Journal of the Operations Research
Society of Japan
2004, Vol. 47, No. 2, 123-128

LINEAR TIME APPROXIMATION ALGORITHM FOR MULTICOLORING LATTICE GRAPHS WITH DIAGONALS

Yuichiro Miyamoto
Sophia University

Tomomi Matsui[1]
University of Tokyo

(Received September 30, 2003)

Abstract Let P be a subset of 2-dimensional integer lattice points $P = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \subseteq \mathrm{Z}^2$. We consider the graph G_P with vertex set P satisfying that two vertices in P are adjacent if and only if Euclidean distance between the pair is less than or equal to $\sqrt{2}$. Given a non-negative vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^P$, a multicoloring of $(G_P, \boldsymbol{w})$ is an assignment of colors to P such that each vertex $v \in P$ admits $w(v)$ colors and every adjacent pair of two vertices does not share a common color.

We show the NP-completeness of the problem to determine the existence of a multicoloring of $(G_P, \boldsymbol{w})$ with strictly less than $(4/3)\omega$ colors where ω denotes the weight of a maximum weight clique. We also propose an $\mathrm{O}(mn)$ time approximation algorithm for multicoloring $(G_P, \boldsymbol{w})$. Our algorithm finds a multicoloring with at most $(4/3)\omega + 4$ colors

Our algorithm based on the property that when $n = 3$, we can find a multicoloring of $(G_P, \boldsymbol{w})$ with ω colors easily, since an undirected graph associated with $(G_P, \boldsymbol{w})$ becomes a perfect graph.

Keywords: Graph theory, coloring, multicoloring, lattice graph, perfect graph

1. Introduction

Given a pair of positive integers m and n, P denotes the subset of 2-dimensional integer lattice points defined by $P \stackrel{\text{def.}}{=} \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \subseteq \mathrm{Z}^2$. Let G_P be an undirected graph with vertex set P satisfying that two vertices are adjacent if and only if Euclidean distance between the pair is less than or equal to $\sqrt{2}$. Given a non-negative vertex weights $\boldsymbol{w} \in \mathrm{Z}_+^P$, the pair $(G_P, \boldsymbol{w})$ is called a *weighted lattice graph with diagonals* and abbreviated by WLGD.

Given an undirected graph H and a non-negative integer vertex weight $\boldsymbol{w}'$ of H, a *multicoloring* of $(H, \boldsymbol{w}')$ is an assignment of colors to vertices of H such that each vertex v admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A *multicoloring problem* on $(H, \boldsymbol{w}')$ finds a multicoloring of $(H, \boldsymbol{w}')$ which minimizes the required number of colors. The multicoloring problem is also known as weighted coloring [2], minimum integer weighted coloring [7] or w-coloring [6]. A vertex subset V' of an undirected graph is called a *clique* if every pair of vertices in V' are adjacent. The weight of a clique is the sum total of all the weights of vertices in the clique. We denote the weight of a maximum weight clique in $(H, \boldsymbol{w}')$ by $\omega(H, \boldsymbol{w}')$. It is clear that for any multicoloring of $(H, \boldsymbol{w}')$, the required number of colors is greater than or equal to $\omega(H, \boldsymbol{w}')$.

In this paper, we study a fundamental class of graphs: lattice graphs with diagonals G_P. We show the NP-completeness of the problem to determine the existence a multicoloring

[1] Supported by Superrobust Computation Project of the 21st Century COE Program "Information Science and Technology Strategic Core."

of $(G_P, \boldsymbol{w})$ with strictly less than $(4/3)\omega(G_P, \boldsymbol{w})$ colors. We also propose an $\mathrm{O}(mn)$ time algorithm for multicoloring $(G_P, \boldsymbol{w})$ with at most $(4/3)\omega(G_P, \boldsymbol{w}) + 4$ colors.

