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REMARKS ON THE CONCURRENT CONVERGENCE METHOD FOR A TYPICAL MUTUAL EVALUATION SYSTEM

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Abstract A mutual evaluation system deals with a decision making problem including the feedback structure between alternatives and criteria. By incorporating decision makers' intuitive judgments from alternatives to criteria into overall evaluations for alternatives, the mutual evaluation system may be effective to make a consensus among the decision makers of the problem. An analyzing tool of the mutual evaluation system, Concurrent Convergent Method (CCM) by Kinoshita and Nakanishi, has practical advantage. This paper introduces an overall weight vector for criteria into CCM. By using the overall weight vector not for alternatives but for criteria, we demonstrate irrationality of CCM under the Pareto principle.

Keywords: AHP, decision making, analytic network process, concurrent convergent method, pareto principle

1. Introduction

Saaty [4] extends a hierarchical structure of criteria and alternatives into a network one, and proposes Analytic Network Process (ANP). When evaluation values between criteria and alternatives are derived by intuitive judgments of a decision maker, ANP provides overall weights of criteria and alternatives.

In practice, it is not easy for the decision maker to evaluate criteria from alternatives. So evaluation values from alternatives to criteria often appear to be very unstable. In order to overcome this difficulty, Kinoshita and Nakanishi [2] develop an iterative method, the Concurrent Convergence Method (CCM). Takahashi [9] explains that CCM not only stabilizes evaluation values from alternatives to criteria but also provides the overall weights of alternatives. Furthermore, the convergence of CCM is proved by Kinoshita, Sekitani and Shi [3]. In spite of the practical advantage of CCM and the guarantee of its convergence, there has never been any case study using CCM. That is the reason why properties of the stable evaluation values has not been investigated sufficiently. Furthermore, Kinoshita et al. [2] and [3] have never paid attention to overall weights of criteria in CCM.

In ANP, the overall weights of criteria are well defined. They are the linear combination of evaluation values from alternatives to criteria with overall weights of alternatives, and overall weights of alternatives are the linear combination of evaluation values from criteria to alternatives with them since the overall weights of criteria and alternatives correspond to a principal eigenvector of the so-called supermatrix of ANP (see [5, 9] for the details). The two linear systems with overall weights of alternatives and criteria imply in terms of the regression analysis that the overall weights of criteria are dependent variables of the independent variables, overall weights of alternatives, and vice versa. This outer dependence between the overall weights of criteria and the overall weights of alternatives may increase accountability for each other effectively.

By introducing overall weights of criteria into CCM, this study shows that CCM has their dependence on the overall weights of alternatives. Since the overall weights of criteria can be regarded as what represents all preference relationships of criteria from each alternative, they must satisfy some natural and reasonable requirements such as the Pareto principle [7]. For example, if each alternative has the same ranking for criteria as the others, then the overall weights of criteria should also mean the same one. This study reports that the overall weights of criteria by CCM may violate the Pareto principle while the overall weights of criteria by ANP always satisfies it. This irrational relationship leads to lack of accountability for overall weights of alternatives by CCM. Under the Pareto principle we can evaluate whether the overall weights of alternatives by CCM is acceptable or not.

This paper is organized as follows: In section 2 we summarize CCM and introduce the concept of overall weights of criteria into CCM. Furthermore, section 2 discusses the properties of overall weights of criteria such as its existence and uniqueness. In section 3 we show a numerical example of CCM such that the overall weights of criteria does not satisfy the Pareto principle. Section 4 shows that the irrational overall weights of criteria in the example of section 3 is not essentially caused by the sum-one column-wise standardization of an evaluation matrix. Hence, we report that CCM provides irrational overall weights of criteria when the evaluation matrix is standardized by some column-wise standardization other than that of section 4. Section 5 gives appearance frequency of occurrence of the irrational overall weights of criteria by numerical experiments. Finally, we devote section 6 to discussions.

2. Overall Weights for Criteria in CCM

First we introduce CCM [2] according to [3] briefly. Let I and J be a set of alternatives and that of criteria, respectively, then a decision maker evaluates alternative $i \in I$ from criteria $j \in J$ and the evaluation value is denoted by a_{ij}. A matrix $A = [a_{ij}]$ is called the evaluation matrix. In CCM the decision maker specifies some alternatives that play the role of a yardstick in the evaluation process (see [2, 3] for details of the regulating alternatives). Let K be a set of regulating alternatives and let A_k be a diagonal matrix whose (j, j) component is a_{kj} for all $k \in K$, then CCM regards AA_k^{-1} as the evaluation matrix of alternative k when regulating alternative k is a yardstick in the evaluation process.

CCM requires the decision maker to evaluate criteria from the viewpoint of each regulating alternative $k \in K$. The evaluation value from regulating alternative k to criterion j is denoted by b_j^k and $\boldsymbol{b}^k = \left[b_1^k, \ldots, b_{|J|}^k\right]^\top$ is called the weight vector for criteria from alternative k, where $^\top$ stands for the transpose operation. All $\boldsymbol{b}^k$ are normalized, that is, $\boldsymbol{e}^\top \boldsymbol{b}^k = 1$ for all $k \in K$, where $\boldsymbol{e}$ is an appropriate dimensional vector whose component is all one. The set $\left\{ \boldsymbol{b}^k | \, k \in K\right\}$ is transformed into $\{ \hat{\boldsymbol{b}}^k | \, k \in K\}$ by the following iterative procedure:

Algorithm 0

Step 0: For the given set $\left\{\boldsymbol{b}^k \mid k \in K\right\}$ of the weight vectors for criteria, let

$$\boldsymbol{b}_0^k := \boldsymbol{b}^k \tag{1}$$

for all $k \in K$. Let $t := 0$ and go to **Step 1**.

Step 1: Let

$$\boldsymbol{b}_{t+1}^i := \frac{1}{|K|} \sum_{k \in K} \frac{A_i A_k^{-1} \boldsymbol{b}_t^k}{\boldsymbol{e}^\top A_i A_k^{-1} \boldsymbol{b}_t^k} \tag{2}$$

for all $i \in I$.

Step 2: If $\max_{i\in I} \|\boldsymbol{b}_{t+1}^i - \boldsymbol{b}_t^i\| \leq \epsilon$ then set $\hat{\boldsymbol{b}}^i := \boldsymbol{b}_{t+1}^i$ for all $i \in I$ and stop. Otherwise, update $t := t+1$ and go to **Step 1**.

