand functions are given for each node. These assumptions are not only useful in calculations, but they seem practical in many situations because cities are located discretely and unevenly in the real world. Since Economides employs a continuous model, he obtains a result only for a symmetric case. On the other hand, our discrete model enables us to obtain a necessary and sufficient condition for the existence of Bertrand-Nash equilibrium in a general setting. Furthermore we present an algorithm determining whether or not equilibrium exist and finding *all* equilibrium prices of practical interest. Also obtained is the market area, that is the set of nodes captured by each firm at equilibrium. In particular, if the number of peaks in each profit function is polynomial in n, the algorithm terminates within polynomial time.

The remainder of this paper is organized as follows. Section 2 introduces our discrete spatial duopoly model based on Hotelling's original work, and formulates a Bertrand-Nash equilibrium. Section 3 gives the necessary and sufficient condition for the existence of equilibrium and the algorithm to compute equilibrium with an evaluation of the computational complexity. In Section 4, we summarize our findings.

2. The Model

Consider two firms A and B providing a homogeneous service. They have nonnegative constant marginal production costs c_A and c_B, and choose pricing strategies $t_A \geq c_A$ and $t_B \geq c_B$, respectively. Customers are discretely located at nodes $1, 2, \ldots, n$. For each node $k \in N = \{1, 2, \ldots, n\}$, let $c_{kA} > 0$ and $c_{kB} > 0$ be transportation cost per customer. Without loss of generality, let each node be numbered such that $c_{1B} - c_{1A} > c_{2B} - c_{2A} > \cdots > c_{nB} - c_{nA}$.* To exclude a trivial case, assume that there exists at least one node k' such that $c_{k'A} + c_A \neq c_{k'B} + c_B$.

Let $p_k \equiv p_k(t_A, t_B) = min(c_{kA} + t_A, c_{kB} + t_B)$ for node k. Customers' demand at node k is denoted by function $q_k(p_k)$. We make a reasonable assumption that q_k is a nonnegative-valued, continuous and nonincreasing function of p_k. In addition, we assume $q_k(c_{kA}+c_A) > 0$ and $q_k(c_{kB} + c_B) > 0$. Given t_A and t_B, we define the market areas of firms A and B as follows:

$$Q_A(t_A, t_B) = \{k | c_{kA} + t_A < c_{kB} + t_B\},$$
$$Q_B(t_A, t_B) = \{k | c_{kA} + t_A > c_{kB} + t_B\},$$
$$Q_{AB}(t_A, t_B) = \{k | c_{kA} + t_A = c_{kB} + t_B\}.$$

Firm A captures all customers located at node $k \in Q_A(t_A, t_B)$ and firm B captures all customers at node $k \in Q_B(t_A, t_B)$. We assume that firms A and B share customers located at node $k \in Q_{AB}(t_A, t_B)$ at the ratio of $\alpha : 1 - \alpha$, where $0 < \alpha < 1$. With these notations, the profit functions of A and B are:

*If nodes have the same value of $c_{kB} - c_{kA}$, we aggregate the corresponding demands.

$$\pi_A(t_A, t_B) = (t_A - c_A)\left\{\sum_{k\in Q_A(t_A,t_B)} q_k(c_{kA} + t_A) + \alpha \sum_{k\in Q_{AB}(t_A,t_B)} q_k(c_{kA} + t_A)\right\},$$
$$\pi_B(t_A, t_B) = (t_B - c_B)\left\{\sum_{k\in Q_B(t_A,t_B)} q_k(c_{kB} + t_B) + (1-\alpha) \sum_{k\in Q_{AB}(t_A,t_B)} q_k(c_{kB} + t_B)\right\}.$$

Definition 2.1 The *Bertrand-Nash equilibrium* (t_A^*, t_B^*) *for* π_A *and* π_B is defined as follows:

$$\pi_A(t_A^*, t_B^*) \geq \pi_A(t_A, t_B^*), \ \textit{for all}\ t_A \in [c_A, \infty), \ \textit{and}$$
$$\pi_B(t_A^*, t_B^*) \geq \pi_A(t_A^*, t_B), \ \textit{for all}\ t_B \in [c_B, \infty).$$

We now give two examples:

Example 1. Let $n = 2$. Figure 1 describes locations of firms and customers. Circles correspond to firms and squares correspond to customers. Transportation costs between firm and customer are described in the figure. Let $c_A = c_B = 1$. We consider the Bertrand-Nash equilibrium regarding the following two cases. Since the procedures to obtain equilibria are involved, we here show only an outline and relegate the details in Section 3.2.

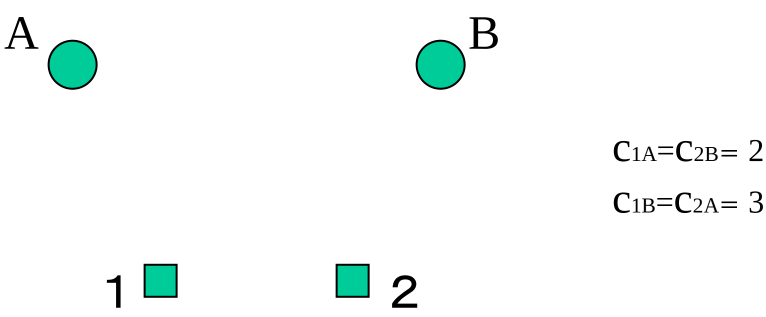

Figure 1: Example 1

Example 1(a). The demand function at node k is given by $q_k(p_k) = \max(-p_k + 4, 0)$. Then $(t_A^*, t_B^*) = (1.5, 1.5)$ is a unique equilibrium, where $Q_A(t_A^*, t_B^*) = \{1\}$, $Q_B(t_A^*, t_B^*) = \{2\}$ and $Q_{AB}(t_A^*, t_B^*) = \emptyset$. This result is obtained as follows:

First, we necessarily have $Q_A(t_A, t_B) = \{1\}$ and $Q_B(t_A, t_B) = \{2\}$ for an equilibrium (t_A, t_B), since otherwise one of the firms who captures no customer (say firm A) can increase his profit by charging $t_A - \varepsilon$ appropriately, a contradiction. In a similar manner, we have that $(t_A^*, t_B^*) = (1.5, 1.5)$, a pair of a unique maximal point of $(t_A - c_A)q_1(c_{1A} + t_A)$ and that of $(t_B - c_B)q_2(c_{2B} + t_B)$, is the only candidate for the equilibrium. Finally, given $t_B^* = 1.5$, firm A maximizes his profit π_A at $t_A^* = 1.5$ and firm B vice versa. Therefore $(t_A^*, t_B^*) = (1.5, 1.5)$ is a unique equilibrium.