The multicoloring problem has been studied in several context. On triangular lattice graphs it corresponds to the radio channel (frequency) assignment problem. McDiarmid and Reed [5] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [5, 6] independently gave approximation algorithms for this problem. In case that a given graph H is a square lattice graph (without diagonal) and/or a hexagonal lattice graph, the graph becomes bipartite and so we can obtain an optimal multicoloring of $(H, \boldsymbol{w}')$ in polynomial time (see [5] for example). Halldórsson and Kortsarz [3] studied planar graphs and partial k-trees. For both classes, they gave a polynomial time approximation scheme (PTAS) for variations of multicoloring problem with min-sum objectives. These objectives appear in the context of multiprocessor task scheduling.

There is a natural graph $H(\boldsymbol{w}')$ associated with a pair $(H, \boldsymbol{w}')$ as above, obtained by replacing each vertex v of H by a complete graph on $w'(v)$ vertices. Multicolorings of the pair $(H, \boldsymbol{w}')$ correspond to usual vertex colorings of the graph $H(\boldsymbol{w}')$, and the multicoloring number of $(H, \boldsymbol{w}')$ is equivalent to the coloring number of $H(\boldsymbol{w}')$. Here we note that the input size of the graph $H(\boldsymbol{w}')$ is bounded by a pseudo polynomial of that of $(H, \boldsymbol{w}')$ in general. We also show that when $n = 3$, we can exactly solve the multicoloring problem on $(G_P, \boldsymbol{w})$ in $\mathrm{O}(m)$ time. It based on the property that the associated graph $G_P(\boldsymbol{w})$ becomes a perfect graph. For (general) perfect graphs, Grötschel, Lovász, and Schrijver [2] gave a polynomial time exact algorithm for the coloring problem. Their algorithm based on the ellipsoid method.

2. Approximation Algorithm

In this section, we propose a linear time approximation algorithm for multicoloring a WLGD $(G_P, \boldsymbol{w})$. For any vertex $(x, y) \in P$, we denote the corresponding vertex weight by $w(x, y)$.

Theorem 1 *There exists an $\mathrm{O}(mn)$ time algorithm for finding a multicoloring of $(G_P, \boldsymbol{w})$ which uses at most $(4/3)\omega(G, \boldsymbol{w}) + 4$ colors.*

Before giving a proof of Theorem 1, let us consider a well-solvable case.

Lemma 1 *When $P = \{1, \ldots, m\} \times \{1, 2, 3\}$, there exists an $\mathrm{O}(m)$ time (exact) algorithm for multicoloring $(G_P, \boldsymbol{w})$ with $\omega(G_P, \boldsymbol{w})$ colors.*

Proof: In the following, we express a multicoloring by an assignment of integers $c : P \to 2^{\mathrm{Z}_+}$ such that $[\forall v \in P,\ w(v) = |c(v)|]$ and [for every adjacent pair of vertices $v, w \in P$, $c(v) \cap c(w) = \emptyset$]. We describe an $\mathrm{O}(m)$ time algorithm explicitly.

First, we compute $\omega(G_P, \boldsymbol{w})$ in $\mathrm{O}(m)$ time. For each odd number $x \in \{1, \ldots, m\}$, we set

$$\begin{aligned} c(x,1) &= \{i \in \mathrm{Z} : w(x,2) < i \le w(x,2) + w(x,1)\}, \\ c(x,2) &= \{i \in \mathrm{Z} : 1 \le i \le w(x,2)\}, \\ c(x,3) &= \{i \in \mathrm{Z} : w(x,2) < i \le w(x,2) + w(x,3)\}, \end{aligned}$$

and for each even number $x \in \{1, \ldots, m\}$, we set

$$\begin{aligned} c(x,1) &= \{i \in \mathrm{Z} : \omega(G, \boldsymbol{w}) - w(x,2) \ge i > \omega(G, \boldsymbol{w}) - w(x,2) - w(x,1)\}, \\ c(x,2) &= \{i \in \mathrm{Z} : \omega(G, \boldsymbol{w}) \ge i > \omega(G, \boldsymbol{w}) - w(x,2)\}, \\ c(x,3) &= \{i \in \mathrm{Z} : \omega(G, \boldsymbol{w}) - w(x,2) \ge i > \omega(G, \boldsymbol{w}) - w(x,2) - w(x,3)\}. \end{aligned}$$

Obviously, the above procedure requires $\mathrm{O}(m)$ time.

