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EXISTENCE CONDITIONS OF THE OPTIMAL STOPPING TIME: THE CASES OF GEOMETRIC BROWNIAN MOTION AND ARITHMETIC BROWNIAN MOTION

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Abstract A type of optimal investment problem can be regarded as an optimal stopping problem in the field of applied stochastic analysis. This study derives the existence conditions of the optimal stopping time when the stochastic process is a geometric Brownian motion or an arithmetic Brownian motion. The conditions concern the intrinsic value function and are natural extensions of the certainty case. Additionally, they are essential for a well-known result in recent investment theory. They are also applied to an optimal land development problem. The analyses give existing studies rigorous foundations and generalize them.

Keywords: Stochastic optimization, stopping time, existence conditions, geometric Brownian motion, arithmetic Brownian motion, land development timing.

1. Introduction

Recently, many researchers have studied optimal investment problems under uncertainty using the continuous-time option theory.[1] They first derive a partial differential equation that the option value function should satisfy, and then use the value-matching condition and the smooth-pasting condition to derive the optimal solution. Almost all of the studies, however, ignore sufficiency of the solution.

A type of optimal investment problem can be regarded as a version of an optimal stopping problem in the field of applied stochastic analysis. The conditions required for optimal stopping time when the stochastic process is Ito diffusion were derived by Dynkin [8]. His theorem gives a general solution of optimal stopping problems, but it is not necessarily useful for specific economic problems. Recently, Brekke and Øksendal [3] derived a relation between optimal stopping time and the smooth-pasting condition, that is often used in economic analysis. The smooth-pasting condition is essentially considered as a first-order condition in the optimization of the stopping time (e.g. Merton [10], p.171, Øksendal [12]). Brekke and Øksendal [3] derived the second-order conditions that guarantee the optimality of the solutions that satisfy the smooth-pasting condition. However, their conditions are very complex.

An aim of this study is to derive more simple existence conditions of the optimal stopping time by limiting the stochastic process to a *geometric Brownian motion* (GBM) or an *arithmetic Brownian motion* (ABM), which are tractable and are often used in investment problems. The results show that the existence conditions are natural extensions of

[1]The continuous-time model for financial-option pricing was developed by Merton [10], and its application to a real-option problem was studied by McDonald and Siegel [9]. Dixit [6] and Dixit and Pindyck [7] are basic textbooks for this area.

the certainty case and are essentially related to a well-known result in recent investment theory, i.e., *optimal investment time is delayed when uncertainty increases*. This is the first contribution of this article.

The second contribution is related to an optimal land development problem. Clarke and Reed [5] analyzed an optimal land development problem as an optimal stopping problem, in which they set development time and capital stock (i.e., building size) as controlled variables, and have shown that uncertainty delays development and increases capital stock.[2] However, their assumptions are not necessarily verified. First, they assume a second-order condition for a *deterministic* version of their model, which was analyzed by Arnott and Lewis [1] (so we call the condition 'AL condition'). Second, they assume that the process of net land rent after investment is a GBM and the one before investment is constant (zero). Some studies insist that an ABM is better for the land rent process (Capozza and Li [4]) and that rent from undeveloped land (e.g. parking lot, old and low buildings) is also stochastic (Williams [16]). We generalize the Clarke=Reed model from this point of view and apply the existence conditions of the optimal stopping time. The results show that the AL condition guarantees the existence of the optimal development time and the main result in Clarke and Reed [5] in more general setting. This fact is very important in an empirical sense, since the AL condition is testable and indeed Arnott and Lewis [1] showed an empirical result supporting this condition.

This article is organized as follows. We first derive the existence conditions of the optimal stopping time when the stochastic process is a geometric Brownian motion or an arithmetic Brownian motion using the Brekke=Øksendal theorem (Section 2). Second, we apply the result to an optimal land development problem (Section 3). From the analyses, we can give the Clarke=Reed model rigorous foundations in more general settings.

2. Existence Conditions for an Optimal Stopping Problem

We specify an optimal investment problem as follows:

$$\sup_{\tau,X} E_0[\int_0^\tau CF_A(Y_t)e^{-rt}dt + \int_\tau^\infty CF(Y_t,X)e^{-rt}dt - c_\tau(X)e^{-r\tau}], \tag{1}$$

where E_0 is the expectation conditional on the present (time 0) information, CF_A is the cash-flow function before investment, CF is the cash-flow function after investment, Y_t is a one-dimensional stochastic process influencing cash flow, X is a vector of investment characteristics (including capital stock), c_t is the investment cost function at t, and r is the real interest rate. Problem (1) implies that a risk-neutral agent can choose the timing and characteristics of the investment, his decisions are one-time, and the investment lasts forever. We should notice that τ is a F_t-stopping time, where F_t is the σ-algebra generated by Y_s, $s \leq t$.