Example 1(b). The demand function at node k is given by $q_k(p_k) = \max(-p_k + 20, 0)$. There exists no equilibrium in this case. This result is obtained as follows:

Following an argument similar to Example 1(a), a point $(t_A^*, t_B^*) = (9.5, 9.5)$ is the only candidate for the equilibrium. However, we have $\pi_A(8.5, 9.5) = 135 > 72.25 = \pi_A(9.5, 9.5)$. Hence this point is not an equilibrium and there exists no equilibrium in this case.

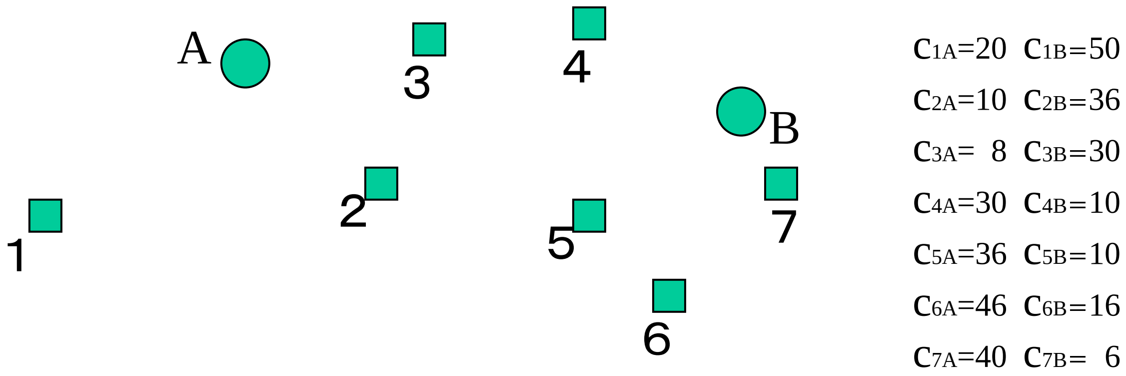

Figure 2: Example 2

As seen in this example, an equilibrium does not necessarily exist even in a simple setting.

Example 2. Let $n = 7$. Figure 2 describes locations of firms and customers. Transportation costs between firm and customer are described in the figure. Let $c_A = c_B = 0$. The demand function at node k is given by

$$q_k(p_k) = \begin{cases} -p_k + 60 & : \quad 0 \leq p_k \leq 36 \\ -\frac{1}{2}p_k + 42 & : \quad 36 \leq p_k \leq 42 \\ 0 & : \quad p_k \geq 42. \end{cases}$$

In this case, $(t_A^*, t_B^*) = (35.7, 36.8)$ is a unique equilibrium, where $Q_A(t_A^*, t_B^*) = \{1, 2, 3\}$, $Q_B(t_A^*, t_B^*) = \{4, 5, 6, 7\}$ and $Q_{AB}(t_A^*, t_B^*) = \emptyset$.

This result can be obtained by examining all candidates for the equilibria as in Example 1. Since the process is lengthy, we describe it in Section 3.2.

3. The Bertrand-Nash Equilibrium

3.1. The equilibrium condition

Given t_B, let $\bar{t}_A^k(t_B) \equiv c_{kB} - c_{kA} + t_B$ $(k = 1, \ldots, n)$. Define $\bar{t}_B^k(t_A)$ similarly. Note that $\bar{t}_A^1(t_B) > \cdots > \bar{t}_A^n(t_B)$ and $\bar{t}_B^1(t_A) < \cdots < \bar{t}_B^n(t_A)$. Let $i^*(t_B) \equiv \max\{i|\bar{t}_A^i(t_B) \geq c_A\}$ and $j^*(t_A) \equiv \min\{j|\bar{t}_B^j(t_A) \geq c_B\}$. Unless ambiguity arises, we denote them hereafter as $\bar{t}_A^i$, $\bar{t}_B^j$, i^*, and j^*. We obtain the following propositions.

Proposition 3.1 *Let $t_A \geq c_A$ and $t_B \geq c_B$. The market is shared by the firms as in Table 1.*

Proof 1.(a): It is immediate that $\bar{t}_A^1 < c_A \Leftrightarrow \bar{t}_A^i = c_{iB} - c_{iA} + t_B < c_A$ for all $i = 1, \ldots, n$. Hence $\bar{t}_A^1 < c_A \Leftrightarrow Q_A(t_A, t_B) = \emptyset$ and $Q_B(t_A, t_B) = N$.
1.(b): $c_A \leq t_A < \bar{t}_A^{i^*} \Leftrightarrow t_A < \bar{t}_A^i = c_{iB} - c_{iA} + t_B$ for all $i \leq i^*$ and $t_A \geq c_A > \bar{t}_A^i = c_{iB} - c_{iA} + t_B$ for all $i > i^*$ by definition of i^*. Hence, $c_A \leq t_A < \bar{t}_A^{i^*} \Leftrightarrow Q_A(t_A, t_B) = \{1, \ldots, i^*\}$, $Q_B(t_A, t_B) = \{i^* + 1, \ldots, n\}$ and $Q_{AB}(t_A, t_B) = \emptyset$.
1.(c): $c_A \leq t_A = \bar{t}_A^{i'}$ for some $i^* \geq i' \geq 2 \Leftrightarrow t_A = c_{i'B} - c_{i'A} + t_B$, $t_A < c_{iB} - c_{iA} + t_B$ for $i = 1, \ldots, i' - 1$ and $t_A > c_{iB} - c_{iA} + t_B$ for $i = i' + 1, \ldots, n$. Hence, $c_A \leq t_A = \bar{t}_A^{i'}$ for some i' such that $i^* \geq i' \geq 2 \Leftrightarrow Q_A(t_A, t_B) = \{1, \ldots, i' - 1\}$, $Q_{AB}(t_A, t_B) = \{i'\}$, and $Q_B(t_A, t_B) = \{i' + 1, \ldots, n\}$.
1.(d)–2.(f): Similar to the above and omitted. **Q.E.D.**