It remains to show that every adjacent pair of two vertices does not share a common color. First, assume on the contrary that the edge between $(x, 1)$ and $(x+1, 1)$ violates the condition, i.e., $c(x, 1) \cap c(x+1, 1) \neq \emptyset$. It follows that $w(x, 1)+w(x, 2)+w(x+1, 1)+w(x+1, 2) > \omega(G_P, \boldsymbol{w})$. Since the set of four vertices $\{(x, 1), (x, 2), (x+1, 1), (x+1, 2)\}$ forms a clique of G_P, it is a contradiction. For other edges, the correctness is proved analogously. ∎

From Lemma 1, the following result is now immediate.

Corollary 1 *If* $P = \{1, \ldots, m\} \times \{1, 2, 3\}$, *the undirected graph* $G_P(\boldsymbol{w})$ *associated with* $(G_P, \boldsymbol{w})$ *is perfect.*

Proof: Every vertex induced subgraph G' of $G_P(\boldsymbol{w})$ is associated with a WLGD $(G_P, \boldsymbol{w}')$, satisfying that $w'(v)$ denotes the number of vertices in G' corresponding to the vertex v. ∎

In case that every vertex weight is a multiple of 3, there exists a simple (4/3)-approximation algorithm. In the following, we describe an outline of the algorithm. First, we construct four vertex weights $\boldsymbol{w}'_k$ for $k \in \{0, 1, 2, 3\}$ by setting

$$w'_k(x, y) = \begin{cases} 0, & y = k \pmod 4, \\ w(x, y)/3, & \text{otherwise.} \end{cases}$$

Next, we exactly solve four multicoloring problems defined on four WLGDs $(G_P, \boldsymbol{w}'_k)$ $(k \in \{0, 1, 2, 3\})$ and obtain four multicolorings. We can solve the problems independently by applying the procedure in the proof of Lemma 1 (we will describe later in detail). Here we assume that four multicolorings use mutually disjoint sets of colors. Lastly, we output the direct sum of four multicolorings. It is clear that $\max_{k \in \{0,1,2,3\}} \omega(G_P, \boldsymbol{w}'_k) \leq (1/3)\omega(G_P, \boldsymbol{w})$. Thus, the obtained multicoloring uses at most $(4/3)\omega(G_P, \boldsymbol{w})$ colors.

In the following, we consider the general case and describe a proof of Theorem 1.

Proof of Theorem 1: For each $k \in \{0, 1, 2, 3\}$, we introduce a partition $\{A_k, B_k, C_k, D_k\}$ of P defined as follows:

$$\begin{aligned} A_k &= \{(x, y) \in P : y = k \pmod 4\}, \\ B_k &= \{(x, y) \in P : y = k + 2 \pmod 4\}, \\ C_k &= \{(x, y) \in P : y = k + 1 \pmod 4,\ x \text{ is odd}\} \\ &\quad \cup \{(x, y) \in P : y = k + 3 \pmod 4,\ x \text{ is even}\}, \\ D_k &= \{(x, y) \in P : y = k + 1 \pmod 4,\ x \text{ is even}\} \\ &\quad \cup \{(x, y) \in P : y = k + 3 \pmod 4,\ x \text{ is odd}\}. \end{aligned}$$

Then we construct vertex weights $\boldsymbol{w}_k$ for $k \in \{0, 1, 2, 3\}$ by the following procedure. We put the weight of every vertex in A_k to 0. For each vertex $(x, y) \in B_k$, we set $w_k(x, y) = \lfloor w(x, y)/3 \rfloor$. If $(x, y) \in C_k$, we set

$$w_k(x, y) = \begin{cases} \lfloor w(x, y)/3 \rfloor, & w(x, y) = 0 \pmod 3, \\ \lfloor w(x, y)/3 \rfloor + 1, & w(x, y) \in \{1, 2\} \pmod 3, \end{cases}$$

and in case that $(x, y) \in D_k$, we set

$$w_k(x, y) = \begin{cases} \lfloor w(x, y)/3 \rfloor, & w(x, y) \in \{0, 1\} \pmod 3, \\ \lfloor w(x, y)/3 \rfloor + 1, & w(x, y) = 2 \pmod 3. \end{cases}$$

Clearly from the definition, the equality $\boldsymbol{w} = \boldsymbol{w}_0 + \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3$ holds.