The objective function of (1) can be restated as

$$E_0[\int_0^\tau CF_A(Y_t)e^{-rt}dt + \int_\tau^\infty CF(Y_t,X)e^{-rt}dt - c_\tau(X)e^{-r\tau}]$$

[2]Titman [15] first studied such a problem using the financial option theory in a two-period setting. Like Clarke and Reed [5], Williams [16] and Capozza and Li [4] analyzed continuous-time models for the problem but they limited building production function to the Cobb-Douglas type. While all of the above studies assumed that investment decisions are one-time, Pindyck [14] and Bertola [2] analyzed incremental-investment models, in which the agent could add capital stock anytime.

$$= E_0[\int_\tau^\infty \{CF(Y_t,X) - CF_A(Y_t)\}e^{-rt}dt - c_\tau(X)e^{-r\tau} + \int_0^\infty CF_A(Y_t)e^{-rt}dt]$$
$$= E_0[\{P(Y_\tau,X) - P_A(Y_\tau) - c_\tau(X)\}e^{-r\tau}] + P_A(Y_0), \qquad (2)$$

where $E_s\int_s^\infty CF(Y_t,X)e^{-r(t-s)}dt$ and $E_s\int_s^\infty CF_A(Y_t)e^{-r(t-s)}dt$ are assumed to have the form $P(Y_s,X)$ and $P_A(Y_s)$, respectively.

2.1. Constant cost case

When the investment cost only depends on X, problem (1) can be rewritten as

$$\sup_{\tau,X} E_0[\{P(Y_\tau,X) - P_A(Y_\tau) - c(X)\}e^{-r\tau}]$$
$$= \sup_\tau E_0[V(Y_\tau)e^{-r\tau}], \qquad (3)$$

where $V(Y) \equiv \max_X\{P(Y,X) - P_A(Y) - c(X)\}$ and we call it the *intrinsic value* of the warrant to invest the land when $Y_t = Y$. Furthermore, the *reward function* v and the *optimal reward function* v^* are defined by $v(s,y) \equiv V(y)e^{-rs}$, $v^*(s,y) \equiv \sup_\tau E_s[V(Y_\tau)e^{-r\tau}]$, respectively, where $Y_s = y$.

Problem (3) is well-known as a type of *optimal stopping problem* in the field of applied stochastic analysis. Brekke and Øksendal [3] assumed Y_t is a multi-dimensional Ito diffusion and proved a theorem giving a relation among the optimal stopping time, the optimal reward function, and the smooth-pasting condition that is often used in economic analysis (see Appendix 1). In this section, we assume Y_t is a *geometric Brownian motion* (GBM) or an *arithmetic Brownian motion* (ABM) and derive the conditions for the existence of optimal stopping time using their theorem. The conditions concern the intrinsic value function and are simple and meaningful.

2.1.1. GBM case

We set the following basic assumptions:

- **(A1)** $(t,Y_t) \in U \equiv \Re_+ \times \Re_{++}$ and $dY_t = gY_tdt + \sigma Y_tdB_t$, where both g and σ are positive constants, $\frac{1}{2}\sigma^2 < g < r$, and B_t is a one-dimensional standard Brownian motion.
- **(A2)** We can find nonnegative y^o such that the intrinsic value function $V : \Re_+ \to \Re$ is positive and belongs to C^2 in (y^o,∞) and V is nonpositive and continuous in $[0,y^o]$.

(A1) says that Y_t is a geometric Brownian motion and that the inequality $\frac{1}{2}\sigma^2 < g$ guarantees that any *first exit time* $\inf\{t > 0 : Y_t \geq u, 0 < u < \infty\}$ is finite a.s. (almost surely) (e.g. Øksendal [13], p.63.). (A2) says that we have at most one break-even point (y^o) except for zero. This is a natural assumption in the real world. Differentiability of V is a technical assumption.