Table 1: The market areas of A and B

No.	The relation between t_A and t_B	$Q_A(t_A,t_B)$	$Q_{AB}(t_A,t_B)$	$Q_B(t_A,t_B)$
1.(a)	$\bar{t}_A^1 < c_A$	$\emptyset$	$\emptyset$	N
1.(b)	$c_A \leq t_A < \bar{t}_A^{i^*}$	$\{1,\dots,i^*\}$	$\emptyset$	$\{i^*+1,\dots,n\}$
1.(c)	$c_A \leq t_A = \bar{t}_A^{i'}$ for some i' s.t. $i^* \geq i' \geq 2$	$\{1,\dots,i'-1\}$	$\{i'\}$	$\{i'+1,\dots,n\}$
1.(d)	$c_A \leq \bar{t}_A^{i'} < t_A < \bar{t}_A^{i'-1}$ for some i' s.t. $i^* \geq i' \geq 2$	$\{1,\dots,i'-1\}$	$\emptyset$	$\{i',\dots,n\}$
1.(e)	$c_A \leq t_A = \bar{t}_A^1$	$\emptyset$	$\{1\}$	$\{2,\dots,n\}$
1.(f)	$c_A \leq \bar{t}_A^1 < t_A$	$\emptyset$	$\emptyset$	N
2.(a)	$\bar{t}_B^n < c_B$	N	$\emptyset$	$\emptyset$
2.(b)	$c_B \leq t_B < \bar{t}_B^{j^*}$	$\{1,\dots,j^*-1\}$	$\emptyset$	$\{j^*,\dots,n\}$
2.(c)	$c_B \leq t_B = \bar{t}_B^{j'}$ for some j' s.t. $j^* \leq j' \leq n-1$	$\{1,\dots,j'-1\}$	$\{j'\}$	$\{j'+1,\dots,n\}$
2.(d)	$c_B \leq \bar{t}_B^{j'} < t_B < \bar{t}_B^{j'+1}$ for some j' s.t. $j^* \leq j' \leq n-1$	$\{1,\dots,j'\}$	$\emptyset$	$\{j'+1,\dots,n\}$
2.(e)	$c_B \leq t_B = \bar{t}_B^n$	$\{1,\dots,n-1\}$	$\{n\}$	$\emptyset$
2.(f)	$c_B \leq \bar{t}_B^n < t_B$	N	$\emptyset$	$\emptyset$

We now define the following single-variable functions:

$$\pi_A^i(t_A) = \sum_{k=1}^{i} \tilde{\pi}_A^k(t_A), \quad i = 1,\dots,n,$$
$$\pi_B^j(t_B) = \sum_{k=j}^{n} \tilde{\pi}_B^k(t_B), \quad j = 1,\dots,n,$$

where

$$\tilde{\pi}_A^k(t_A) = (t_A - c_A)q_k(c_{kA} + t_A), \quad k = 1,\dots,n,$$
$$\tilde{\pi}_B^k(t_B) = (t_B - c_B)q_k(c_{kB} + t_B), \quad k = 1,\dots,n.$$

By 1.(a) and 2.(a) in Proposition 3.1, if $\bar{t}_A^1 < c_A$ then $\pi_A(t_A,t_B) = 0$ for all $t_A \geq c_A$. Similarly if $\bar{t}_B^n < c_B$ then $\pi_B(t_A,t_B) = 0$ for all $t_B \geq c_B$. We now assume that $\bar{t}_A^1 \geq c_A$ and $\bar{t}_B^n \geq c_B$. Then, from 1.(b)–(f) and 2.(b)–(f) in Proposition 3.1 we can rewrite $\pi_A(t_A,t_B)$ and $\pi_B(t_A,t_B)$ as the following piecewise continuous functions (see Figure 3):

$$\pi_A(t_A,t_B) = \begin{cases} \pi_A^{i^*}(t_A) & : \ c_A \leq t_A < \bar{t}_A^{i^*} \\ \pi_A^{i-1}(t_A) + \alpha\tilde{\pi}_A^i(t_A) & : \ t_A = \bar{t}_A^i, \ i^* \geq i \geq 2 \\ \pi_A^{i-1}(t_A) & : \ \bar{t}_A^i < t_A < \bar{t}_A^{i-1}, \ i^* \geq i \geq 2 \\ \alpha\tilde{\pi}_A^1(t_A) & : \ t_A = \bar{t}_A^1 \\ 0 & : \ t_A > \bar{t}_A^1, \end{cases} \tag{1}$$

$$\pi_B(t_A,t_B) = \begin{cases} \pi_B^{j^*}(t_B) & : \ c_B \leq t_B < \bar{t}_B^{j^*} \\ \pi_B^{j+1}(t_B) + (1-\alpha)\tilde{\pi}_B^j(t_B) & : \ t_B = \bar{t}_B^j, \ j^* \leq j \leq n-1 \\ \pi_B^{j+1}(t_B) & : \ \bar{t}_B^j < t_B < \bar{t}_B^{j+1}, \ j^* \leq j \leq n-1 \\ (1-\alpha)\tilde{\pi}_B^n(t_B) & : \ t_B = \bar{t}_B^n \\ 0 & : \ t_B > \bar{t}_B^n. \end{cases} \tag{2}$$