For each WLGD $(G_P, \boldsymbol{w}_k)$ $(k \in \{0, 1, 2, 3\})$, we delete all the vertices in A_k and decompose the graph into $\mathrm{O}(n)$ connected components. Then each connected component satisfies the condition in Lemma 1 and so the procedure in the proof of Lemma 1 finds a multicoloring of $(G_P, \boldsymbol{w}_k)$ using $\omega(G_P, \boldsymbol{w}_k)$ colors in $\mathrm{O}(mn)$ time. Here we assume that four multicolorings use mutually disjoint sets of colors. Then the direct sum of four multicoloring becomes a multicoloring of original WLGD $(G_P, \boldsymbol{w})$.

Lastly, we show that the algorithm finds a multicoloring with at most $(4/3)\omega(G_P, \boldsymbol{w}) + 4$ colors. We only need to show the inequality $\omega(G_P, \boldsymbol{w}_k) \leq (1/3)\omega(G_P, \boldsymbol{w}) + 1$ for all $k \in \{0, 1, 2, 3\}$. Let V' be a clique of G_P and $V''_k \stackrel{\text{def.}}{=} \{(x, y) \in V' : w_k(x, y) = \lfloor w(x, y)/3 \rfloor + 1\}$. The definition of weights $\boldsymbol{w}_k$ directly implies that $|V''_k| \leq 2$, since $|V' \cap C_k| \leq 1$ and $|V' \cap D_k| \leq 1$. We denote the weight of the clique V' with respect to $\boldsymbol{w}_k$ or $\boldsymbol{w}$ by $w_k(V')$ or $w(V')$, respectively. If $V''_k = \emptyset$, we have done. When $|V''_k| = 1$, the inequality $w(V') \geq 3(w_k(V') - 1) = 3w_k(V') - 3$ holds. In case that $|V''_k| = 2$, $|V' \cap C_k| = |V' \cap D_k| = 1$ and so we have $w(V') \geq 3(w_k(V') - 2) + 1 + 2 = 3w_k(V') - 3$. Thus we have the desired result. ∎

3. Hardness Result

In this section, we discuss the hardness of our problem.

Theorem 2 *Given a WLGD $(G_P, \boldsymbol{w})$, it is NP-complete to determine whether $(G_P, \boldsymbol{w})$ is multicolorable with strictly less than $(4/3)\omega(G_P, \boldsymbol{w})$ colors or not.*

Proof: It is known to be NP-complete to determine the 3-colorability of a given planar graph H with each vertex of degree either 3 or 4 (see [1] e.g.). We show a procedure to construct a WLGD $(G_P, \boldsymbol{w})$ such that $(G_P, \boldsymbol{w})$ is 3-multicolorable if and only if H is 3-colorable. In the following, we identify a WLGD $(G_P, \boldsymbol{w})$ with the $n \times m$ integer matrix $\boldsymbol{w} \in \mathrm{Z}_+^{n \times m}$ such that rows and columns are indexed by $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$ respectively.

First, we introduce 3 special WLGDs defined by the following matrices:

$$L_0 = \begin{bmatrix} 0\,0\,1\,0\,0 \\ 0\,2\,0\,2\,0 \\ 1\,0\,0\,0\,1 \\ 0\,2\,0\,2\,0 \\ 0\,0\,1\,0\,0 \end{bmatrix}, L_1 = \begin{bmatrix} 0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,2\,0\,2\,0\,0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,1\,2\,1\,2\,1\,2\,1 \\ 0\,2\,0\,2\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}, L_2 = \begin{bmatrix} 0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,2\,0\,2\,0\,0 \\ 1\,1\,0\,0\,1\,0\,1 \\ 0\,0\,0\,0\,0\,2\,0 \\ 0\,0\,0\,0\,0\,0\,0 \end{bmatrix}.$$

The four elements of L_0 indexed by $\{(1, 3), (3, 1), (3, 5), (5, 3)\}$ are the "contact points" of L_0. Observe that in any 3-multicoloring of L_0, all the contact points must have the same color. Similarly, four elements of L_1 indexed by $\{(1, 3), (3, 1), (5, 3), (3, 11)\}$ are the "contact points" such that in any 3-multicoloring of L_1, the contact points must have the same color. The "contact pair" of L_2 indexed by $\{(3, 1), (3, 7)\}$ satisfies that in any 3-multicoloring of L_2, the contact points have different colors.