In this case, the Brekke=Øksendal theorem could be restated as follows (see Appendix 1). If the following conditions are also satisfied, then τ_D is an optimal stopping time and w^* is the optimal reward function, where $w^*(s,y) \equiv w(s,y)$ if $(s,y) \in D$, and $w^*(s,y) \equiv v(s,y)$ otherwise:

- **(C1)** An open set $D \subset U$ with a C^1 boundary exists, $\tau_D \equiv \inf\{t > 0 : (t,Y_t) \notin D\} < \infty$ a.s., and, for each $s \in \Re_+$, the set $\{y : (s,y) \in \partial D\}$ has a zero one-dimensional Lebesgue measure, where ∂D is the boundary of D.
- **(C2)** A function $w : \overline{D} \to \Re$ exists, and $w \in C^1(\overline{D}) \cap C(D)$, where $\overline{D}$ is the closure of D.

(C3) $v \in C^1(\partial D \cap U)$ and $Lv \leq 0$ outside $\overline{D}$, where L is the characteristic operator of (t, Y_t) and

$$L = \frac{\partial}{\partial s} + gy\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2}. \tag{4}$$

(C4) $w \geq v$ in D.

(C5) D and w satisfy (a), (b), and (c):

(a) $Lw = 0$ in D.

(b) (value-matching condition)

$$w(s, y) = v(s, y) for (s, y) \in \partial D \cap U. \tag{5}$$

(c) (smooth-pasting condition)

$$\frac{\partial}{\partial y} w(s, y) = \frac{\partial}{\partial y} v(s, y) \text{ for } (s, y) \in \partial D \cap U. \tag{6}$$

These conditions seem to be complex, but they can be roughly interpreted as follows: When D is given, (C5)(a) and (C5)(b) determine w. (C5)(c) is a first-order condition for determining optimal D. (C2) and the first part of (C3) guarantee Lv, Lw, $\frac{\partial w}{\partial y}$, and $\frac{\partial v}{\partial y}$ to exist in each specified region. The second part of (C3) and (C4) are the second-order conditions for the optimality of D and w.

(C1) is a technical condition. By (C5)(a), (C5)(b), and the theorem of the stochastic Dirichlet problem (e.g. Øksendal [13], p.172), we have $w^*(s, y) = E_s[V(Y_{\tau_D})e^{-r\tau_D}]$ for a given D, which means $w^* \leq v^*$. By the Dynkin theorem of optimal stopping, v^* must be the *least superharmonic majorant* of v. On the other hand, w^* is a majorant of v by (C3) and (C4), so $w^* = v^*$ only if w^* is superharmonic. We can easily show this if $w^* \in C^2$, but (C5)(c) only guarantees $w^* \in C^1$ on $\partial D \cap U$. (C1) is a condition that guarantees the double differentiability. For details, see Brekke and Øksendal [3].

The next proposition tells us that some conditions for the intrinsic value function V verify (C1) - (C5):

Proposition 1 ***(Existence of an optimal stopping time: GBM case)***. *Define* $h(y) \equiv \frac{yV'(y)}{V(y)}$ *in* (y^o, ∞) *and let* β *be a positive root of the equation* $\frac{1}{2}\sigma^2\beta^2 + (g - \frac{1}{2}\sigma^2)\beta - r = 0$. *If* $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \beta$, *and* $\lim_{y\downarrow y^o} h(y) > \beta$, *then a unique optimal stopping time* τ_D *exists, where* $D = \{(s, y) : s \in \Re_+, 0 < y < y^*\}$ *and* $y^* = h^{-1}(\beta)$. *Furthermore, if we let* $w^*(s, y) \equiv V(y^*)\left(\frac{y}{y^*}\right)^\beta e^{-rs}$ *for* $y \in [0, y^*)$ *and* $w^* \equiv v$ *for* $y \geq y^*$, *then* w^* *is the optimal reward function.*

Proof. See Appendix 2(1). □

Remarks. (i) The set of conditions, $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \beta$, and $\lim_{y\downarrow y^o} h(y) > \beta$, is a *natural extension of the certainty case.* In the certainty case, problem (3) can be rewritten as $\sup_t V(Y_t)e^{-rt}$, and the first-order condition and the second-order condition are as follows:

$$\text{(f.o.c.)}\ V(y^c) = \frac{g}{r} y^c V'(y^c), \tag{7}$$

$$\text{(s.o.c.)}\ g^2 y^{c2} V''(y^c) + (g^2 - 2rg)y^c V'(y^c) + r^2 V(y^c) < 0, \tag{8}$$

where y^c is the optimal stopping time in this case. From (7) and (8), we have

$$\begin{aligned} y^{c2}V''(y^c) + \left(1 - \frac{2r}{g}\right) y^c V'(y^c) + \left(\frac{r}{g}\right)^2 V(y^c) &< 0 \Longleftrightarrow \\ y^{c2}V''(y^c) + \left(1 - \frac{2y^cV'(y^c)}{V(y^c)}\right) y^c V'(y^c) + \left(\frac{y^cV'(y^c)}{V(y^c)}\right)^2 V(y^c) &< 0 \Longleftrightarrow \\ \{V'(y^c) + y^cV''(y^c)\}V(y^c) - y^cV'(y^c)^2 &< 0. \end{aligned} \tag{9}$$