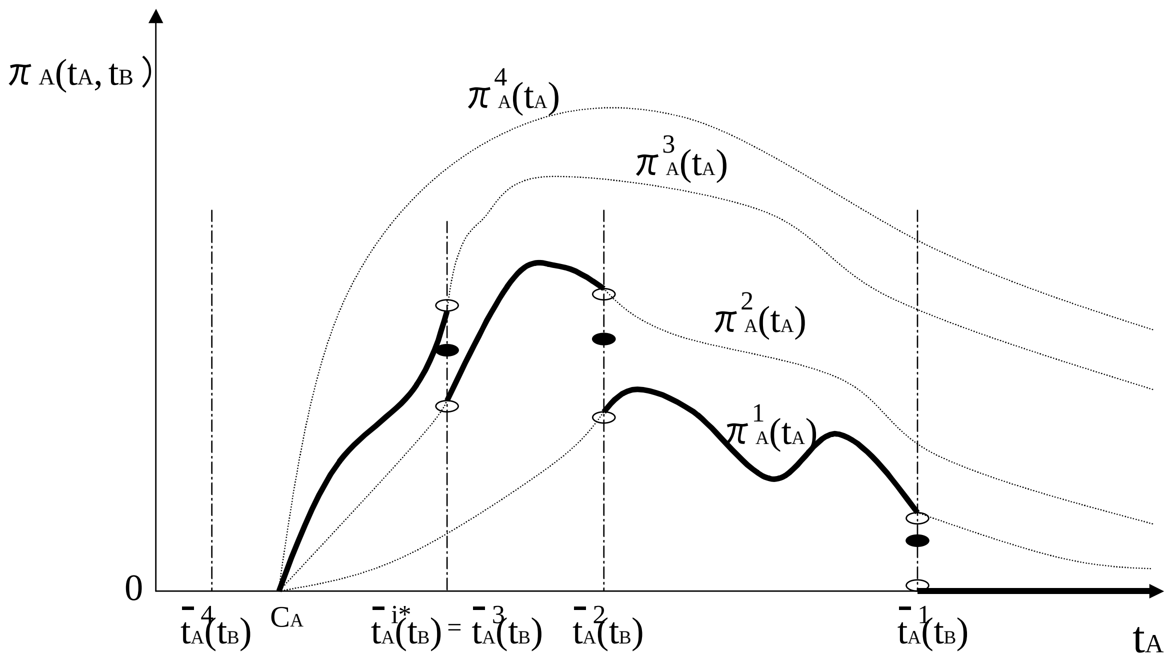

Figure 3: $\pi_A(t_A, t_B)$ for given t_B ($n = 4$ and $i^*(t_B) = 3$)

Note that $\pi_A(t_A, t_B)$ and $\pi_B(t_A, t_B)$ are discontinuous at $\bar{t}_A^i$ and $\bar{t}_B^j$ respectively.

Next, we define peaks of π_A^i and π_B^j.

Definition 3.1 The point $\hat{t}$ is a *peak of* π_h^i ($h = A, B,\ i = 1, \ldots, n$) if $\pi_h^i(\hat{t}) > 0$ and there exists $\delta > 0$ such that $\pi_h^i(\hat{t}) \geq \pi_h^i(t)$ for all $t \in (\hat{t} - \delta,\ \hat{t} + \delta)$.

Let T_A^i and T_B^i ($i = 1, \ldots, n$) be the sets of peaks of π_A^i and π_B^i, respectively. We define $\bar{\pi}_{A,t_B}^i$ and $\bar{\pi}_{B,t_A}^j$ as follows:

$$\bar{\pi}_{A,t_B}^i = \begin{cases} \max\{\pi_A^{i^*}(t_A) |\ c_A \leq t_A \leq \bar{t}_A^{i^*}(t_B)\} & : \ i = i^* \\ \max\{\pi_A^i(t_A) |\ \bar{t}_A^{i+1}(t_B) \leq t_A \leq \bar{t}_A^i(t_B)\} & : \ i^* - 1 \geq i \geq 1, \end{cases}$$

$$\bar{\pi}_{B,t_A}^j = \begin{cases} \max\{\pi_B^{j^*}(t_B) |\ c_B \leq t_B \leq \bar{t}_B^{j^*}(t_A)\} & : \ j = j^* \\ \max\{\pi_B^j(t_B) |\ \bar{t}_B^{j-1}(t_A) \leq t_B \leq \bar{t}_B^j(t_A)\} & : \ j^* + 1 \leq j \leq n. \end{cases}$$

We are ready to state a necessary and sufficient condition for the existence of Bertrand-Nash equilibrium in our model.

Theorem 3.1

1. *If A and B are located at the same point and* $c_A = c_B = c$*, then* $t_A^* = t_B^* = c$ *is the unique equilibrium.*
2. *Otherwise,* $(t_A^*,\ t_B^*)$ *is an equilibrium iff* $(t_A^*,\ t_B^*)$ *satisfies one of conditions* (a)–(c).
   (a) i. $Q_A(t_A^*, c_B) = N$, *and* $Q_B(t_A^*, c_B) = \emptyset$,
   ii. $t_A^* \in T_A^n$,
   iii. $\pi_A^n(t_A^*) = \max\ (\bar{\pi}_{A,t_B^*}^1, \ldots, \bar{\pi}_{A,t_B^*}^n)$.
   (b) i. $Q_A(c_A, t_B^*) = \emptyset$, *and* $Q_B(c_A, t_B^*) = N$,
   ii. $t_B^* \in T_B^1$,
   iii. $\pi_B^1(t_B^*) = \max\ (\bar{\pi}_{B,t_A^*}^1, \ldots, \bar{\pi}_{B,t_A^*}^n)$.
   (c) i. $Q_A(t_A^*, t_B^*) = \{1, \ldots, l\}$, *and* $Q_B(t_A^*, t_B^*) = \{l + 1, \ldots, n\}$ ($l \leq n - 1$), *and* $Q_{AB}(t_A^*, t_B^*) = \emptyset$,

ii. $t^*_A \in T^l_A$ *and* $t^*_B \in T^{l+1}_B$,

iii. $\pi^l_A(t^*_A) = \max\ (\bar{\pi}^1_{A,t^*_B}, \ldots, \bar{\pi}^{i^*}_{A,t^*_B})$ *and* $\pi^{l+1}_B(t^*_B) = \max\ (\bar{\pi}^{j^*}_{B,t^*_A}, \ldots, \bar{\pi}^n_{B,t^*_A})$.

Proof Statement 1 is obvious. We first assume that $(t^*_A,\ t^*_B)$ satisfies condition (a). Since $Q_A(t^*_A, c_B) = N$ implies that $c_{kA} + t^*_A \le c_{kB} + t_B$ for any $t_B \ge c_B$, $\pi_B(t^*_A, t_B) = \pi_B(t^*_A, t^*_B) = 0$ for any $t_B \ge c_B$. On the other hand, it follows from (1) that $\pi_A(t^*_A, t^*_B) = \max\ (\bar{\pi}^1_{A,t^*_B}, \ldots, \bar{\pi}^{i^*}_{A,t^*_B}) = \sup_{t_A} \pi_A(t_A, t^*_B)$. We therefore have (t^*_A, t^*_B) is an equilibrium. We can show that condition (b) or (c) implies (t^*_A, t^*_B) is an equilibrium in a similar manner and omit the proof.