Next, we embed the planar graph H (with each vertex degree is either 3 or 4) on the x–y plane and obtain a plane graph H' such that (1) H' is a subdivision of H (H' is homeomorphic to H), (2) every vertex of H' is an integer lattice point in $\{1, 2, \ldots, m'\} \times \{1, 2, \ldots, n'\}$, (3) every edge of H' is either a vertical or horizontal edge with unit length, and (4) m' and n' are bounded by a polynomial of the number of vertices of H. Figure 1 shows an embedding H' of a subdivision of K_4. For each edge of H', we insert 9 vertices and obtain a finer subdivision H'' of H'. Figure 2 shows the finer subdivision $H''$of H' appearing in Figure 1. We put $P = \{1, 2, \ldots, 10m'\} \times \{1, 2, \ldots, 10n'\}$ and construct G_P (a lattice graph with diagonals) from P. It is easy to see that H'' is a subgraph of G_P. Since there is a linear time algorithm for finding a planar embedding of a given graph or deciding that it is not planar [4], the computational effort of the above procedure is obviously bounded by a polynomial of the number of vertices in H.

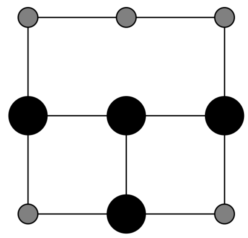

Figure 1: An embedding of H' which is a subdivision of K_4

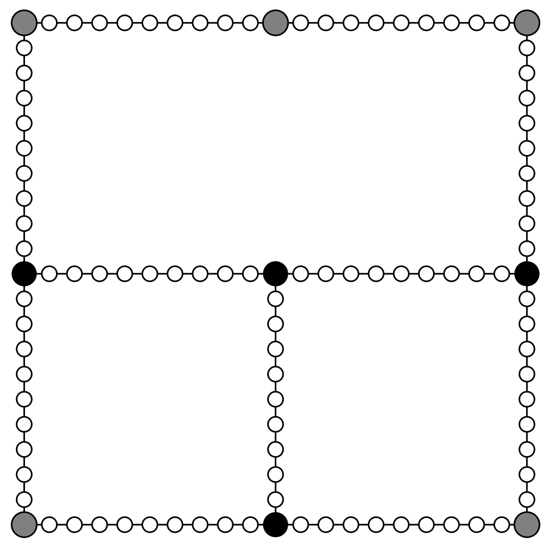

Figure 2: The finer subdivision H'' of H' in Figure 1

Lastly, we construct the vertex weights $\boldsymbol{w}$ of G_P as follows. Initially, we put all the vertex weights to 0. For each vertex v of H'' whose degree is greater than 2, we replace the weights of vertices in G_P whose Euclidean distances from v are less than or equal to $2\sqrt{2}$ by matrix L_0. For each edge e in the original graph H, there exists a corresponding path P_e in H''. We denote the path P_e by a sequence of vertices $(v_0, v_1, \ldots, v_{10k})$. Then we replace the weights of vertices near the vertices in the subpath $(v_2, v_3, \ldots, v_8)$ with the matrix L_2 or its rotated image satisfying that $\{v_2, v_8\}$ becomes the contact pair of L_2. Here we note that the copies of L_0 and L_2 share five vertices. In case $k \geq 2$, we apply the following. For every $k' \in \{1, 2, \ldots, k-1\}$, we replace the weights of vertices near the vertices in the subpath $(v_{10k'-2}, v_{10k'-1}, \ldots, v_{10k'+8})$ by a copy of L_1 or its rotated image satisfying that $v_{10k'-2}$ corresponds to one of the elements of L_1 indexed by (1,3),(3,1),(5,3) and $v_{10k'+8}$ corresponds to the element indexed by (3,11). Similarly to the above, consecutive pair of matrices shares five elements. For example, the above procedure transforms H'' appearing in Figure 2 to a matrix in Figure 3. (We omit the vertices whose weights are 0.)

