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COMPARISON OF CYCLIC AND DELAYED MAINTENANCES FOR A PHASED ARRAY RADAR

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Abstract An antenna of phased array radar (PAR) consists of a large number of individual radiating elements and steers its electromagnetic wave direction by shifting wave phases of these elements. Failed elements have to be detected, diagnosed, localized and replaced at scheduled times to hold a required performance of a radar system. However, the maintenance of an antenna should not be made so frequently, because it interrupts the radar operation and degrades the radar availability. This paper considers two typical policies of cyclic and delayed maintenances for a PAR where a certain number of survival elements is needed to hold a required performance : Failed elements in cyclic maintenance are replaced at periodic times, and they in delayed maintenance are done when their number has exceeded a predesignated one. When failures of a PAR antenna elements occur at a Poisson process, the expected costs are obtained and optimal policies for two maintenances which minimize them are analytically discussed.

Keywords: Phased array, maintenance, diagnosis, failure detection, optimal policy.

1. Introduction

A phased array radar (PAR) is the radar which steers the electromagnetic wave direction electrically. Comparing with conventional radars which steer their electromagnetic wave direction by moving their antenna mechanically, a PAR has no mechanical portion to steer its wave direction, and hence, it can steer very rapidly. Most anti-aircraft missile systems and early warning systems have presently adopted PARs because they can acquire and track multiple targets simultaneously.

A PAR antenna consists of a large number of small and homogeneous elements which are arranged flatly and regularly, and steers its electromagnetic wave direction by shifting signal phases of waves which are radiated from these individual elements [2, 8, 9]. A schematic diagram of PAR antenna is drawn in Figure 1.

The increase in the number of failed elements degrades the radar performance, and at last, this may cause an undesirable situation such as the omission of targets [1]. The detection, diagnosis, localization and replacement of failed elements of a PAR antenna are indispensable to hold a certain required level of radar performance. A digital computer system controls a whole PAR system, and detects, diagnoses and localizes failed elements. However, such maintenance actions intermit the radar operation and decrease its availability. So that, the maintenance should not be made so frequently. From the above reasons, it would be important to decide an optimal maintenance policy for a PAR antenna, by comparing the downtime loss caused by its maintenance with the degradational loss caused by its performance downgrade.

Recently, a new method of failure detection for PAR antenna elements has been proposed

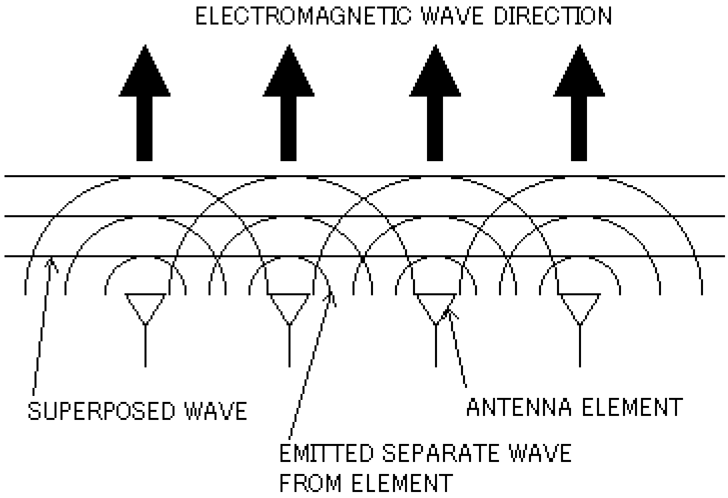

Figure 1: Schematic diagram of PAR antenna.

by measuring the electromagnetic wave pattern [3]. This method could detect some failed elements even when a radar system is operating, *i.e.*, it could be applied to the detection of confined failure modes such as power on-off failures. However, it would be generally necessary to stop the PAR operation for the detection of all failed elements.

Keithley [6] showed by Monte Carlo simulation that the maintenance time of PAR with 1024 elements had a strong influence on its availability. Hevesh [5] discussed the following three maintenances of PAR in which all failed elements could be detected immediately, and calculated the average times to failures of its equipments, and its availability in immediate maintenance :

1) *Immediate maintenance*: Failed elements are detected, localized and replaced immediately.

2) *Cyclic maintenance*: Failed elements are detected, localized and replaced periodically.

3) *Delayed maintenance*: Failed elements are detected and localized periodically, and replaced when their number has exceeded a predesignated one.