Here, ϵ in Step 2 is given as a tolerance of the convergence. Algorithm 0 makes $\{\boldsymbol{b}^i | i \in K\}$ into $\{\hat{\boldsymbol{b}}^i | i \in I\}$ such that

$$\frac{A_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top A_k^{-1}\hat{\boldsymbol{b}}^k} = \frac{A_l^{-1}\hat{\boldsymbol{b}}^l}{\boldsymbol{e}^\top A_l^{-1}\hat{\boldsymbol{b}}^l} \tag{3}$$

for all $k, l \in I$. It follows from (3) that $AA_k^{-1}\hat{\boldsymbol{b}}^k$ coincides (up to scalar multiples) with $AA_l^{-1}\hat{\boldsymbol{b}}^l$ for all $l \in I$. This means that

$$\frac{AA_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AA_k^{-1}\hat{\boldsymbol{b}}^k} = \frac{AA_l^{-1}\hat{\boldsymbol{b}}^l}{\boldsymbol{e}^\top AA_l^{-1}\hat{\boldsymbol{b}}^l} \tag{4}$$

for all $k, l \in I$, that is called the consistency property [3]. Hereafter, if a vector $\boldsymbol{a}$ coincides up to scalar multiples with a vector $\boldsymbol{b}$, we say that $\boldsymbol{a}$ has the same direction as $\boldsymbol{b}$. It follows from (4) that $AA_k^{-1}\hat{\boldsymbol{b}}^k$ has the same direction as $AA_l^{-1}\hat{\boldsymbol{b}}^l$ for all $l \neq k$. Therefore, Kinoshita and Nakanishi [2] call

$$\frac{AA_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AA_k^{-1}\hat{\boldsymbol{b}}^k} \tag{5}$$

the overall weight vector for alternatives, which is denoted by $\boldsymbol{p}$. The pair of Algorithm 0 and (5) is called CCM by Kinoshita and Nakanishi [2].

We let $I = \{1, \dots, m\}$ and assume $I = K$ without loss of generality. For the the overall weight vector for alternatives $\boldsymbol{p}$, we consider a vector $\boldsymbol{q}$ such that

$$\boldsymbol{q} = \left[\ \hat{\boldsymbol{b}}^1 \ \ \cdots \ \ \hat{\boldsymbol{b}}^m\ \right] \boldsymbol{p}. \tag{6}$$

The equation (6) means that $\boldsymbol{p}$ are dependent variables of $\boldsymbol{q}$. Since $\boldsymbol{e}^\top \boldsymbol{p}$ and $\boldsymbol{e}^\top \hat{\boldsymbol{b}}^i = 1$ for all $i = 1, \dots, m$, it follows from (6) that $\boldsymbol{e}^\top \boldsymbol{q} = 1$ and that $\boldsymbol{q}$ is a convex combination of $\{\hat{\boldsymbol{b}}^1, \dots, \hat{\boldsymbol{b}}^m\}$ with $\boldsymbol{p}$. In ANP, the overall weight vector for criteria is defined by a convex combination of all weight vectors for criteria from each alternative with coefficients that are components of the overall weight vector for alternatives. In the same way as ANP, we define $\boldsymbol{q}$ of (6) as the overall weight vector for criteria by CCM.

Since it follows from (3) that $A_i^{-1}\hat{\boldsymbol{b}}^i/(\boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i)$ is constant independent of the choice of $i \in I$, the overall weight vector for criteria $\boldsymbol{q}$ is associated with (3) by the following lemma:

Lemma 1 *Suppose that $K = I = \{1, \dots, m\}$. Let $\boldsymbol{r}$ be $A_i^{-1}\hat{\boldsymbol{b}}^i/(\boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i)$ satisfying (3) for some $i \in I$, then the overall weight vector $\boldsymbol{q}$ for criteria has the same direction as $\left(\sum_{i\in I} A_i\right)\boldsymbol{r}$.*

Proof: Since $\boldsymbol{r} = A_i^{-1}\hat{\boldsymbol{b}}^i/(\boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i)$ for all $i \in I$, we have

$$\boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i A_i \boldsymbol{r} = \hat{\boldsymbol{b}}^i \tag{7}$$

for all $i \in I$. This means from (2) that

$$1 = \boldsymbol{e}^\top \hat{\boldsymbol{b}}^i = \boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i \boldsymbol{e}^\top A_i \boldsymbol{r} \tag{8}$$

for all $i \in I$. From (7) we have

$$\boldsymbol{p} = \frac{AA_i^{-1}\hat{\boldsymbol{b}}^i}{\boldsymbol{e}^\top AA_i^{-1}\hat{\boldsymbol{b}}^i} = \frac{A\boldsymbol{r}(\boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i)}{\boldsymbol{e}^\top A\boldsymbol{r}(\boldsymbol{e}^\top A_i^{-1}\hat{\boldsymbol{b}}^i)} = \frac{A\boldsymbol{r}}{\boldsymbol{e}^\top A\boldsymbol{r}} = \frac{1}{\boldsymbol{e}^\top A\boldsymbol{r}}\begin{bmatrix} \boldsymbol{e}^\top A_1\boldsymbol{r} \\ \vdots \\ \boldsymbol{e}^\top A_m\boldsymbol{r} \end{bmatrix}. \tag{9}$$

It follows from (6), (7), (9) and (8) that

$$\begin{aligned}
\boldsymbol{q} &= \begin{bmatrix} \hat{\boldsymbol{b}}^1 & \cdots & \hat{\boldsymbol{b}}^m \end{bmatrix}\boldsymbol{p} = \frac{1}{\boldsymbol{e}^\top A\boldsymbol{r}}\begin{bmatrix} \boldsymbol{e}^\top A_1^{-1}\hat{\boldsymbol{b}}^1 A_1\boldsymbol{r} & \cdots & \boldsymbol{e}^\top A_m^{-1}\hat{\boldsymbol{b}}^m A_m\boldsymbol{r} \end{bmatrix}\begin{bmatrix} \boldsymbol{e}^\top A_1\boldsymbol{r} \\ \vdots \\ \boldsymbol{e}^\top A_m\boldsymbol{r} \end{bmatrix} \\
&= \frac{1}{\boldsymbol{e}^\top A\boldsymbol{r}}\begin{bmatrix} A_1\boldsymbol{r} & \cdots & A_m\boldsymbol{r} \end{bmatrix}\begin{bmatrix} \boldsymbol{e}^\top A_1^{-1}\hat{\boldsymbol{b}}^1 & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{e}^\top A_m^{-1}\hat{\boldsymbol{b}}^m \end{bmatrix}\begin{bmatrix} \boldsymbol{e}^\top A_1\boldsymbol{r} \\ \vdots \\ \boldsymbol{e}^\top A_m\boldsymbol{r} \end{bmatrix} \\
&= \frac{1}{\boldsymbol{e}^\top A\boldsymbol{r}}\begin{bmatrix} A_1\boldsymbol{r} & \cdots & A_m\boldsymbol{r} \end{bmatrix}\begin{bmatrix} \boldsymbol{e}^\top A_1^{-1}\hat{\boldsymbol{b}}^1\boldsymbol{e}^\top A_1\boldsymbol{r} \\ \vdots \\ \boldsymbol{e}^\top A_m^{-1}\hat{\boldsymbol{b}}^m\boldsymbol{e}^\top A_m\boldsymbol{r} \end{bmatrix} = \frac{1}{\boldsymbol{e}^\top A\boldsymbol{r}}\begin{bmatrix} A_1\boldsymbol{r} & \cdots & A_m\boldsymbol{r} \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= \frac{1}{\boldsymbol{e}^\top A\boldsymbol{r}}\left(\sum_{i\in I} A_i\right)\boldsymbol{r}. \qquad \square
\end{aligned}$$