From the definitions of L_0, L_1, L_2, it is obvious that the WLGD $(G_P, \boldsymbol{w})$ defined above satisfies $\omega(G_P, \boldsymbol{w}) = 3$ and 4-colorable. The above procedure directly implies that the given graph H is 3-colorable if and only if $(G_P, \boldsymbol{w})$ is 3-multicolorable. Thus, NP-completeness of the original problem implies that it is NP-complete to determine whether a given WLGD $(G_P, \boldsymbol{w})$ is multicolorable with strictly less than $(4/3)w(G_P, \boldsymbol{w})$ colors. ∎

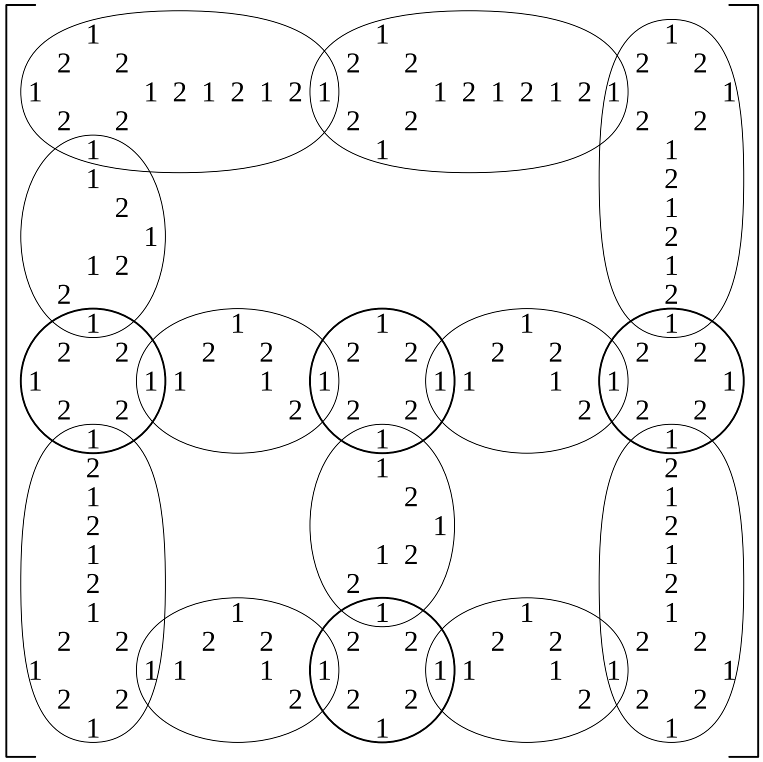

Figure 3: A matrix of G_P transformed from H'' in Figure 2

References

[1] M. R. Garey and D. S. Johnson: *Computers and intractability, a guide to the theory of NP-completeness* (W. H. Freeman and Company, 1979).

[2] M. Grötschel, L. Lovász and A. Schrijver: *Geometric algorithms and combinatorial optimization* (Springer-Verlag, 1988).

[3] M. M. Halldórsson and G. Kortsarz: Tools for multicoloring with applications to planar graphs and partial k-trees. *Journal of Algorithms*, **42** (2002) 334–366.

[4] J. E. Hopcroft and R. E. Tarjan: Efficient planarity testing. *Journal of the Association for Computing Machinery*, **21** (1974) 549–568.

[5] C. McDiarmid and B. Reed: Channel assignment and weighted coloring. *Networks*, **36** (2000) 114–117.

[6] L. Narayanan and S. M. Shende: Static frequency assignment in cellular networks. *Algorithmica*, **29** (2001) 396–409.

[7] J. Xue: Solving the minimum weighted integer coloring problem. *Combinatorial Optimization and Application*, **11** (1998) 53–64.

Yuichiro Miyamoto
Department of Mechanical Engineering
Faculty of Science and Technology
Sophia University
7-1 Kioi-cho, Chiyoda-ku, Tokyo 102-8554
Japan.
E-mail: y-miyamo@sophia.ac.jp