Further, Hesse [4] analysed the field maintenance data of U.S. Army prototype PAR, and clarified that the repair times have a log-normal distribution. In the actual maintenance, the immediate maintenance is rarely adopted because frequent maintenances degrade a radar system availability. Either cyclic or delayed maintenances is commonly adopted.

In this paper, we perform the periodic detection of failed elements of a PAR where it is consisted of N_0 elements and failures are detected at scheduled time T: If the number of failed elements has exceeded a specified number N $(0 < N \leq N_0)$, a PAR cannot hold a required level of radar performance, and it causes the operational loss such as the target oversight to a PAR. We assume that failed elements occur at a Poisson process, and consider two maintenances of cyclic and delayed. Applying the method of Nakagawa [7] to such maintenances, the expected costs per unit of time are obtained, and optimal policies which minimize them are analytically discussed. In a numerical example, we decide which maintenance is better, by comparing two expected costs.

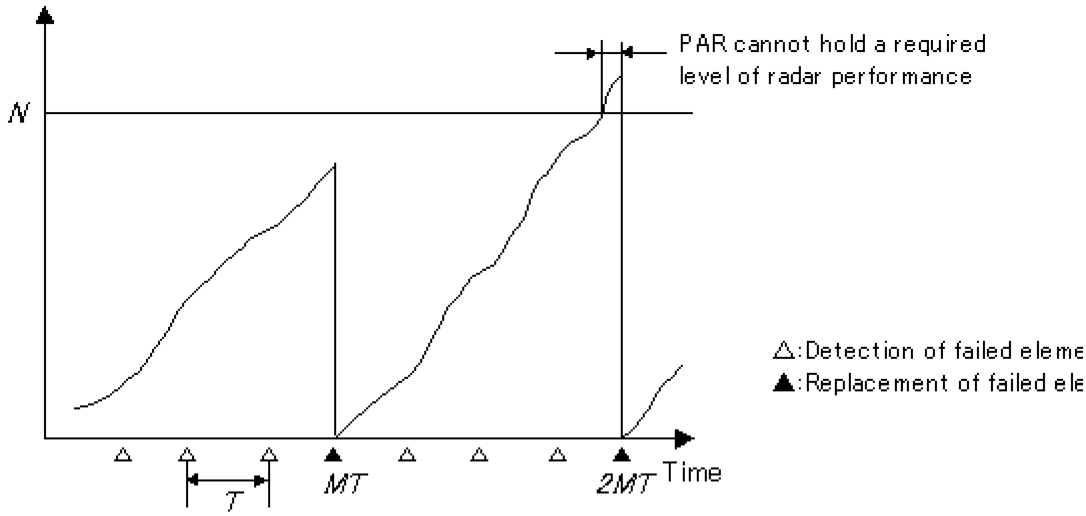

Figure 2: Schematic diagram of cyclic maintenance.

2. Cyclic Maintenance

2.1. Analysis

We consider the following cyclic maintenance of a PAR :

1) A PAR is consisted of N_0 elements which are independent and homogeneous on all plains of PAR, and have an identical constant hazard rate λ_0. The number of failed elements at time t has a binomial distribution with mean $N_0[1 - exp(-\lambda_0 t)]$. Since N_0 is large and λ_0 is very small, it might be assumed that failures of elements occur approximately at a Poisson process with mean $\lambda \equiv N_0\lambda_0$. That is, the probability that j failures occur during $(0, t]$ is

$$P_j(t) \equiv \frac{(\lambda t)^j e^{-\lambda t}}{j!} \qquad (j = 0, 1, 2, \cdots) .$$

2) When the number of failed elements has exceeded a specified number N, a PAR cannot hold a required level of radar performance such as maximum detection range and resolution.
3) Failed elements cannot be detected during operation and can be ascertained only according to the diagnosis software executed by a PAR system computer. The detection of failed elements is usually performed at periodic interval T and its time would be neglected because it is very short.
4) All failed elements are replaced by new ones at time $MT\,(M = 1, 2, \cdots)$ or at the time when failed elements have exceeded N, whichever occurs first.

A conceptual diagram of cyclic maintenance is drawn in Figure 2.