From Equation (22) of Appendix 2(1), (7) and (9) mean $h(y^c) = \frac{r}{g}$ and $h'(y^c) < 0$. Therefore, the set of conditions, $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \frac{r}{g}$, and $\lim_{y\downarrow y^o} h(y) > \frac{r}{g}$, is sufficient for the existence of y^c.

(ii) The condition $h'(y) < 0$ is meaningful. Since we have $h(y^*) = \beta$, $\frac{\partial\beta}{\partial\sigma^2} < 0$, $\lim_{\sigma^2\to 0} \beta = \frac{r}{g} > 1$, and $\lim_{\sigma^2\to 2g} \beta = \sqrt{\frac{r}{g}}$, this condition shows that *the optimal stopping time is delayed when uncertainty (σ^2) increases* (Figure 1). In addition, this condition guarantees

$$h(y) > 0 \text{ in } (y^o, \infty). \tag{10}$$

If y such as $h(y) \le 0$ exists in (y^o, ∞), then we have $\lim_{y\to\infty} h(y) < 0$, which means $\lim_{y\to\infty} V'(y) < 0$ by the definition of $h(y)$. This contradicts $V(y) > 0$ in (y^o, ∞); thus, (10) is satisfied, and (10) implies that $V(y)$ is *strictly increasing* in (y^o, ∞) by the definition of $h(y)$.

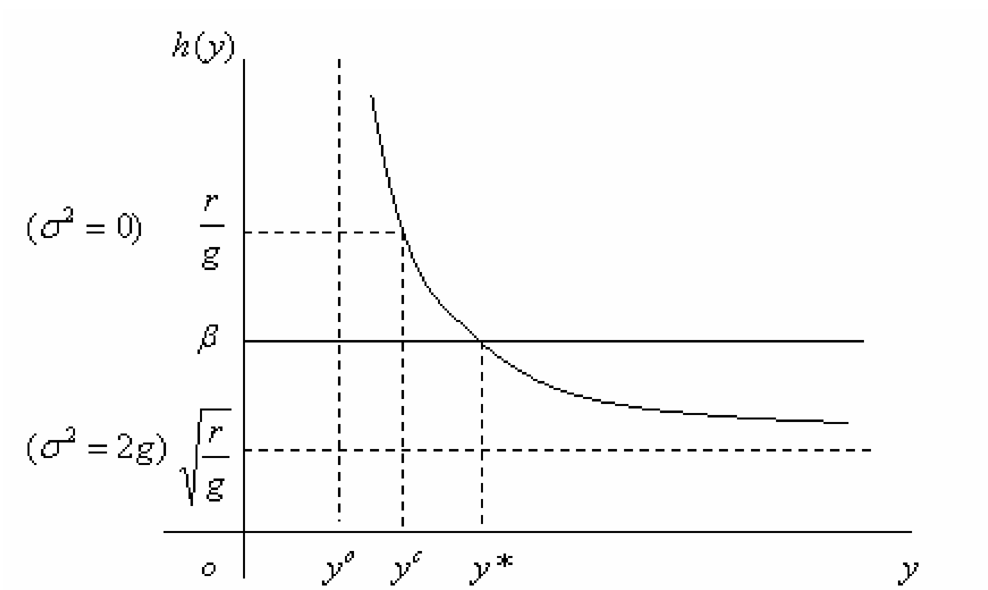

Figure 1: The graph of $h(y)$

(iii) The conditions $\lim_{y\to\infty} h(y) < \beta$ and $\lim_{y\downarrow y^o} h(y) > \beta$ do not guarantee that the optimal stopping time exists for any level of uncertainty. If we assume $\lim_{y\to\infty} h(y) < \sqrt{\frac{r}{g}}$ and $\lim_{y\downarrow y^o} h(y) > \frac{r}{g}$ instead of them, then the optimal stopping time exists for any level of uncertainty, where we should notice that $0 < \sigma^2 < 2g$ from (A1).