From Lemma 1 we have the following outer dependence between the overall weight vector for alternatives and that for criteria:

Lemma 2 *Suppose that $K = I$, then we have*

$$\boldsymbol{p} = A\left(\sum_{i\in I} A_i\right)^{-1}\boldsymbol{q}. \tag{10}$$

Proof: It follows from Lemma 1 that $\boldsymbol{q} = \left(\boldsymbol{e}^\top AA_i^{-1}\hat{\boldsymbol{b}}^i\right)^{-1}\left(\sum_{l\in I} A_l\right)A_i^{-1}\hat{\boldsymbol{b}}^i$ for all $i \in I$. This means from (5) that

$$A\left(\sum_{l\in I} A_l\right)^{-1}\boldsymbol{q} = \frac{1}{\boldsymbol{e}^\top AA_i^{-1}\hat{\boldsymbol{b}}^i}AA_i^{-1}\hat{\boldsymbol{b}}^i = \boldsymbol{p}.$$

$\square$

A meaning of the equation (10) is as follows: Since $\sum_{i\in I} A_i$ is the diagonal matrix whose (j, j) component is a sum of the jth column of A, $A\left(\sum_{i\in I} A_i\right)^{-1}$ has all column-sums equal to 1, which is a typical column-wise standardization evaluation matrix in Analytic Hierarchy Process (AHP) and ANP (e.g., see (2.9) of [4] for the evaluation matrix of AHP). AHP and ANP define the overall weight vector for alternatives as multiplication of the typical column-wise standardization evaluation matrix by the given overall weight vector for criteria. This multiplication coincides with the right-hand side of (10) and it is equal to the overall weight vector for alternatives $\boldsymbol{p}$. Hence, it follows from (10) that the definition (6) of the overall weight vector for criteria $\boldsymbol{q}$ is consistent with the way of definition of the overall weight vector for alternatives in AHP and ANP. The fact is summarized as the following theorem:

Theorem 3 *Let $I = \{1, \ldots, m\}$ and suppose $I = K$. Suppose that $\{\hat{\boldsymbol{b}}^1, \ldots, \hat{\boldsymbol{b}}^m\}$ satisfies (3), then the overall weight vector for criteria $\boldsymbol{q}$ defined by (6) is a positive principal eigenvector of*

$$\left[\begin{array}{ccc} \hat{\boldsymbol{b}}^1 & \cdots & \hat{\boldsymbol{b}}^m \end{array} \right] A(\textstyle\sum_{i \in I} A_i)^{-1}. \tag{11}$$

Hence, $[\boldsymbol{q}^\top, \boldsymbol{p}^\top]^\top$ is a positive principal eigenvector of a supermatrix

$$\left[\begin{array}{cc} 0 & \hat{\boldsymbol{b}}^1 \cdots \hat{\boldsymbol{b}}^m \\ A(\sum_{i \in I} A_i)^{-1} & 0 \end{array} \right], \tag{12}$$

where $\boldsymbol{p}$ is defined by (5).

Proof: It follows from (10) and (6) that

$$\left[\begin{array}{ccc} \hat{\boldsymbol{b}}^1 & \cdots & \hat{\boldsymbol{b}}^m \end{array} \right] A(\textstyle\sum_{i \in I} A_i)^{-1} \boldsymbol{q} = \left[\begin{array}{ccc} \hat{\boldsymbol{b}}^1 & \cdots & \hat{\boldsymbol{b}}^m \end{array} \right] \boldsymbol{p} = \boldsymbol{q},$$

which implies that $\boldsymbol{q}$ is an eigenvector of (11). Since the matrix (11) is irreducible and $\boldsymbol{q}$ is a positive vector, it follows from Perron-Frobenius Theorem (e.g., see Theorem 4 of [5]) that $\boldsymbol{q}$ is a principal eigenvector of (11).

In the same way, we can show that $\left[\boldsymbol{q}^\top, \boldsymbol{p}^\top\right]^\top$ is a principal eigenvector of (12). □

By Theorem 3 we have the unique pair of $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying (6) and (10).

Takahashi [8] implicitly assumes the assertion of Theorem 3 and then he defines CCM as the pair of Algorithm 0 and applying the eigenvalue method to (12). Therefore, this study contributes an explicit proof of Theorem 3.

Takahashi-type CCM seems different from the original CCM, the pair of Algorithm 0 and (5) by Kinoshita and Nakanishi [2]. However Theorem 3 guarantees that the Takahashi-type CCM is equivalent to the pair of the original CCM and (6). Namely, (6) may be a natural definition of the overall weight vector of criteria by CCM.

3. A Numerical Example of a Paradox

This section demonstrates a potential irrationality of CCM by using the overall weight vector not for alternatives but for criteria. We considers that the overall weight vector $\boldsymbol{q}$ for criteria should satisfy the three following requirements:

1. $q_j \leq q_l$ if $b_j^k \leq b_l^k$ for all $k \in K$
2. $q_j < q_l$ if $b_j^k < b_l^k$ for all $k \in K$
3. $q_j = q_l$ if $b_j^k = b_l^k$ for all $k \in K$,

where q_j is the jth component of $\boldsymbol{q}$. These requirements for $\boldsymbol{q}$ is called Pareto principle. If the overall weight vector for criteria violates at least one of three requirements, we say that the overall weight vector for criteria is irrational.