We introduce the following costs : A cost c_1 is the constant operation loss cost required for replacement, a cost c_2 is the replacement cost of one failed element, and a cost c_3 is the operation loss cost per unit of time caused by the degradation of radar performance. Then, the expected cost when the number of failed elements is less than N at the M-th failure detection is

$$\sum_{j=0}^{N-1} (c_1 + jc_2) P_j(MT) . \tag{1}$$

When the number of failed elements has exceeded N at the i-th failure detection ($i = 1, 2, \cdots M$), the expected cost is

$$\sum_{i=1}^{M}\sum_{j=0}^{N-1} P_j[(i-1)T] \sum_{k=N-j}^{\infty} \left\{P_k(T)[c_1+(j+k)c_2] + c_3\int_{(i-1)T}^{iT}(iT-t)dP_k[t-(i-1)T]\right\}. \quad (2)$$

Thus, the total expected cost until replacement is, from (1) and (2),

$$c_1 + c_2\lambda T\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT) + \frac{c_3}{\lambda}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T). \quad (3)$$

Next, the mean time to replacement when the number of failed elements is less than N at the M-th failure detection is

$$\sum_{j=0}^{N-1} MT P_j(MT). \quad (4)$$

When the number of failed elements has exceeded N at the i-th failure detection ($i = 1, 2, \cdots M$), the mean time to replacement is

$$\sum_{i=1}^{M}\sum_{j=0}^{N-1} P_j[(i-1)T]\sum_{k=N-j}^{\infty} iT\, P_k(T). \quad (5)$$

Thus, the mean time to the replacement is, from (4) and (5),

$$T\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT). \quad (6)$$

Therefore, by dividing (3) by (6), the expected cost per unit of time $C_1(M)$ is

$$C_1(M) = c_2\lambda + \frac{c_1 + \frac{c_3}{\lambda}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)}{T\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT)} \quad (M = 1, 2, \cdots). \quad (7)$$

Note that the replacement cost c_2 of a failed element does not affect an optimal M^* which minimizes $C_1(M)$.

2.2. Optimal policy

We seek an optimal number M^* which minimizes $C_1(M)$ in (7). Forming the inequality $C_1(M+1) - C_1(M) \geq 0$, we have

$$\frac{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT)\sum_{k=0}^{N-1} P_k(MT)\sum_{l=N-k}^{\infty}(l-N+k)P_l(T)}{\sum_{i=0}^{N-1} P_i(MT)} - \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} P_j(iT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T) \geq \frac{\lambda c_1}{c_3} \quad (M = 1, 2, \cdots). \quad (8)$$

Denoting the left-hand side of (8) by $Q(M)$, it is shown from Appendix 1 that $Q(M)$ is an increasing function of M from $Q(1)$ to $Q(\infty) \equiv \lim_{M\to\infty} Q(M)$. Thus, if there exists a finite M^* which satisfies (8), it is unique.

A necessary and sufficient condition that there exists a finite M^* which satisfies (8) is

$$Q(\infty) > \frac{\lambda c_1}{c_3}. \tag{9}$$

Further, from Appendix 2, since

$$Q(M) > \frac{\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1} P_i(MT)} - \sum_{i=0}^{\infty} iP_{N+i}(T) \quad (M=2,3,\cdots), \tag{10}$$

we have

$$\begin{aligned} Q(\infty) \quad \geq \quad & \lim_{M\to\infty}\left[\frac{\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1} P_i(MT)} - \sum_{i=0}^{\infty} iP_{N+i}(T)\right] \\ & = N - \sum_{i=0}^{N}(N-i)P_i(T) - (1-e^{-\lambda T}) \\ & = (N-1)(1-e^{-\lambda T}) - \sum_{i=1}^{N}(N-i)P_i(T). \end{aligned} \tag{11}$$

Thus, a sufficient condition that there exists a finite M^* which satisfies (8) is

$$(N-1)(1-e^{-\lambda T}) - \sum_{i=1}^{N}(N-i)P_i(T) > \frac{\lambda c_1}{c_3}. \tag{12}$$

Note that the left-hand side of (12) is strictly increasing in both T and N.

3. Delayed Maintenance

3.1. Analysis

We consider the delayed maintenance of a PAR :

4) All failed elements are replaced by new ones only when failed elements have exceeded a number $N_c\,(\leq N)$ at diagnosis.

The other assumptions are the same as ones in Section 2.1. A conceptual diagram of delayed maintenance is drawn in Figure 3.