Next, we suppose that (t^*_A, t^*_B) is an equilibrium and $Q_{AB}(t^*_A, t^*_B) = \{i'\}$. We furthermore suppose $t^*_A = c_A$ and $t^*_B = c_B$. From the assumption, there exists k' such that $c_{k'A} + c_A \ne c_{k'B} + c_B$. Without loss of generality, we assume $c_{k'A} + c_A < c_{k'B} + c_B$. Since $q_{k'}$ is continuous and $q_{k'}(c_{k'A} + c_A) > 0$, firm A can increase his profit from $\pi_A(c_A, c_B) = 0$ by charging $t_A = c_A + \varepsilon$ such that $t_A < c_{k'B} + c_B - c_{k'A}$. Thus, it contradicts $(t^*_A, t^*_B) = (c_A, c_B)$ being an equilibrium. Hence $t^*_A > c_A$ or $t^*_B > c_B$. Without loss of generality, we suppose $t^*_A > c_A$. Since $t^*_A = \bar{t}^{i'}_A(t^*_B)$, firm A can capture customers at i' by charging $t^*_A - \varepsilon$ and can increase his profit, a contradiction. Hence $Q_{AB}(t^*_A, t^*_B) = \emptyset$. We now assume $Q_A(t^*_A, t^*_B) = N$. Since $\pi_B(t^*_A, t^*_B) = 0$ and (t^*_A, t^*_B) is an equilibrium, we have $\pi_B(t^*_A, t_B) = 0$ for all $t_B \ge c_B$, which implies $Q_A(t^*_A, t_B) = N$ for all $t_B \ge c_B$. Hence, (a)-i holds and this implies $i^* = n$. Then we have $c_A \le t^*_A < \bar{t}^n_A(t^*_B)$ by 1.(b) in Proposition 3.1. Since $\pi_A(t_A, t^*_B) = \pi^n_A(t_A)$ for any $c_A \le t_A < \bar{t}^n_A(t^*_B)$ from (1), $\pi_A(t^*_A, t^*_B) = \pi^n_A(t^*_A)$ also holds. Note that $t^*_A > c_A$, since otherwise firm A can increase his profit. Furthermore, $t^*_A \in T^n_A$, since otherwise firm A can increase his profit by charging $t^*_A + \varepsilon$ or $t^*_A - \varepsilon$ appropriately. Hence (a)-ii holds. We next show (a)-iii holds. We have $\max\ (\bar{\pi}^1_{A,t^*_B}, \ldots, \bar{\pi}^n_{A,t^*_B}) = \sup_{t_A} \pi_A(t_A, t^*_B) \ge \pi_A(t^*_A, t^*_B)$. On the other hand, $\pi_A(t^*_A, t^*_B) \ge \max\ (\bar{\pi}^1_{A,t^*_B}, \ldots, \bar{\pi}^n_{A,t^*_B})$, since otherwise there exists $\tilde{t}_A$ such that $\pi_A(t^*_A, t^*_B) < \pi_A(\tilde{t}_A, t^*_B) \le \max\ (\bar{\pi}^1_{A,t^*_B}, \ldots, \bar{\pi}^n_{A,t^*_B})$, which contradicts the fact that (t^*_A, t^*_B) is an equilibrium. Hence, $\pi_A(t^*_A, t^*_B) = \pi^n_A(t^*_A) = \max\ (\bar{\pi}^1_{A,t^*_B}, \ldots, \bar{\pi}^n_{A,t^*_B})$.

The statement (b) is symmetric to (a). The case (c) follows from 1.(b), 1.(d) and 2.(b), 2.(d) in Proposition 3.1. **Q.E.D.**

Theorem 3.1 describes the properties which an equilibrium satisfies. The equilibrium (t^*_A, t^*_B) is a point such that: (i) $Q_{AB}(t^*_A, t^*_B) = \emptyset$, that is, $t^*_A \ne \bar{t}^i_A(t^*_B)$ and $t^*_B \ne \bar{t}^j_B(t^*_A)$, (ii) t^*_A and t^*_B are peaks of $\pi_A(t_A, t^*_B)$ and $\pi_B(t^*_A, t_B)$ respectively, and (iii) t^*_A and t^*_B attain the maximum values of $\pi_A(t_A, t^*_B)$ and $\pi_B(t^*_A, t_B)$ respectively (see Figure 4). Note that we require fairly weak assumptions regarding a demand function, as described in Section 2. We develop an algorithm for finding all equilibria (if any) based on Theorem 3.1.

3.2. The algorithm of finding equilibrium points

In this section, we present an algorithm to find equilibrium points. We assume that T^i_A and T^i_B $(i = 1, \ldots, n)$ are known and finite. Our algorithm can determine the existence of equilibrium and find *all* equilibria in which each firm earns a positive profit. Furthermore, not only the equilibrium prices but also market areas of both firms at each equilibrium are determined simultaneously.

The following notation is useful to describe the algorithm:

$$T^i_{A,t_B} \equiv \{t \in T^i_A |\ \bar{t}^{i+1}_A(t_B) \le t \le \bar{t}^i_A(t_B)\} \quad \text{for} \quad i = 1, \ldots, i^*,$$

where $\bar{t}^{i^*+1}_A(t_B) = c_A$. Define T^j_{B,t_A} similarly.

Our algorithm is presented as follows:

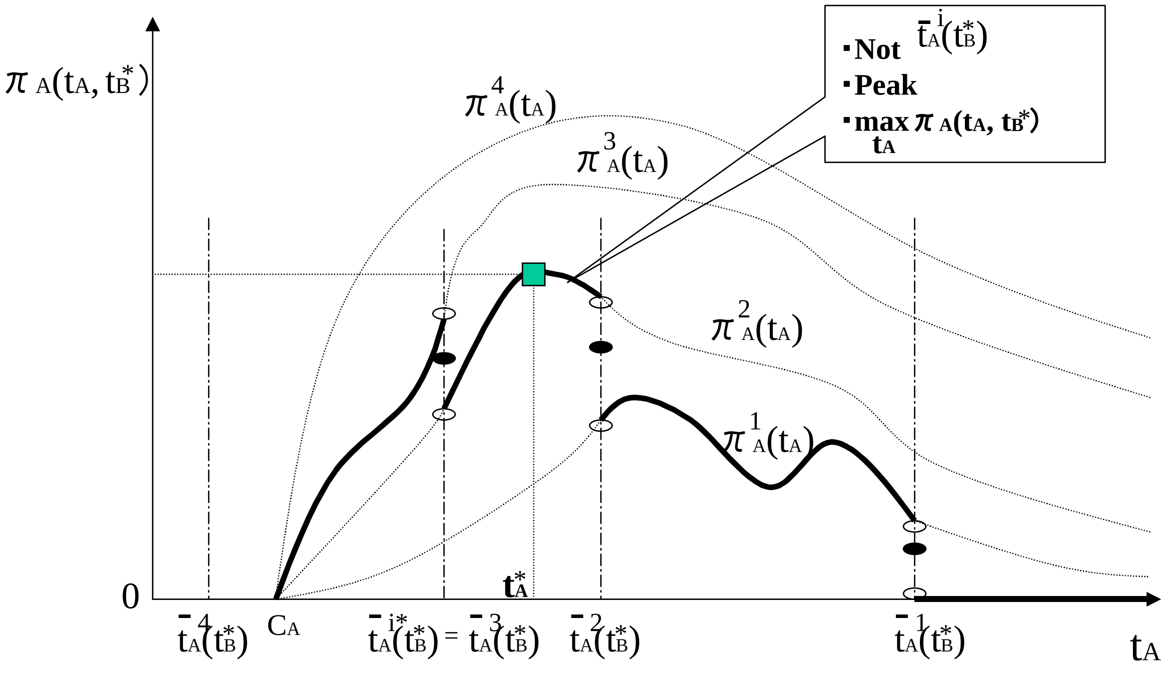

Figure 4: Equilibrium point (t^*_A, t^*_B) ($n = 4$ and $i^*(t_B) = 3$)