(iv) Dixit and Pindyck [7] (pp.103-104, 128-130) also discuss a sufficient condition for the uniqueness of the optimal stopping time, in other words, a sufficient condition of *clean* *division* in the range of the continuation region and the stopping region. In our case, their condition is that $\frac{1}{2}\sigma^2 y^2 V''(y) + gyV'(y) - rV(y)$ is monotonically decreasing (i.e., $Lv(s,y)$ is monotonically decreasing with respect to y). In contrast to our condition, this condition require more information about the intrinsic value function V, that is, V'''. We only require $V \in C^2$ in (y^o, ∞).

2.1.2. ABM case

We set the following basic assumptions instead of (A1) and (A2) for the GBM case:

- **(A3)** $(t, Y_t) \in U \equiv \Re_+ \times \Re$ and $dY_t = gdt + \sigma dB_t$, where both g and σ are positive constants and B_t is a one-dimensional standard Brownian motion.
- **(A4)** We can find $y^o \in [-\infty, \infty)$ such that the intrinsic value function $V : \Re \to \Re$ is positive and belongs to C^2 in (y^o, ∞) and V is nonpositive and continuous in $[-\infty, y^o]$.

In addition, the required conditions are the same as (C1) - (C5) for the GBM case except for Equation (4). Instead of (4), we use

$$L = \frac{\partial}{\partial s} + g\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2}. \tag{11}$$

In this case, we have the following proposition, which resembles Proposition 1:

Proposition 2 ***(Existence of an optimal stopping time: ABM case)***. *Define $h_A(y) \equiv \frac{V'(y)}{V(y)}$ in (y^o, ∞) and let α be a positive root of the equation $\frac{1}{2}\sigma^2\alpha^2 + g\alpha - r = 0$. If $h'_A(y) < 0$, $\lim_{y\to\infty} h_A(y) < \alpha$, and $\lim_{y \downarrow y^o} h(y) > \alpha$, then a unique optimal stopping time τ_D exists, where $D = \{(s,y) : s \in \Re_+, -\infty < y < y^*\}$ and $y^* = h_A^{-1}(\alpha)$. Furthermore, if we let $w^*(s,y) \equiv V(y^*)e^{\alpha(y-y^*)-rs}$ for $y \in (-\infty, y^*)$ and $w^* \equiv v$ for $y \ge y^*$, then w^* is the optimal reward function.*

Proof. See Appendix 2(2). □

Remarks. Almost all the remarks for the conditions of Proposition 1 are effective. If we assume $\lim_{y\to\infty} h_A(y) = 0$ and $\lim_{y\downarrow y^o} h_A(y) > \frac{r}{g}$ instead of the condition $\lim_{y\to\infty} h_A(y) < \alpha$ and $\lim_{y\downarrow y^o} h_A(y) > \alpha$, then the optimal stopping time exists for any level of uncertainty.

2.2. Stochastic cost case

Next, we turn to a stochastic investment-cost case. We assume that the investment cost at t is $C_t X$, where C_t is a one-dimensional stochastic process and X is an investment characteristic (e.g. capital stock). Then, problem (1) can be rewritten as

$$\begin{aligned} & \sup_{\tau, X} E_0[\{P(Y_\tau, X) - P_A(Y_\tau) - C_\tau X\}e^{-r\tau}] \\ = & \sup_{\tau} E_0[V(Y_\tau, C_\tau)e^{-r\tau}], \end{aligned} \tag{12}$$

where $V(Y,C) \equiv \max_X\{P(Y,X) - P_A(Y) - CX\}$ (the intrinsic value of the warrant to invest when $Y_t = Y$ and $C_t = C$). Furthermore, the reward function v and the optimal reward function v^* are defined by $v(s,y,c) \equiv V(y,c)e^{-rs}$, $v^*(s,y,c) \equiv \sup_\tau E_s[V(Y_\tau, C_\tau)e^{-r\tau}]$, where $Y_s = y$ and $C_s = c$, respectively. We also set the following assumptions in this case:

- **(A5)** $(t, Y_t, C_t) \in U \equiv \Re_+ \times \Re_{++} \times \Re_{++}$, $dY_t = g_y Y_t dt + \sigma_y Y_t dB_t^y$ and $dC_t = g_c C_t dt + \sigma_c Y_t dB_t^c$, where g_y, g_c, σ_y, and σ_c are all positive constants, $\frac{1}{2}(\sigma_y^2 - \sigma_c^2) < g_y - g_c$ and

$g_c < g_y < r$. B_t^y and B_t^c are correlated one-dimensional standard Brownian motions and the correlation coefficient ρ is less than $\min(\frac{\sigma_y}{\sigma_c}, \frac{\sigma_c}{\sigma_y})$.