Firstly, we show a numerical example with two criteria and three alternatives. Suppose that $I = K = \{1, 2, 3\}$, $J = \{1, 2\}$,

$$A = \left[\begin{array}{cc} 1 & 1 \\ 3/8 & 32 \\ 1/8 & 1/16 \end{array} \right], \boldsymbol{b}^1 = \left[\begin{array}{c} 0.520 \\ 0.480 \end{array} \right], \boldsymbol{b}^2 = \left[\begin{array}{c} 0.510 \\ 0.490 \end{array} \right] \text{ and } \boldsymbol{b}^3 = \left[\begin{array}{c} 0.530 \\ 0.470 \end{array} \right]. \tag{13}$$

Since $b_1^j > b_2^j$ for all $j = 1, 2, 3$, all alternatives prefer criterion 1 to criterion 2. From the input data $A, \boldsymbol{b}^1, \boldsymbol{b}^2$ and $\boldsymbol{b}^3$ of (13), CCM provides

$$\hat{\boldsymbol{b}}^1 = \left[\begin{array}{c} 0.775 \\ 0.225 \end{array} \right], \hat{\boldsymbol{b}}^2 = \left[\begin{array}{c} 0.039 \\ 0.961 \end{array} \right], \hat{\boldsymbol{b}}^3 = \left[\begin{array}{c} 0.865 \\ 0.135 \end{array} \right] \text{ and } \boldsymbol{p} = \left[\begin{array}{c} 0.116 \\ 0.871 \\ 0.013 \end{array} \right], \tag{14}$$

where $\epsilon = 10^{-3}$ in Step 2 in Algorithm 0. The numerical results of three iterates $\boldsymbol{b}_t^1, \boldsymbol{b}_t^2$ and $\boldsymbol{b}_t^3$ of Algorithm 0 are given in Appendix 1. From (6) we have

$$\boldsymbol{q} = \begin{bmatrix} 0.135 \\ 0.865 \end{bmatrix}. \tag{15}$$

Since the first component of $\boldsymbol{q}$ of (15) is less than the second one, the overall weight vector $\boldsymbol{q}$ means that criterion 2 is preferred to 1 in the aggregate. However, no alternative prefers criterion 2 to 1. This does not satisfy the Pareto principle, that is, though each regulating alternative has the same ranking of criteria as the others, the overall ranking of criteria is against the same one.

In order to apply ANP to the numerical example (13), we have a supermatrix

$$S = \begin{bmatrix} 0 & 0 & 0.520 & 0.510 & 0.530 \\ 0 & 0 & 0.480 & 0.490 & 0.470 \\ 2/3 & 16/529 & 0 & 0 & 0 \\ 1/4 & 512/529 & 0 & 0 & 0 \\ 1/12 & 1/529 & 0 & 0 & 0 \end{bmatrix}$$

and find a principal eigenvector $\left[\boldsymbol{x}^\top, \boldsymbol{y}^\top\right]^\top$ of S. Since

$$\frac{\boldsymbol{x}}{\boldsymbol{e}^\top \boldsymbol{x}} = \begin{bmatrix} 0.514 \\ 0.486 \end{bmatrix} \text{ and } \frac{\boldsymbol{y}}{\boldsymbol{e}^\top \boldsymbol{y}} = \begin{bmatrix} 0.357 \\ 0.599 \\ 0.044 \end{bmatrix},$$

the overall weight vector for criteria by ANP means that criterion 1 is preferred to 2 in the aggregate. Therefore, the overall weight vector for criteria by ANP satisfies the Pareto principle. Though the ranking of alternatives by ANP is equal to that by CCM, the overall weight vector for alternatives by ANP is more easily accountable than that by CCM under the Pareto principle. Furthermore, the following theorem guarantees that the overall weight vector for criteria by ANP always satisfies the Pareto principle.

Theorem 4 *Let $I = \{1, \ldots, m\}$ and suppose $I = K$. If $[\boldsymbol{x}^\top, \boldsymbol{y}^\top]^\top$ is a positive principal eigenvector of a supermatrix*

$$\begin{bmatrix} 0 & \boldsymbol{b}^1 \cdots \boldsymbol{b}^m \\ A(\sum_{i \in I} A_i)^{-1} & 0 \end{bmatrix}, \tag{16}$$

then $\boldsymbol{x}$ satisfies the Pareto principle.

Proof: Since the supermatrix (16) is an irreducible nonnegative matrix, there exists a positive principle eigenvector $[\boldsymbol{x}^\top, \boldsymbol{y}^\top]^\top$ of (16). If $b_l^i \leq b_j^i$ for all $i \in I$ and $b_l^h < b_j^h$ for some $h \in I$, then we have

$$x_j - x_l = \sum_{i \in I} b_j^i y_i - \sum_{i \in I} b_l^i y_i = \sum_{i \in I} (b_j^i - b_l^i) y_i > \sum_{i \neq h} (b_j^i - b_l^i) y_i \geq 0.$$

This means that the first and second requirement of the Pareto principle are met. In the similar way, we can prove the third requirement of the Pareto principle. □

A paradox is defined by a case where there exists an overall weight vector for criteria violating the Pareto principle. Theorem 4 and the numerical example (13) imply that ANP is free from the paradox but CCM is not.

4. Variation of Column-wise Standardization of the Evaluation Matrix

This section relaxes and generalizes the definition (6) of the overall weight vector for criteria. So we consider a problem of finding a positive vector $\boldsymbol{q}^N$ and a positive number λ such that

$$AN^{-1}\boldsymbol{q}^N = \lambda\boldsymbol{p}, \tag{17}$$

where N is an $m \times m$ diagonal matrix whose diagonal component n_{jj} is positive for all $j = 1, \ldots, m$. Replacing N and $\boldsymbol{q}^N$ of (17) with $\sum_{i\in I} A_i$ and $\boldsymbol{q}$, respectively, and setting $\lambda = 1$ in (17), (17) is equivalent to (10). Since $\boldsymbol{q}$ of (6) satisfies (10), a pair of $\boldsymbol{q}^N = \boldsymbol{q}$ and $\lambda = 1$ satisfies (17) replacing N with $\sum_{i\in I} A_i$. In other words, the overall weight vector of criteria defined by (6) satisfies (17) with $N = \sum_{i\in I} A_i$ and $\lambda = 1$. Hence, (6) is relaxed and generalized into (17). We call $\boldsymbol{q}^N$ of (17) the overall weight vector of criteria with respect to N.

The diagonal matrix N of (17) plays the role of the column-wise standardization of the evaluation matrix A. For example, when the diagonal matrix N has each diagonal component $n_{jj} = \sum_{i\in I} a_{ij}$ for all $j \in J$, AN^{-1} is the sum-one column-wise standardization evaluation matrix. When $n_{jj} = \max_{i\in I} a_{ij}$ for all $j \in J$, each column of AN^{-1} has at least one maximum value 1 and it is so-called ideal mode [4]. The diagonal component n_{jj} can be chosen independent of the jth column of A. Hence, any column-wise standardization corresponds to a diagonal matrix N and AN^{-1} is any type of a column-wise standardization evaluation matrix. We can consider $\boldsymbol{q}^N$ with respect to any column-wise standardization.