When the number of failed elements is less than N_c at the $(i-1)$-th diagnosis and has exceeded N_c at the i-th diagnosis, the expected cost is given by

$$\sum_{i=1}^{\infty}\sum_{j=0}^{N_c-1} P_j[(i-1)T]\sum_{k=N_c-j}^{N-j-1}[c_1+(j+k)c_2]P_k(T). \tag{13}$$

When the number of failed elements is less than N_c at the $(i-1)$-th diagnosis and has exceeded N at the i-th diagnosis, the expected cost is

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=0}^{N_c-1} P_j[(i-1)T]\sum_{k=N-j}^{\infty}\Big\{[c_1 \quad + \quad & (j+k)c_2]P_k(T) \\ + \quad & c_3\int_{(i-1)T}^{iT}(iT-t)dP_k[t-(i-1)T]\Big\}. \end{aligned} \tag{14}$$

Thus, the total expected cost until replacement is, from (13) and (14),

$$c_1 + c_2\lambda T\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1} P_j(iT) + \frac{c_3}{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1} P_j(iT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T). \tag{15}$$

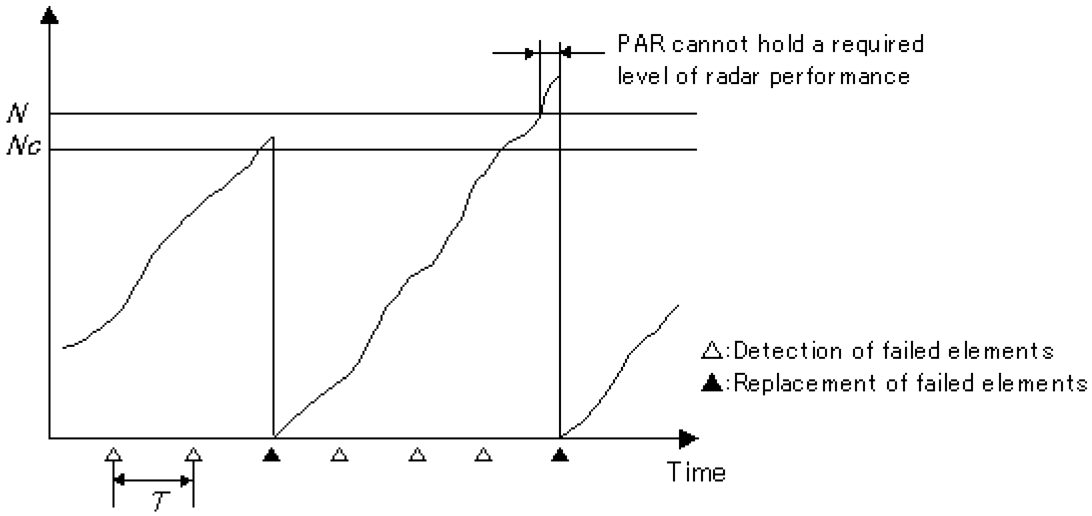

Figure 3: Schematic diagram of delayed maintenance.

Similarly, the mean time to replacement is

$$\sum_{i=1}^{\infty}\sum_{j=0}^{N_c-1}P_j[(i-1)T]\sum_{k=N_c-j}^{N-j-1}iTP_k(T)+\sum_{i=1}^{\infty}\sum_{j=0}^{N_c-1}P_j[(i-1)T]\sum_{k=N-j}^{\infty}iTP_k(T)$$
$$=T\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1}P_j(iT)\,. \tag{16}$$

Therefore, the expected cost per unit of time is, from (15) and (16),

$$C_2(N_c)=c_2\lambda+\frac{c_1+\frac{c_3}{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1}P_j(iT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)}{T\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1}P_j(iT)}\quad(N_c=1,2,\cdots,N)\,. \tag{17}$$

3.2. Optimal policy

We seek an optimal number N_c^* which minimizes $C_2(N_c)$ in (17). Forming the inequality $C_2(N_c+1)-C_2(N_c)\geq 0$, we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1}P_j(iT)\sum_{k=1}^{\infty}k[P_{N-N_c+k}(T)-P_{N-j+k}(T)]\geq\frac{\lambda c_1}{c_3}\quad(N_c=1,2,\cdots,N)\,. \tag{18}$$

Denoting the left-hand side of (18) by $L(N_c)$, it is shown from Appendix 3 that $L(N_c)$ is an increasing function of N_c. Therefore, if there exists a finite N_c^* which satisfies (18), it is unique.