FINDEQ

Step 0 $E = \emptyset$.

Step 1

1. For all $i = 1, \ldots, n$, let $\bar{t}^i_A := c_{iB} - c_{iA} + c_B$.
2. Construct T^i_{A,c_B} for all $1 \le i \le n$.
3. **(Check 2.(a)-i and iii in Theorem 3.1)** For each $\hat{t}_A \in T^n_A$, if $\hat{t}_A < c_{nB} - c_{nA} + c_B$, and $\pi^n_A(\hat{t}_A) = \max_{1\le i\le n}\{\pi^i_A(t) |\ t \in T^i_{A,c_B} \text{ or } t = \bar{t}^i_A\}$, then add $((\hat{t}_A, c_B), (N, \emptyset))$ to E.
4. If $E = \emptyset$, then go to Step 2. Otherwise, go to Step 4.

Step 2

1. For all $j = 1, \ldots, n$, let $\bar{t}^j_B := c_{jA} - c_{jB} + c_A$.
2. Construct T^j_{B,c_A} for all $1 \le j \le n$.
3. **(Check 2.(b)-i and iii in Theorem 3.1)** For each $\hat{t}_B \in T^1_B$, if $\hat{t}_B < c_{1A} - c_{1B} + c_A$, and $\pi^1_B(\hat{t}_B) = \max_{1\le j\le n}\{\pi^j_B(t) |\ t \in T^j_{B,c_A} \text{ or } t = \bar{t}^j_B\}$, then add $((c_A, \hat{t}_B), (\emptyset, N))$ to E.
4. If $E = \emptyset$, then go to Step 3. Otherwise, go to Step 4.

Step 3 For all $(\hat{t}_A, \hat{t}_B) \in T^l_A \times T^{l+1}_B$, $l = 1, \ldots, n-1$, do Steps 3-1 to 3-4.

1. For all $i = 1, \ldots, n$ and $j = 1, \ldots, n$, let $\bar{t}^i_A := c_{iB} - c_{iA} + \hat{t}_B$ and $\bar{t}^j_B := c_{jA} - c_{jB} + \hat{t}_A$
2. Let $i^* := \max\{i|\bar{t}^i_A \ge c_A\}$, $j^* := \min\{j|\bar{t}^j_B \ge c_B\}$ and construct $T^i_{A,\hat{t}_B}$ for all $i \le i^*$ and $T^j_{B,\hat{t}_A}$ for all $j^* \le j$.

3. **(Check 2.(c)-i in Theorem 3.1)**
   If $\bar{t}_A^{l+1} < \hat{t}_A < \bar{t}_A^l$ and $\bar{t}_B^l < \hat{t}_B < \bar{t}_B^{l+1}$,
   then go to Step 3-4. Otherwise, skip Step 3-4.
4. **(Check 2.(c)-iii in Theorem 3.1)**
   If $\pi_A^l(\hat{t}_A) = \max_{1\le i\le i^*} \{\pi_A^i(t)|\ t \in T_{A,\hat{t}_B}^i \text{ or } t = \bar{t}_A^i\}$ and
   $\pi_B^{l+1}(\hat{t}_B) = \max_{j^*\le j\le n} \{\pi_B^j(t)|\ t \in T_{B,\hat{t}_A}^j \text{ or } t = \bar{t}_B^j\}$,
   then add $((\hat{t}_A, \hat{t}_B), (\{1,\dots,l\}, \{l+1,\dots,n\}))$ to E.

Step 4 All elements of E are equilibrium points, each representing both prices and the market areas of each A and B. If $E = \emptyset$, there exists no equilibrium in our model.

The flow chart of **FINDEQ** is described in Figure 5. We note that Theorem 3.1 implies that an equilibrium arises only in $T_A^i \times T_B^j$ for some i and j. In condition 2.(a)-iii in Theorem 3.1, max $(\bar{\pi}^1_{A,t_B^*}, \dots, \bar{\pi}^n_{A,t_B^*})$ can be attained at either a peak or at some $\bar{t}_A^i$. This fact is employed in Step 1-3. Step 2 is symmetric to Step 1. Steps 3-3 and 3-4 check the equilibrium condition 2.(c) in Theorem 3.1 for each possible market segmentation. Equilibria not obtained by **FINDEQ** are trivial ones. For example, when the condition 2.(a) is satisfied, (t_A^*, t_B') is an equilibrium for all t_B' with $c_B \le t_B' \le t_B^*$.

In what follows, we describe how the equilibria in Section 2 are obtained.

Example 1(a). We have $T_A^1 = \{t_A^*\} = \{1.5\}$, $T_A^2 = \{\hat{t}_A\} = \{1.25\}$, $T_B^1 = \{\hat{t}_B\} = \{1.25\}$ and $T_B^2 = \{t_B^*\} = \{1.5\}$. We first calculate $\bar{t}_A^1 = 2$ and $\bar{t}_A^2 = 0$ in Step 1-1. Since $\hat{t}_A \ge 0$, E remains $\emptyset$ at the end of Step 1-3. Similarily, E remains $\emptyset$ at the end of Step 2. In Step 3, it is sufficient to check only $(t_A^*, t_B^*) = (1.5, 1.5) \in T_A^1 \times T_B^2$. We have $i^* = 1$ and $j^* = 2$. It can be directly constructed that $T^1_{A,t_B^*} = \{t_A^*\}$, $T^2_{A,t_B^*} = \emptyset$, $T^1_{B,t_A^*} = \emptyset$ and $T^2_{B,t_A^*} = \{t_B^*\}$. The value of π_A^i at each $\bar{t}_A^i$ and peaks in T^i_{A,t_B^*} (also the value of π_B^j at $\bar{t}_B^j$ and peaks in T^j_{B,t_A^*}) are shown in Table 2, so that $((t_A^*, t_B^*), (\{1\}, \{2\}))$ is added to E from Table 2. Hence, it is proved that (t_A^*, t_B^*) is the unique equilibrium.