Note that the output $\left\{\hat{\boldsymbol{b}}^i | i \in I\right\}$ of Algorithm 0 remains unchanged if the input data A is replaced with AN^{-1} for any standardizing matrix N. In other words, the output $\left\{\boldsymbol{b}^i | i \in I\right\}$ of Algorithm 0 is invariant under the column-wise standardization. Suppose that A is replaced with AN^{-1}, then the overall weight vector for alternatives is

$$\frac{AN^{-1}\left(A_k N^{-1}\right)^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AN^{-1}\left(A_k N^{-1}\right)^{-1}\hat{\boldsymbol{b}}^k}. \tag{18}$$

It follows from (18) and (5) that

$$\frac{AN^{-1}\left(A_k N^{-1}\right)^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AN^{-1}\left(A_k N^{-1}\right)^{-1}\hat{\boldsymbol{b}}^k} = \frac{AA_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AA_k^{-1}\hat{\boldsymbol{b}}^k} = \boldsymbol{p}.$$

Therefore, the overall weight vector $\boldsymbol{p}$ for alternatives is also invariant under the column-wise standardizations. From the invariance of overall weight vector for alternatives, it is natural that the right-hand side of (17) is $\boldsymbol{p}$.

Though the definition of (17) is a generalized version of (6), it has a drawback that $\boldsymbol{q}^N$ is not uniquely determined. Furthermore, $\boldsymbol{q}^N$ of (17) does not necessarily satisfy (6). Avoiding multiple overall weight vectors for criteria in the definition of (17), we assume that the set of all columns of A is linearly independent. Under the linear independence assumption, we have

$$\boldsymbol{q}^N = \left(\sum_{i\in I} A_i\right)^{-1} N\boldsymbol{q}, \tag{19}$$

where $\boldsymbol{q}$ is defined by (6). Therefore, it follows from (6) that

$$\boldsymbol{q}^N = N\left(\sum_{i\in I} A_i\right)^{-1}\left[\begin{array}{ccc}\hat{\boldsymbol{b}}^1 & \cdots & \hat{\boldsymbol{b}}^m\end{array}\right]\boldsymbol{p}. \tag{20}$$

The pair of $\boldsymbol{q}^N$ and $\boldsymbol{p}$ is characterized as the following corollary :

Corollary 5 *Let $I = \{1, \ldots, m\}$ and suppose $I = K$. Suppose that $\{\hat{\boldsymbol{b}}^1, \ldots, \hat{\boldsymbol{b}}^m\}$ satisfies (3) and assume that the set of all columns of A is linearly independent, then the overall weight vector for criteria $\boldsymbol{q}^N$ defined by (17) is a positive principal eigenvector of*

$$N\left(\sum_{i \in I} A_i\right)^{-1} \left[\ \hat{\boldsymbol{b}}^1 \ \cdots \ \hat{\boldsymbol{b}}^m\ \right] AN^{-1}. \tag{21}$$

Hence, if $[\boldsymbol{x}^\top, \boldsymbol{y}^\top]^\top$ is a positive principal eigenvector of a supermatrix

$$\begin{bmatrix} 0 & N\left(\sum_{i\in I} A_i\right)^{-1} \hat{\boldsymbol{b}}^1 \ \cdots \ N\left(\sum_{i\in I} A_i\right)^{-1} \hat{\boldsymbol{b}}^m \\ AN^{-1} & 0 \end{bmatrix}, \tag{22}$$

then $\boldsymbol{x}$ and $\boldsymbol{y}$ has the same directions as the overall weight vector for criteria $\boldsymbol{q}^N$ and the overall weight vector for alternatives $\boldsymbol{p}$ defined by (5), respectively.

Proof: The principal eigenvalue of (22) is $\sqrt{\lambda}$ and $\left[\boldsymbol{q}^{N\top}, \sqrt{\lambda}\boldsymbol{p}^\top\right]^\top$ is a principal eigenvector of (22). □

We consider the example (13) of section 3 again and show that the paradox occurs for some standardizations of the evaluation matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3/8 & 32 \\ 1/8 & 1/16 \end{bmatrix}. \tag{23}$$

Let $N = \begin{bmatrix} n_{11} & 0 \\ 0 & n_{22} \end{bmatrix}$, then it follows from (15),(23) and (19) that

$$\boldsymbol{q}^N = \begin{bmatrix} \frac{2n_{11}}{3} & 0 \\ 0 & \frac{16n_{22}}{529} \end{bmatrix} \begin{bmatrix} 0.135 \\ 0.865 \end{bmatrix} = \begin{bmatrix} \frac{2\times 0.135}{3} n_{11} \\ \frac{16\times 0.865}{529} n_{22} \end{bmatrix}. \tag{24}$$

Therefore, $\boldsymbol{q}^N$ of (24) implies that criteria 2 is preferred to criteria 1 if and only if $(2 \times 0.135n_{11})/3 < (16\times 0.865n_{22})/529$. If the column-wise standardizing matrix $N = \begin{bmatrix} n_{11} & 0 \\ 0 & n_{22} \end{bmatrix}$ has $n_{11}/n_{22} < 3.440$, then the paradox of CCM occurs. This fact implies that two simple standardizations lead to the paradox as follows: For the column-wise standardizing matrix $N = \begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}$, both the columns of the evaluation matrix AN^{-1} form the ideal modes. Since $n_{11}/n_{22} = 1/32 < 3.440$, the ideal mode standardization causes the paradox. The other standardizing matrix N is given as $n_{11} = 1 \times 0.375 \times 0.125$ and $n_{22} = 1 \times 32 \times 0.0625$. Then we have $\prod_{i\in I} a_{ij}/n_{jj} = 1$ for all $j = 1, 2$ that is a standardization of the logarithmic least squared method of the AHP [10]. Since $n_{11}/n_{22} < 0.0235$, CCM also provides the paradox from the input data (13).