A necessary and sufficient condition that there exists a finite N_c^* which satisfies (18) is

$$L(N)>\frac{\lambda c_1}{c_3}\,, \tag{19}$$

which corresponds to (9). Further, from Appendix 4, since

$$L(N_c)>\frac{N_c\sum_{i=N-N_c}^{\infty}P_i(T)-\sum_{i=N-N_c}^{N}(N-i)P_i(T)}{1-e^{-\lambda T}}\quad(N_c=2,3,\cdots)\,, \tag{20}$$

we have

$$L(N) \geq \lim_{N_c \to N} \left[\frac{N_c \sum_{i=N-N_c}^{\infty} P_i(T) - \sum_{i=N-N_c}^{N} (N-i) P_i(T)}{1 - e^{-\lambda T}} \right]$$

$$= \frac{\lambda T - \sum_{i=0}^{\infty} i P_{N+i}(T)}{1 - e^{-\lambda T}}\,. \tag{21}$$

Thus, a sufficient condition that there exists a finite N_c^* which satisfies (18) is

$$\frac{\lambda T - \sum_{i=0}^{\infty} i P_{N+i}(T)}{1 - e^{-\lambda T}} > \frac{\lambda c_1}{c_3}\,, \tag{22}$$

where note that the left-hand side of (22) is strictly increasing in N from 1 to $\lambda T/(1-e^{-\lambda T})$.

4. Numerical Examples

We show a numerical example in which optimal replacement numbers are computed under suitable conditions. An algorithm to compute optimal numbers M^* and N_c^* is specified as follows:

Step 1: Check sufficient conditions in (12) and (22), and confirm that these conditions are satisfied. Note that even if they are not satisfied, finite M^* and N_c^* may exist.
Step 2: Solve (8) and (18) by the half-interval search and compute M^* and N_c^* numerically.
Step 3: Compute $C_1(M^*)$ from (7) and $C_2(N_c^*)$ from (17).

For simplification, suppose that $c_2 = 0$, because it does not affect M^* and N_c^*. Table 1 gives the optimal numbers M^* and N_c^*, and the total expected costs $C_1(M^*)$ and $C_2(N_c^*)$ for $T = 24, 48, 72, \cdots 168\, hours(1, 2, 3, \cdots 7 days)$ and $\lambda_0 = 1, 2, 3, \cdots 10 \times 10^{-4}/hours$, when $N_0 = 1000$, $N = 100$ and $c_1 = c_3 = 1$. In this case, $\sum_{j=1001}^{\infty} P_j(t) \simeq 0$. All cases in Table 1 satisfy the sufficient conditions (12) and (22).

Table 1 indicates that M^* and N_c^* decrease when T and λ_0 increase. It is of great interest that the values of M^*T are approximately 720 when $\lambda_0 = 1 \times 10^{-4}$. In this calculation, $C_1(M^*)$ is always greater than $C_2(N_c^*)$ and $C_1(M^*)/C_2(N_c^*) \simeq 1.2$. Therefore, we can arbitrate in this case that the delayed maintenance is more efficient than the cyclic one from the economical point of view.

5. Conclusion

We have studied cyclic and delayed maintenances of a PAR which detect failed elements at periodic times. Assuming that the failure of elements approximately occur at a Poisson process, the expected costs per unit of time have been derived and the optimal replacement number M^* of cyclic maintenance and the optimal failed element number N_c^* of delayed maintenance have been analytically discussed. Comparing these two expected costs, the delayed maintenance is better than the cyclic one under some suitable conditions. Such comparison would be very useful at the beginning step when we have to plan a maintenance policy for a PAR.

Table 1. Optimal replacement number M^* and failed element number N_c^*, and expected costs $C_1(M^*)$ and $C_2(N_c^*)$