Table 2: The values of $\pi_A^i(t_A)$ and $\pi_B^j(t_B)$ corresponding to Example 1(a).

t_A	$t_A = \bar{t}_A^i$ or $t_A \in T^i_{A,t_B^*}$	$\pi_A^i(t_A)$	t_B	$t_B = \bar{t}_B^j$ or $t_B \in T^j_{B,t_A^*}$	$\pi_B^j(t_B)$
1.5	$t_A^* \in T^1_{A,t_B^*}$	0.25	1.5	$t_B^* \in T^2_{B,t_A^*}$	0.25
2.5	$\bar{t}_A^1$	0	2.5	$\bar{t}_B^2$	0

Table 3: The values of $\pi_A^i(t_A)$ and $\pi_B^j(t_B)$ corresponding to Example 1(b).

t_A	$t_A = \bar{t}_A^i$ or $t_A \in T^i_{A,t_B^*}$	$\pi_A^i(t_A)$	t_B	$t_B = \bar{t}_B^j$ or $t_B \in T^j_{B,t_A^*}$	$\pi_B^j(t_B)$
8.5	$\bar{t}_A^2$	135	8.5	$\bar{t}_B^1$	135
9.5	$t_A^* \in T^2_{A,t_B^*}$	72.25	9.5	$t_B^* \in T^1_{B,t_A^*}$	72.25
10.5	$\bar{t}_A^1$	71.25	10.5	$\bar{t}_B^2$	71.25

Example 1(b). We have $T_A^1 = \{t_A^*\} = \{9.5\}$, $T_A^2 = \{9.25\}$, $T_B^1 = \{9.25\}$ and $T_A^2 = \{t_B^*\} = \{9.5\}$. As in Example 1(a), there is only one candidate for equilibria, that is, $(t_A^*, t_B^*) = (9.5, 9.5) \in T_A^1 \times T_B^2$. However, it follows that $\pi_A^2(\bar{t}_A^2) > \pi_A^1(t_A^*)$ from Table 3. Therefore (t_A^*, t_B^*) is not an equilibrium, that is, there exists no equilibrium in this example.