We consider another numerical example such that the sum-one column-wise standardization does not cause the paradox but the ideal mode standardization does. Suppose that $I = K = \{1, 2, 3\}$, $J = \{1, 2\}$,

$$A = \begin{bmatrix} 20.000 & 10.000 \\ 10.000 & 20.000 \\ 20.000 & 8.000 \end{bmatrix}, \boldsymbol{b}^1 = \begin{bmatrix} 0.600 \\ 0.400 \end{bmatrix}, \boldsymbol{b}^2 = \begin{bmatrix} 0.500 \\ 0.500 \end{bmatrix} \text{ and } \boldsymbol{b}^3 = \begin{bmatrix} 0.600 \\ 0.400 \end{bmatrix}. \tag{25}$$

From the input data $A, \boldsymbol{b}^1, \boldsymbol{b}^2$ and $\boldsymbol{b}^3$ of (25), CCM provides

$$\hat{\boldsymbol{b}}^1 = \begin{bmatrix} 0.654 \\ 0.346 \end{bmatrix}, \hat{\boldsymbol{b}}^2 = \begin{bmatrix} 0.321 \\ 0.679 \end{bmatrix}, \hat{\boldsymbol{b}}^3 = \begin{bmatrix} 0.703 \\ 0.297 \end{bmatrix} \text{ and } \boldsymbol{p} = \begin{bmatrix} 0.339 \\ 0.345 \\ 0.316 \end{bmatrix}, \tag{26}$$

where $\epsilon = 10^{-3}$ of Step 2 in Algorithm 0. The numerical results of three iterates $\boldsymbol{b}_t^1, \boldsymbol{b}_t^2$ and $\boldsymbol{b}_t^3$ of Algorithm 0 are given in Appendix 2. From (6) and (19) we have

$$\boldsymbol{q} = \begin{bmatrix} 0.554 \\ 0.446 \end{bmatrix} \text{ and } \boldsymbol{q}^N = \begin{bmatrix} 0.486 \\ 0.514 \end{bmatrix}, \tag{27}$$

respectively. It follows from (27) that the overall weight vector $\boldsymbol{q}$ for criteria with respect to the sum-one column-wise standardization satisfies the Pareto principle but $\boldsymbol{q}^N$ with respect to the ideal mode standardization violates it.

5. Appearance Frequency of Paradox Occurrence

This section examines the appearance frequency of the paradox, violation of three conditions 1,2,3 of section 3, by using simple numerical examples. Avoiding complicated analysis of experiment results, all examples of this section are set the simplest mutual evaluation system with 2 criteria and 2 alternatives. Since it is too few actual case studies of CCM to simulate A, $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$, we introduce the standard scale $\{1, 3, 5, 7, 9\}$ of AHP and generate each value of components of A and $\left\{\boldsymbol{b}^1, \boldsymbol{b}^2\right\}$ as follows:

Without loss of generality, the evaluation matrix A of alternatives from criteria is set $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a_{21} & a_{22} \end{bmatrix}$. Each value of a_{21} and a_{22}, is chosen from $\{1/9, 1/7, 1/5, 1/3, 1, 3, 5, 7, 9\}$. To examine occurrence of the paradox, we generate $\boldsymbol{b}^1 = [b_1^1, b_2^1]^\top$ and $\boldsymbol{b}^2 = [b_1^2, b_2^2]^\top$ such that $b_1^k \geq b_2^k$ for $k = 1, 2$. That is, we assume that each $\boldsymbol{b}^k$ is derived from a pairwise comparison matrix $C = \begin{bmatrix} 1 & c_{12} \\ c_{12}^{-1} & 1 \end{bmatrix}$, where $c_{12} = 1, 3, 5, 7, 9$. Analyzing C by the standard method of AHP such as eigenvalue method or the geometric mean method, we have $\boldsymbol{b}^k = \begin{bmatrix} c_{12}/(1 + c_{12}) \\ 1/(1 + c_{12}) \end{bmatrix}$ and $b_1^k \geq b_2^k$. Let c^k be c_{12} of C that is the pairwise comparison matrix with respect to alternative k, then $\boldsymbol{b}^k = \begin{bmatrix} c^k/(1 + c^k) \\ 1/(1 + c^k) \end{bmatrix}$.

In each experiment we choose $a_{21}, a_{22} \in \{1/9, 1/7, 1/5, 1/3, 1, 3, 5, 7, 9\}$ and $c^1, c^2 \in \{1, 3, 5, 7, 9\}$, apply CCM to the numerical example, $\begin{bmatrix} 1 & 1 \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} c^1/(1 + c^1) \\ 1/(1 + c^1) \end{bmatrix}, \begin{bmatrix} c^2/(1 + c^2) \\ 1/(1 + c^2) \end{bmatrix}$ and carry out the paradox test that is to check weather $\boldsymbol{q}$ satisfies three conditions 1,2 and 3. The program used in the experiments was coded in C language and was run on Sun ultra-1 with double precision arithmetic. The tolerance ϵ of the stopping criteria in Step 2 of CCM was 10^{-8} and that of the paradox test was 10^{-3}.

Since there are 81 combinations of the value of a_{12} and that of a_{22} and 25 combinations of the value of c^1 and that of c^2, the total number of the numerical examples generated in the experiments is $2055 (= 9^2 \times 5^2)$. We conduct one paradox test for each numerical example and there exist 78 examples whose $\boldsymbol{q}$ violates at least one of three conditions 1, 2 and 3. The paradox example is called if it has $\boldsymbol{q}$ that violates at least one of three conditions 1, 2 and 3. That is, the total number of the paradox examples is 78 and its appearance frequency is $3.80\% (= 78/2055)$.

For a given pair of a_{21} and a_{22} there exist 25 combinations of the value of c^1 and that of c^2, i.e., $(1,1),(1,3),\cdots,(1,9),(3,1),\cdots,(3,9),\cdots,(9,9)$, and the number of the paradox examples is denoted by $G(a_{21},a_{22})$. All of $G(a_{21},a_{22})$ are listed in Table 1. It follows from

Table 1: Paradox occurrence $G(a_{21},a_{22})$ for $a_{21},a_{22} = 1/9, 1/7, 1/5, 1/3, 1, 3, 5, 7, 9$

a_{21} \ a_{22}	1/9	1/7	1/5	1/3	1	3	5	7	9
1/9	0	1	1	1	1	1	1	1	0
1/7	1	0	1	1	1	1	1	0	1
1/5	1	1	0	1	1	1	0	1	1
1/3	1	1	1	0	1	0	1	1	3
1	5	2	1	1	0	1	1	2	5
3	3	1	1	0	1	0	1	1	1
5	1	1	0	1	1	1	0	1	1
7	1	0	1	1	1	1	1	0	1
9	0	1	1	1	1	1	1	1	0

Table 1 that the paradox always occurs for each (a_{21},a_{22}) with $a_{21} \neq a_{22}$ or $a_{21} \neq 1/a_{22}$. Hence, the paradox occurs uniformly over all (a_{21},a_{22}) with $a_{21} \neq a_{22}$ or $a_{21} \neq 1/a_{22}$.