λ_0	T	M^*	M^*T	$C_1(M^*)$	N_c^*	$C_2(N_c^*)$	$C_1(M^*)/C_2(N_c^*)$
	24 (1)	31	744	1.38×10^{-3}	93	1.07×10^{-3}	1.29
	48 (2)	15	720	1.41×10^{-3}	89	1.10×10^{-3}	1.28
	72 (3)	10	720	1.42×10^{-3}	86	1.13×10^{-3}	1.26
1×10^{-4}	96 (4)	7	672	1.49×10^{-3}	83	1.16×10^{-3}	1.28
	120 (5)	6	720	1.42×10^{-3}	80	1.18×10^{-3}	1.20
	144 (6)	5	720	1.42×10^{-3}	77	1.21×10^{-3}	1.17
	168 (7)	4	672	1.49×10^{-3}	74	1.24×10^{-3}	1.20
2×10^{-4}		16	384	2.75×10^{-3}	90	2.19×10^{-3}	1.26
3×10^{-4}		11	264	4.19×10^{-3}	87	3.34×10^{-3}	1.25
4×10^{-4}		8	192	5.41×10^{-3}	85	4.54×10^{-3}	1.19
5×10^{-4}		6	144	6.98×10^{-3}	82	5.77×10^{-3}	1.21
6×10^{-4}	24 (1)	5	120	8.36×10^{-3}	80	7.03×10^{-3}	1.19
7×10^{-4}		5	120	1.04×10^{-2}	77	8.34×10^{-3}	1.25
8×10^{-4}		4	96	1.06×10^{-2}	75	9.69×10^{-3}	1.09
9×10^{-4}		3	72	1.39×10^{-2}	73	1.10×10^{-2}	1.26
10×10^{-4}		3	72	1.39×10^{-2}	71	1.26×10^{-2}	1.10

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Appendix

1. Prove that $Q(M)$ is an increasing function of M

From (8), we have

$$Q(M+1)-Q(M)=\left[\frac{\sum_{i=0}^{N-1}P_i[(M+1)T]\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1}P_i[(M+1)T]}\right.$$

$$-\frac{\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1} P_i(MT)}\Bigg]\sum_{k=1}^{M+1}\sum_{l=0}^{N-1} P_l[(k-1)T]\,. \tag{A.1}$$

The probability that the number of failed elements is less than N at time MT and exceeds N during the next T hours is given by

$$\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)=\sum_{i=0}^{N-1} P_i(MT)\sum_{j=0}^{\infty} jP_{j+N-i}(T)\,, \tag{A.2}$$

where $\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)$ means the expected number of failed elements during $(MT,(M+1)T]$ and it is an increasing function of i.

Since $\sum_{k=1}^{M+1}\sum_{l=0}^{N-1} P_l[(k-1)T]$ is positive, the braket of (A.1) is rewritten as

$$\begin{aligned}&\frac{\sum_{i=0}^{N-1} P_i[(M+1)T]\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1} P_i[(M+1)T]}\\ &\qquad-\frac{\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1} P_i(MT)}\\ &=\frac{\left[\begin{array}{l}\sum_{i=0}^{N-1} P_i[(M+1)T]\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)\sum_{k=0}^{N-1}P_k(MT)\\ -\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)\sum_{k=0}^{N-1}P_k[(M+1)T]\end{array}\right]}{\sum_{i=0}^{N-1} P_i[(M+1)T]\sum_{j=0}^{N-1}P_j(MT)}\,.\end{aligned} \tag{A.3}$$

The denominator of right-hand side is positive and its numerator is rewritten as

$$\begin{aligned}&\sum_{i=0}^{N-1} P_i[(M+1)T]\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)\sum_{k=0}^{N-1}P_k(MT)\\ &\qquad-\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)\sum_{k=0}^{N-1}P_k[(M+1)T]\\ &=e^{-\lambda T}\sum_{i=0}^{N-1}\sum_{j=0}^{i}P_i(MT)P_j(MT)\left[\left(\frac{M+1}{M}\right)^i-\left(\frac{M+1}{M}\right)^j\right]\\ &\qquad\times\left[\sum_{k=N-i}^{\infty}(k-N+i)P_k(T)-\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)\right],\end{aligned} \tag{A.4}$$

since both $[(M+1)/M]^i$ and $\sum_{k=N-i}^{\infty}(k-N+i)P_k(T)$ are increasing functions of i. Therefore, $Q(M+1)-Q(M)>0$ and $Q(M)$ is an increasing function of M.

2. Proofing of inequality (10)

Supposing that $1\le L<M$, we have, from (8) and (10),

$$\begin{aligned}&\frac{\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)\sum_{k=1}^{L+1}\sum_{l=0}^{N-1}P_l[(k-1)T]}{\sum_{i=0}^{N-1}P_i(MT)}\\ &\qquad-\sum_{i=1}^{L+1}\sum_{j=0}^{N-1}P_j[(i-1)T]\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)\\ &-\Bigg[\frac{\sum_{i=0}^{N-1} P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)\sum_{k=1}^{L}\sum_{l=0}^{N-1}P_l[(k-1)T]}{\sum_{i=0}^{N-1}P_i(MT)}\end{aligned}$$

$$-\sum_{i=1}^{L}\sum_{j=0}^{N-1}P_j[(i-1)T]\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)\Bigg]$$

$$=\left[\frac{\sum_{i=0}^{N-1}P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1}P_i(MT)}\right.$$
$$\left.-\frac{\sum_{i=0}^{N-1}P_i(LT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1}P_i(LT)}\right]\sum_{k=0}^{N-1}P_k(LT)\,. \qquad \text{(A.5)}$$