(A6) V is a homogeneous function of degree one and $\widetilde{V}(z) \equiv \frac{1}{c}V(y,c) = V(\frac{y}{c}, 1)$, where $z \equiv \frac{y}{c}$. We can find nonnegative z^o such that $\widetilde{V} : \Re_+ \to \Re$ is positive and belongs to C^2 in (z^o, ∞) and $\widetilde{V}$ is nonpositive and continuous in $[0, z^o]$.

(A5) says that Y_t and C_t are (weakly) correlated geometric Brownian motions and the inequality $\frac{1}{2}(\sigma_y^2 - \sigma_c^2) < g_y - g_c$ guarantees that any first exit time $\inf\{t > 0 : \frac{Y_t}{C_t} \geq u, 0 < u < \infty\}$ is finite a.s.[3] We can easily show this using Ito's formula for a ratio of Ito processes (e.g. Nielsen [11], p.69). In addition, the required conditions are the same as (C1) - (C5) except for Equations (4), (5), and (6). Instead of them, we use

$$L = \frac{\partial}{\partial s} + g_y y \frac{\partial}{\partial y} + g_c c \frac{\partial}{\partial c} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2} + \frac{1}{2}\sigma_c^2 c^2 \frac{\partial^2}{\partial c^2} + \rho \sigma_y \sigma_c y c \frac{\partial^2}{\partial y \partial c}, \tag{13}$$

$$w(s,y,c) = v(s,y,c) \text{ for } (s,y,c) \in \partial D \cap U, \tag{14}$$

$$\frac{\partial}{\partial y} w(s,y,c) = \frac{\partial}{\partial y} v(s,y,c) \text{ and } \frac{\partial}{\partial c} w(s,y,c) = \frac{\partial}{\partial c} v(s,y,c) \text{ for } (s,y,c) \in \partial D \cap U. \tag{15}$$

L is the characteristic operator of (t, Y_t, C_t). In this case, we have the following proposition:

Proposition 3 ***(Existence of an optimal stopping time: stochastic cost case).*** *Define* $\widetilde{h}(z) \equiv \frac{z\widetilde{V}'(z)}{\widetilde{V}(z)}$ *in* (z^o, ∞) *and let* δ *be a positive root of the equation* $\frac{1}{2}(\sigma_y^2 - 2\rho\sigma_y\sigma_c + \sigma_c^2)\delta(\delta - 1) + (g_y - g_c)\delta - (r - g_c) = 0$. *If* $\widetilde{h}'(z) < 0$, $\lim_{z\to\infty} \widetilde{h}(z) < \delta$, *and* $\lim_{z\downarrow z^o} \widetilde{h}(z) > \delta$, *then a unique optimal stopping time* τ_D *exists, where* $D = \{(s,y,c) : s \in \Re_+, 0 < \frac{y}{c} < z^*\}$ *and* $z^* = \widetilde{h}^{-1}(\delta)$. *Furthermore, if we let* $w^*(s,y,c) \equiv c\left(\frac{y/c}{z^*}\right)^{\delta} \widetilde{V}(z^*) e^{-rs}$ *for* $\frac{y}{c} \in [0, z^*)$ *and* $w^* \equiv v$ *for* $\frac{y}{c} \geq z^*$, *then* w^* *is the optimal reward function.*

Proof. See Appendix 2(3). □

Remarks. Almost all the remarks for the conditions of proposition 1 are effective again. Since we have $\widetilde{h}(z^*) = \delta$, $\frac{\partial\delta}{\partial\sigma_y^2} < 0$, $\frac{\partial\delta}{\partial\sigma_c^2} < 0$, $\lim_{\sigma_y^2, \sigma_c^2 \to 0} \delta = \frac{r - g_c}{g_y - g_c} > 1$, $\lim_{\sigma_y^2 \to\infty} \delta = 1$, and $\lim_{\sigma_c^2\to\infty} \delta = 1$ by using (A5), this condition shows that the optimal stopping time is delayed when uncertainty (σ_y^2 or σ_c^2) increases. Also, if we assume $\lim_{z\to\infty} \widetilde{h}(z) \leq 1$ and $\lim_{z\downarrow z^o} \widetilde{h}(z) > \frac{r-g_c}{g_y - g_c}$ instead of the condition $\lim_{z\to\infty} \widetilde{h}(z) < \delta$ and $\lim_{z\downarrow z^o} \widetilde{h}(z) > \delta$, then the optimal stopping time exists for any level of uncertainty.