For a given pair of c^1 and c^2 there exist 81 combinations of (a_{21},a_{22}), i.e., $(1/9,1/9),(1/9,1/7),\cdots,(1/9,9),(1/7,1/9),\cdots,(1/7,9),\cdots,(9,9)$, and the number of the paradox examples is denoted by $H(c^1,c^2)$. All of $H(c^1,c^2)$ are documented in Table 2.

Table 2: Paradox occurrence $H(c^1,c^2)$ for $c^1,c^2 = 1,3,5,7,9$

c^1 \ c^2	1	3	5	7	9
1	64	3	2	1	1
3	3	0	0	0	0
5	2	0	0	0	0
7	1	0	0	0	0
9	1	0	0	0	0

Table 2 is summarized as the following two points:

- Finding 1: The total number of paradox examples is 78 and 64 paradox examples are in (c^1,c^2) with $c^1 = c^2 = 1$. Hence, 82.1%(= 64/78) of all paradox examples occurs in (c^1,c^2) with $c^1 = c^2 = 1$.
- Finding 2: Each (c^1,c^2) with $\min\{c^1,c^2\} > 1$ has no paradox example and every (c^1,c^2) with $\min\{c^1,c^2\} = 1$ has at least one paradox example. Hence, $H(c^1,c^2) = 0$ if and only if $c^1 > 1$ and $c^2 > 1$.

Note that $c^k > 1$ if and only if $b_1^k > b_2^k$ and that $c^k = 1$ if and only if $b_1^k = b_2^k$. From this fact Finding 1 means that almost paradox examples violate the condition 3:

$$\text{if } b_1^1 = b_2^1 \text{ and } b_1^2 = b_2^2, \text{ then } q_1 = q_2. \tag{28}$$

Finding 2 implies that all examples satisfy the condition 2 that

$$\text{if } b_1^1 > b_2^1 \text{ and } b_1^2 > b_2^2, \text{ then } q_1 > q_2. \tag{29}$$

In $(c^1, c^2) = (1, 9)$ there exists the paradox example $A = \begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix}$, $\boldsymbol{b}^1 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\boldsymbol{b}^2 = \begin{bmatrix} 9/10 \\ 1/10 \end{bmatrix}$ whose $\boldsymbol{q} = \begin{bmatrix} 0.498 \\ 0.502 \end{bmatrix}$ violates the condition 1:

$$\text{if } b_1^1 \geq b_2^1, b_1^2 \geq b_2^2, \text{ then } q_1 \geq q_2. \tag{30}$$

It follows from Tables 2 that it is easy for CCM to satisfy the conditions (29), however, it is hard to satisfy (28). Each sufficient condition of (28), (29) and (30) means similarity among all alternatives' evaluation to criteria. Even if all alternatives' evaluation to criteria are similar to each other, CCM sometimes provides irrational overall weight vector $\boldsymbol{q}$ of criteria. It follows from Table 1 that almost evaluation matrices A of alternatives can not avoid the paradox.

6. Concluding Remarks

For the overall weight vector for criteria that is left out of consideration in [2], this paper introduces the definition (6) and summarizes some properties of the outputs $\left\{ \hat{\boldsymbol{b}}^k \mid k \in K \right\}$ of Algorithm 0 as Lemma 1, Lemma 2 and Theorem 3. By considering the Pareto principle as the desirable requirements for overall weight vector for criteria, we illustrate that there exist some examples such that the overall weight vector for criteria (6) violates the Pareto principle. Furthermore, by the numerical experiments we see that appearance frequency of the violation is 3.8% and the violation occurs in almost evaluation matrices A. On the other hand, Theorem 4 states that the overall weight vector for criteria by ANP always satisfies the Pareto principle. Hence, it is not severe to impose the Pareto principle on the overall weight vectors in the mutual evaluation system. Therefore, the existence of such irrational overall weight vector is a fatal shortage of CCM. These discussion may answer an open problem of (i) in section 7 of [8] that is to find a way of determining whether the estimated values $\left\{ \hat{\boldsymbol{b}}^k \mid k \in K \right\}$ by CCM is reasonable or not.

As stated in section 4, the sum-one column-wise standardizing matrix N of (10) does not necessarily cause the paradox of CCM under the Pareto principle. The essential cause is that there is a pair of an alternative $\hat{k} \in K$ and an iteration $\hat{t}$ of Algorithm 0 such that

$$A_{\hat{k}}^{-1} \boldsymbol{b}_{\hat{t}}^{\hat{k}} \notin \left\{ \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \middle| \begin{array}{lll} q_j \leq q_l & \text{if} & b_j^k \leq b_l^k \text{ for all } k \in K, \\ q_j < q_l & \text{if} & b_j^k < b_l^k \text{ for all } k \in K, \\ q_j = q_l & \text{if} & b_j^k = b_l^k \text{ for all } k \in K, \\ \multicolumn{3}{c}{q_j \geq 0 \text{ for all } j = 1, \ldots, n} \end{array} \right\}. \tag{31}$$

As stated in Theorem 3, if the set $\left\{ \hat{\boldsymbol{b}}^k \mid k \in K \right\}$ satisfies only (3), then $\boldsymbol{p}$ defined by (5) and $\boldsymbol{q}$ defined by (6) have the outer dependence (10). Therefore, a sufficient condition for avoiding (31) and satisfying (10) is to modify Algorithm 0 such that each iterate $\boldsymbol{b}_t^k$ belongs to the convex hull of $\left\{ \boldsymbol{b}^k \mid k \in K \right\}$ for any iteration t.

Finally, it is hoped that this study makes a small contribution of the future research development of CCM and ANP, especially actual applications of CCM.

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Appendix 1

In order to report calculation results of each iteration in Algorithm 0, we visualize calculation of each iteration in table as follows:

Table 3: Calculation in each iteration

Iteration t	$\boldsymbol{b}_t^1$	$\frac{A_2A_1^{-1}\boldsymbol{b}_t^1}{\boldsymbol{e}^\top A_2A_1^{-1}\boldsymbol{b}_t^1}$	$\frac{A_3A_1^{-1}\boldsymbol{b}_t^1}{\boldsymbol{e}^\top A_3A_1^{-1}\boldsymbol{b}_t^1}$
	$\frac{A_1A_2^{-1}\boldsymbol{b}_t^2}{\boldsymbol{e}^\top A_1A_2^{-1}\boldsymbol{b}_t^2}$	$\boldsymbol{b}_t^2$	$\frac{A_3A_2^{-1}\boldsymbol{b}_t^2}{\boldsymbol{e}^\top A_3A_2^{-1}\boldsymbol{b}_t^2}$
	$\frac{A_1A_3^{-1}\boldsymbol{b}_t^3}{\boldsymbol{e}^\top A_1A_3^{-1}\boldsymbol{b}_t^3}$	$\frac{A_2A_3^{-1}\boldsymbol{b}_t^3}{\boldsymbol{e}^\top A_2A_3^{-1}\boldsymbol{b}_t^3}$	$\boldsymbol{b}_t^3$
	$\frac{1}{3}\sum_{i=1}^3 \frac{A_iA_1^{-1}\boldsymbol{b}_t^1}{\boldsymbol{e}^\top A_iA_1^{-1}\boldsymbol{b}_t^1}$	$\frac{1}{3}\sum_{i=1}^3 \frac{A_iA_2^{-1}\boldsymbol{b}_t^2}{\boldsymbol{e}^\top A_iA_2^{-1}\boldsymbol{b}_t^2}$	$\frac{1}{3}\sum_{i=1}^3 \frac{A_iA_3^{-1}\boldsymbol{b}_t^3}{\boldsymbol{e}^\top A_iA_3^{-1}\boldsymbol{b}_t^3}$