The bracket of right-hand side is rewritten as

$$\frac{\sum_{i=0}^{N-1}P_i[(L+1)T]\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1}P_i[(L+1)T]} \qquad \text{(A.6)}$$
$$-\frac{\sum_{i=0}^{N-1}P_i(LT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1}P_i(LT)}$$
$$=\frac{\left[\begin{array}{l}\sum_{i=0}^{N-1}P_i(LT)\sum_{j=0}^{N-1}P_j[(L+1)T]\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)\\ -\sum_{i=0}^{N-1}P_i[(L+1)T]\sum_{j=0}^{N-1}P_j(LT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)\end{array}\right]}{\sum_{i=0}^{N-1}P_i(LT)\sum_{j=0}^{N-1}P_j[(L+1)T]}\,.$$

The denominator of right-hand side is positive, and its numerator is

$$\sum_{i=0}^{N-1}P_i(LT)\sum_{j=0}^{N-1}P_j[(L+1)T]\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)$$
$$-\sum_{i=0}^{N-1}P_i[(L+1)T]\sum_{j=0}^{N-1}P_j(LT)\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)$$
$$=e^{-\lambda T}\sum_{i=0}^{N-1}\sum_{j=0}^{i}P_i(LT)P_j(LT)\left[\left(\frac{L+1}{L}\right)^i-\left(\frac{L+1}{L}\right)^j\right]$$
$$\times\left[\sum_{k=N-i}^{\infty}(k-N+i)P_k(T)-\sum_{k=N-j}^{\infty}(k-N+j)P_k(T)\right]. \qquad \text{(A.7)}$$

Noting that both $[(L+1)/L]^i$ and $\sum_{k=N-i}^{\infty}(k-N+i)P_k(T)$ are also increasing functions of i from Appendix 1, (A.7) is positive. Therefore, (A.5) is also positive, and hence, we have

$$Q(M)>\frac{\sum_{i=0}^{N-1}P_i(MT)\sum_{j=N-i}^{\infty}(j-N+i)P_j(T)}{\sum_{i=0}^{N-1}P_i(MT)}-\sum_{i=1}^{\infty}iP_{N+i}(T)\,. \qquad \text{(A.8)}$$

3. Prove that $L(N_c)$ is an increasing function of N_c

From (18), we have

$$\begin{aligned}L(N_c+1)-L(N_c) &= \sum_{i=0}^{\infty}\sum_{j=0}^{N_c}P_j(iT)\sum_{k=0}^{\infty}k[P_{N-N_c+k+1}(T)-P_{N-j+k}(T)]\\ &\quad -\sum_{i=0}^{\infty}\sum_{j=0}^{N_c-1}P_j(iT)\sum_{k=0}^{\infty}k[P_{N-N_c+k}(T)-P_{N-j+k}(T)]\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{N_c}P_j(iT)\sum_{k=N-N_c}^{\infty}P_k(T)>0\,. \qquad \text{(A.9)}\end{aligned}$$

4. Proof of inequality (20)

We define

$$L_l(N_c) \equiv \sum_{i=0}^{\infty}\sum_{j=0}^{l} P_j(iT)\sum_{k=0}^{\infty} k[P_{N-N_c+k}(T) - P_{N-j+k}(T)]\,, \tag{A.10}$$

for $0 \le l < N_c$. Then, we can easily prove

$$L_{N_c-1}(N_c) - L_{N_c-2}(N_c) = \sum_{i=0}^{\infty} P_{N_c}(iT) \sum_{k=N-N_c+1}^{\infty} P_k(T) > 0\,. \tag{A.11}$$

Therefore, $L_l(N_c)$ is an increasing of l, and hence, we have

$$L(N_c) > \lim_{l\to 0} L_l(N_c) = \frac{N_c \sum_{i=N-N_c}^{\infty} P_i(T) - \sum_{i=N-N_c}^{N}(N-i)P_i(T)}{1-e^{-\lambda T}}\,. \tag{A.12}$$

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