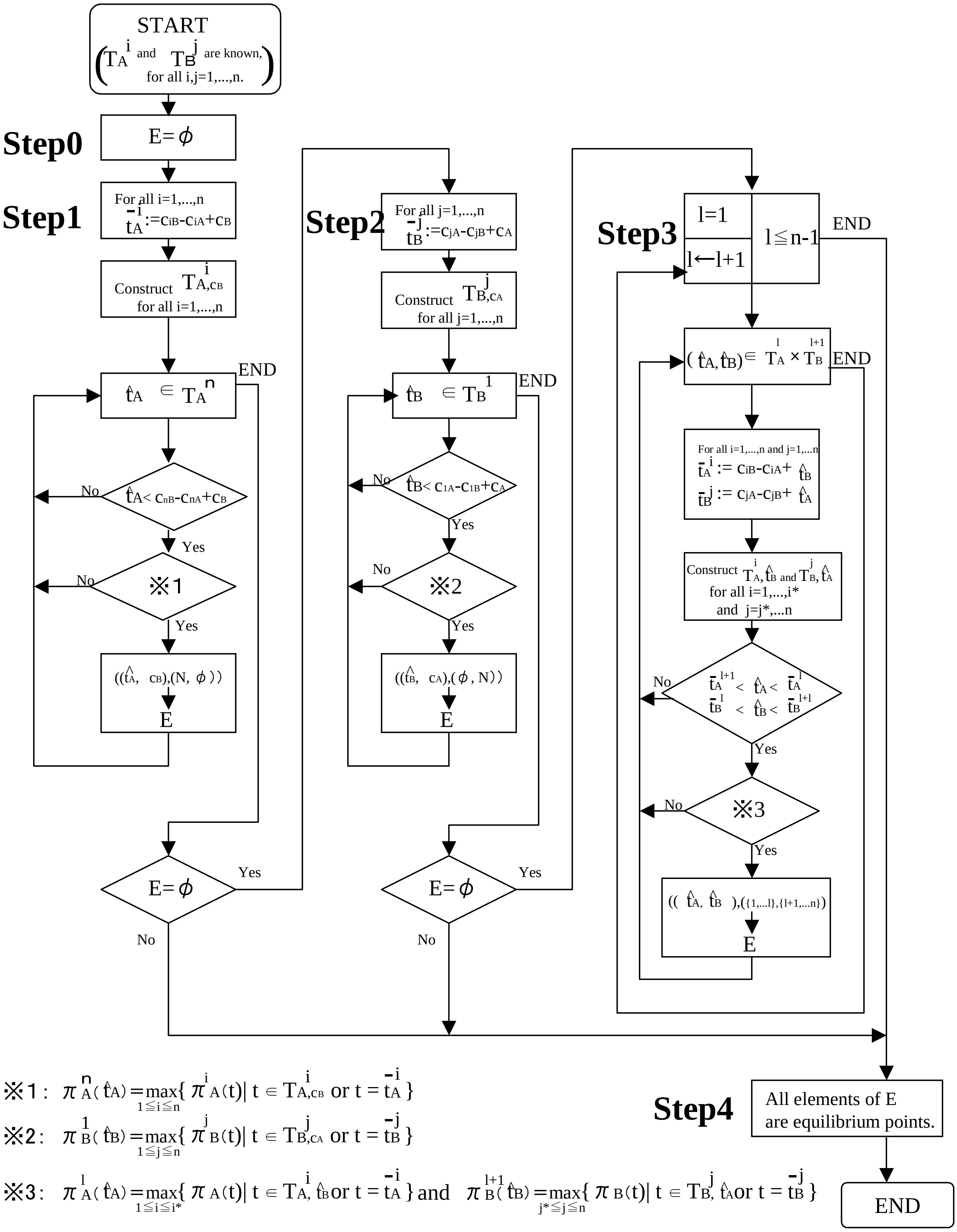

Figure 5: The flow chart of FINDEQ

Example 2. In this example, the sets of peaks are as follows: $T_A^1 = \{32\}$, $T_A^2 = \{34.5\}$, $T_A^3 = \{35.7\}$, $T_A^4 = \{33.5\}$, $T_A^5 = \{31.6\}$, $T_A^6 = \{25.5, 27.3, 29.5\}$, $T_A^7 = \{25.1, 26.6, 28.4\}$, $T_B^1 = \{25.6, 26.8, 30.7\}$, $T_B^2 = \{33\}$, $T_B^3 = \{34.8\}$, $T_B^4 = \{36.8\}$, $T_B^5 = \{36.7\}$, $T_B^6 = \{29.3, 36.5\}$ and $T_B^7 = \{27, 29\}$. It is verified that $E = \emptyset$ at the end of Step 1 and Step 2. In Step 3, we need check each pair of peaks which belongs to $T_A^l \times T_B^{l+1}$. We can now verify that only $(t_A^*, t_B^*) = (35.7, 36.8) \in T_A^3 \times T_B^4$ is added to E from Table 4. Hence, it is proved that (t_A^*, t_B^*) is the unique equilibrium.

Table 4: The values of $\pi_A^i(t_A)$ and $\pi_B^j(t_B)$ corresponding to Example 2.

t_A	$t_A = \bar{t}_A^i$ or $t_A \in T_{A,t_B^*}^i$	$\pi_A^i(t_A)$	t_B	$t_B = \bar{t}_B^j$ or $t_B \in T_{B,t_A^*}^j$	$\pi_B^j(t_B)$
1.5	$\bar{t}_A^7$	343.1	5.7	$\bar{t}_B^1$	1338.4
5.5	$\bar{t}_A^6$	1029.9	9.7	$\bar{t}_B^2$	1944.9
9.5	$\bar{t}_A^5$	1472.5	13.7	$\bar{t}_B^3$	2100.9
16.8	$\bar{t}_A^4$	1858.1	36.8	$t_B^* \in T_{B,t_A^*}^4$	2701.1
35.7	$t_A^* \in T_{A,t_B^*}^3$	1908.2	55.7	$\bar{t}_B^4$	1982.9
58.8	$\bar{t}_A^3$	1105.4	61.7	$\bar{t}_B^5$	1076.7
62.8	$\bar{t}_A^2$	389.4	65.7	$\bar{t}_B^6$	479.6
66.8	$\bar{t}_A^1$	0	69.7	$\bar{t}_B^7$	289.3

3.3. The computational complexity of FINDEQ

In this section, we evaluate the computational complexity of **FINDEQ**. Informally speaking, the "time complexity" of an algorithm is the maximum amount of time necessary for the algorithm to solve a problem instance of a specific size. A "polynomial algorithm" is the one whose time complexity is bounded by a polynomial function of the problem size. If the time complexity is not bounded by any polynomial function, then it is called an "exponential algorithm". As one may imagine, an exponential time algorithm is virtually intractable if the problem size is even modestly large. For more rigorous treatment for the topics, the readers are referred to the classic [7].

We now show that if the numbers of peaks $|T_A^i|$ and $|T_B^i|$ are polynomial in n, within polynomial time **FINDEQ** can determine the existence of equilibrium and find *all* equilibria in which both firms earn a positive profit. Before we state this result as a theorem, we formally define some terms used in the theorem as follows:

Definition 3.2 Let $f(n)$ and $g(n)$ be any functions. Then $f(n)$ *is* $O(g(n))$ if there exists a constant c such that $|f(n)| \leq c|g(n)|$ for all values of $n \geq 0$.

Definition 3.3 Let p be any positive number. An algorithm can solve a problem *within polynomial time* if the time complexity of the algorithm is $O(n^p)$.

Now, we are ready to state our main theorem.

Theorem 3.2 *If T_A^i and T_B^i are known and the numbers of peaks $|T_A^i|$ and $|T_B^i|$ are polynomial in n for $i = 1, \ldots, n$, we can determine within polynomial time the existence of equilibrium and find all equilibrium prices in which both A and B earn a positive profit. The market areas of each firm at each equilibrium are also determined simultaneously.*

Proof Suppose that $|T_A^i|$ and $|T_B^i|$ are polynomial in n, that is, $O(n^p)$ for $i = 1, \ldots, n$. Step1-3 requires at most $|T_A^n|$ iterations and each iteration involves a comparison of at most $\sum_{i=1}^{n} |T_A^i|$ values. Hence the complexity of Step 1 is $O(n^p \times (n \times n^p)) = O(n^{2p+1})$. Similarly, Step 2 can be performed in $O(n^{2p+1})$ time. Finally, Step 3 requires at most $\sum_{l=1}^{n-1}(|T_A^l| \times |T_B^{l+1}|)$ iterations, which is the complexity of $O(n \times n^{2p})$. Each iteration involves comparisons of at most $\sum_{i=1}^{i^*}(|T_A^i|+1) + \sum_{j=j^*}^{n}(|T_B^j|+1)$ values which take $O(n \times n^p)$ time. Thus Step 3 can be performed in $O(n^{3p+2})$ time. **Q.E.D.**

The condition regarding the number of peaks is satisfied for a wide class of demand functions. In particular, if q_k is given by $q_k(p_k) = \max({q'}_k(p_k), 0)$ where ${q'}_k$ is strictly decreasing and concave, then each $\tilde{\pi}_A^i$ and $\tilde{\pi}_B^i$ are quasi-concave functions which are single-peaked and thus $|T_A^i|$ and $|T_B^i|$ are less than n.

4. Concluding Remarks

In this paper we explore a spatial duopoly model which is an extension of Hotelling [9]. We assume that customers are discretely distributed at nodes and the demand functions are given for each node, while Economides [5] assumes that customers are uniformly distributed. A necessary and sufficient condition for the existence of Bertrand-Nash equilibrium is derived, and the polynomial time algorithm to compute all equilibrium points is presented for a wider class of demand functions. Economides analyzes Bertrand-Nash equilibrium only for a symmetric location of firms, which is a convenient assumption reducing analytical difficulties. Our discrete model solves these difficulties and succeeds in obtaining results for more general settings.

Generally speaking, it is not easy to determine an existence of Nash equilibrium and compute the equilibrium points. For example, given a finite two-person game in normal form, it is not necessarily possible to determine within polynomial time whether there exists a *unique* Nash equilibrium in the game, see [8]. Our algorithm solves both problems at the same time in polynomial time. This fact has a significant implication that the players themselves can easily determine a rational price strategy in a discrete spatial duopoly setting.

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