The iteration t of Algorithm 0 provides

$$\boldsymbol{b}_{t+1}^1 = \frac{1}{3}\sum_{i=1}^{3} \frac{A_i A_1^{-1} \boldsymbol{b}_t^1}{\boldsymbol{e}^\top A_i A_1^{-1} \boldsymbol{b}_t^1}, \boldsymbol{b}_{t+1}^2 = \frac{1}{3}\sum_{i=1}^{3} \frac{A_i A_2^{-1} \boldsymbol{b}_t^2}{\boldsymbol{e}^\top A_i A_2^{-1} \boldsymbol{b}_t^2}, \boldsymbol{b}_{t+1}^3 = \frac{1}{3}\sum_{i=1}^{3} \frac{A_i A_3^{-1} \boldsymbol{b}_t^3}{\boldsymbol{e}^\top A_i A_3^{-1} \boldsymbol{b}_t^3}$$

from the input data $\left\{\boldsymbol{b}_t^1, \boldsymbol{b}_t^2, \boldsymbol{b}_t^3\right\}$. The jth row and $(i+1)$st column of Table 3 is given $\frac{A_i A_j^{-1} \boldsymbol{b}_t^j}{\boldsymbol{e}^\top A_i A_j^{-1} \boldsymbol{b}_t^j}$ for all $j = 1, 2, 3$ and all $i = 1, 2$. Therefore, input data of the iteration t are in $(j, j+1)$ position of Table 3. The bottom line of Table 3 is given output of the iteration t, that is input data of the iteration $t+1$. For $A = \begin{bmatrix} 1 & 1 \\ 3/8 & 32 \\ 1/8 & 1/16 \end{bmatrix}$, $\boldsymbol{b}^1 = \begin{bmatrix} 0.520 \\ 0.480 \end{bmatrix}$, $\boldsymbol{b}^2 = \begin{bmatrix} 0.510 \\ 0.490 \end{bmatrix}$ and $\boldsymbol{b}^3 = \begin{bmatrix} 0.530 \\ 0.470 \end{bmatrix}$ Algorithm 0 stops within 6 iterations under the tolerance $\epsilon = 10^{-4}$. The numerical results of calculations of each iteration is represented according to the format of Table 3 and the convergence behavior from the iteration 0 to the iteration 5 is in Table 4.

Table 4: Convergence behavior of the example in Section 3

	$\boldsymbol{b}_t^1$		$\boldsymbol{b}_t^2$		$\boldsymbol{b}_t^3$	
Iteration 0	0.520	0.480	0.013	0.987	0.667	0.333
	0.989	0.011	0.510	0.490	0.994	0.006
	0.378	0.622	0.007	0.993	0.530	0.470
	0.629	0.371	0.177	0.823	0.730	0.270
Iteration 1	0.629	0.371	0.019	0.981	0.759	0.241
	0.948	0.052	0.177	0.823	0.971	0.029
	0.594	0.406	0.017	0.983	0.730	0.270
	0.724	0.276	0.071	0.929	0.820	0.180
Iteration 2	0.724	0.276	0.030	0.970	0.829	0.171
	0.867	0.133	0.071	0.929	0.923	0.077
	0.711	0.289	0.028	0.972	0.820	0.180
	0.767	0.233	0.043	0.957	0.858	0.142
Iteration 3	0.767	0.233	0.037	0.963	0.859	0.141
	0.793	0.207	0.043	0.957	0.876	0.124
	0.765	0.235	0.037	0.963	0.858	0.142
	0.775	0.225	0.039	0.961	0.864	0.136
Iteration 4	0.775	0.225	0.039	0.961	0.864	0.136
	0.776	0.224	0.039	0.961	0.865	0.135
	0.775	0.225	0.039	0.961	0.864	0.136
	0.775	0.225	0.039	0.961	0.865	0.135
Iteration 5	0.775	0.225	0.039	0.961	0.865	0.135
	0.775	0.225	0.039	0.961	0.865	0.135
	0.775	0.225	0.039	0.961	0.865	0.135
	0.775	0.225	0.039	0.961	0.865	0.135

Appendix 2

For the input data

$$A = \begin{bmatrix} 20 & 10 \\ 10 & 20 \\ 20 & 8 \end{bmatrix}, \boldsymbol{b}^1 = \begin{bmatrix} 0.600 \\ 0.400 \end{bmatrix}, \boldsymbol{b}^2 = \begin{bmatrix} 0.500 \\ 0.500 \end{bmatrix} \text{ and } \boldsymbol{b}^3 = \begin{bmatrix} 0.600 \\ 0.400 \end{bmatrix},$$

Algorithm 0 stops within 3 iterations under the tolerance $\epsilon = 10^{-3}$. The convergence behavior from the iteration 0 to the iteration 2 is in Table 5.

Table 5: Convergence behavior of the example in Section 4

	$\boldsymbol{b}_t^1$		$\boldsymbol{b}_t^2$		$\boldsymbol{b}_t^3$	
Iteration 0	0.600	0.400	0.273	0.727	0.652	0.348
	0.800	0.200	0.500	0.500	0.833	0.167
	0.545	0.455	0.231	0.769	0.600	0.400
	0.648	0.352	0.334	0.666	0.695	0.305
Iteration 1	0.648	0.352	0.316	0.684	0.698	0.302
	0.668	0.332	0.334	0.666	0.715	0.285
	0.646	0.354	0.313	0.687	0.695	0.305
	0.654	0.346	0.321	0.679	0.703	0.297
Iteration 2	0.654	0.346	0.321	0.679	0.703	0.297
	0.654	0.346	0.321	0.679	0.703	0.297
	0.654	0.346	0.321	0.679	0.703	0.297
	0.654	0.346	0.321	0.679	0.703	0.297

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