3. Application to an Optimal Land Development Problem

In this section, we consider an optimal land development problem, that is, a special case of the problem in Section 2. We set Y and X in problem (1) to be the net rent R yielded by the unit floor and the capital stock K allocated per unit land when it is developed, respectively, and assume that the investment cost at t is $C_t K$. If we let $Q(K)$ be the output of the

[3] C_t must be positive, so we could not assume that C_t is an ABM. Y_t can be negative, but we suppose that both Y_t and C_t are GBMs, since the ratio of the two processes is also a GBM in the case and it is convenient for analysis.

floor on land developed with capital K, then we have $CF(R,K) = Q(K)R$. We suppose $Q \in C^2(\Re_+)$, $Q(0) = 0$, $Q' > 0$, and $Q'' < 0$.

Clarke and Reed [5] have derived the optimal development time and shown that uncertainty delays development and increases capital stock. However, their assumptions are not necessarily verified.

First, they assume a second-order condition for a *deterministic* version ($\sigma^2 = 0$) of their model, i.e., $\varepsilon'(K) < 0$, where the output elasticity of capital ε is defined by $\varepsilon(K) \equiv \frac{Q'(K)K}{Q(K)}$. Arnott and Lewis [1] analyzed the deterministic model and have derived the condition (so we call the condition 'AL condition'). Furthermore, they gave an empirical support to this condition. They suppose a CES and constant-returns-to-scale production function $Q(K) = [\lambda + (1-\lambda)K^\rho]^{\frac{1}{\rho}}$, where $0 < \lambda < 1$, $\rho = \frac{\sigma-1}{\sigma}$, and σ is elasticity of the substitution between land and capital, and estimated $\sigma = 0.372, 0.342$, employing data on Canadian cities (1975, 1976). This result implies $\varepsilon'(K) < 0$, since $\varepsilon'(K) = \frac{\lambda(1-\lambda)\rho K^{\rho-1}}{[\lambda+(1-\lambda)K^\rho]^2}$ has a negative value if $\rho < 0$, that is, $\sigma < 1$. Nevertheless, Clarke and Reed [5] are not verified since their model is *stochastic*.

Second, they assume that $CF_A(R) = 0$, R_t, and C_t are geometric Brownian motions. However, Capozza and Li [4] (p.893, footnote14) insist an ABM (normal diffusion) is better than a GBM (lognormal diffusion) for land rent processes in several points of view. At first, the empirical evidence on the behavior of real estate is more consistent with the ABM since the variance of the growth rate tends to decline as urban areas increase in size. The ABM also permits negative cash flows, which are common in real estate. In addition, we often encounter situations that cash flows from undeveloped land (e.g. parking lot, old and low buildings) is also stochastic (Williams [16]).

Considering these problems in Clarke and Reed [5], we generalize their model and apply the existence conditions of the optimal stopping time obtained in Section 2. Especially, we set $CF_A(R) = aR + b$, where $a \geq 0$ and $b \geq 0$, and R_t can be both of a GBM and an ABM. The results show that the AL condition guarantees the existence of the optimal development time and the main result in Clarke and Reed [5] in our general setting. This fact is very important in an empirical sense, since the AL condition is given with an empirical support.

3.1. Constant cost case

3.1.1. GBM case

In this case, $C_t = C$ and the value of a unit floor at s, $E_s \int_s^\infty R_t e^{-r(t-s)}dt$, is $\frac{R_s}{r-g}$, since $E_s[R_t] = R_s e^{g(t-s)}$. Therefore we have $P(R,K) = \frac{Q(K)R}{r-g}$, $P_A(R) = \frac{aR}{r-g} + \frac{b}{r}$, and the intrinsic value function

$$V(R) = \max_K \left[\frac{Q(K)R}{r-g} - (\frac{aR}{r-g} + \frac{b}{r}) - CK \right]. \tag{16}$$

We can show that $V(R)$ satisfies (A2) (see Appendix 3(1)); therefore, we can apply Proposition 1. The conditions in Proposition 1 can be restated as conditions for the building-production technology:

Proposition 4 ***(Existence of an optimal development time: GBM case)****. Suppose (A1). Define $\widetilde{\varepsilon}(K) \equiv \frac{Q'(K)(\frac{b}{rC}+K)}{Q(K)-a}$ in (K^a, ∞) and $K^o \equiv Q'^{-1}\left(\frac{(r-g)C}{R^o}\right)$, where $K^a \equiv Q^{-1}(a)$ and R^o is defined as y^o in (A2). If $\widetilde{\varepsilon}'(K) < 0$, $\lim_{K\to\infty} \widetilde{\varepsilon}(K) < \frac{\beta-1}{\beta}$, and $\lim_{K \downarrow K^o} \widetilde{\varepsilon}(K) > \frac{\beta-1}{\beta}$, where β is defined in Proposition 1, then a unique optimal development time τ_D exists, where $D = \{(s,R) : s \in \Re_+, 0 < R < R^*\}$, $R^* = \frac{(r-g)C}{Q'(K^*)}$, and $K^* = \widetilde{\varepsilon}^{-1}\left(\frac{\beta-1}{\beta}\right)$. Furthermore, if we* *let* $w^*(s, R) \equiv V(R^*) \left(\frac{R}{R^*}\right)^\beta e^{-rs}$ *for* $R \in [0, R^*)$ *and* $w^*(s, R) \equiv V(R)e^{-rs}$ *for* $R \geq R^*$*, then* w^* *is the optimal reward function.*

Proof. See Appendix 3(1). □

Remarks. (i) If $a > 0$ or $b > 0$, then the condition $\lim_{K \downarrow K^o} \widetilde{\varepsilon}(K) > \frac{\beta-1}{\beta}$ is not necessary since $\lim_{K \downarrow K^o} \widetilde{\varepsilon}(K) = 1 > \frac{\beta-1}{\beta}$. Also, if $a = b = 0$ and $Q'' > -\infty$, the condition is not necessary either. Otherwise, when $a = b = 0$ and $Q''(0) = -\infty$, the condition is sufficient (see Appendix 3(1)).

(ii) The AL condition ($\varepsilon'(K) < 0$) is also effective, since $\varepsilon'(K) < 0 \Rightarrow \widetilde{\varepsilon}'(K) < 0$ (see Appendix 3). If we assume $\lim_{K \to \infty} \widetilde{\varepsilon}(K) < \frac{\sqrt{r}-\sqrt{g}}{\sqrt{r}}$ instead of the condition $\lim_{K \to \infty} \widetilde{\varepsilon}(K) < \frac{\beta-1}{\beta}$, then the optimal stopping time exists for any levels of uncertainty, where we should notice that $0 < \sigma^2 < 2g$ from (A1).

(iii) When we assume a Cobb-Douglas production function $Q(K) = K^\gamma$ $(0 < \gamma < 1)$, we have $\varepsilon'(K) = 0$. If we, furthermore, suppose $a = b = 0$, we also have $\widetilde{\varepsilon}'(K) = 0$, that is, $h'(R) = 0$ (see Appendix 3(1)). This implies that R^*, which is the value satisfying the value-matching and smooth-pasting conditions that are necessary for optimal stopping, does not exist; therefore, we could not find the optimal development time. This fact is also referred to by Williams [16] (p.204, note 12).[4] In a case with $a > 0$ or $b > 0$, we have $\widetilde{\varepsilon}'(K) < 0$ and $\lim_{K \to \infty} \widetilde{\varepsilon}(K) = \gamma$. Therefore, if $\gamma < \frac{\beta-1}{\beta}$, then the optimal development time exists.

3.1.2. ABM case

In this case, the value of a unit floor at s is $\frac{R_s}{r} + \frac{g}{r^2}$, since $E_s[R_t] = R_s + (t-s)g$. Thus, we have $P(R, K) = Q(K)\left(\frac{R}{r} + \frac{g}{r^2}\right)$, $P_A(R) = a\left(\frac{R}{r} + \frac{g}{r^2}\right) + \frac{b}{r}$, and the intrinsic value function

$$V(R) = \max_K \left[Q(K)\left(\frac{R}{r} + \frac{g}{r^2}\right) - a\left(\frac{R}{r} + \frac{g}{r^2}\right) - \frac{b}{r} - CK \right] \tag{17}$$

It should be noted that the value of a unit floor must be negative if the net rent R is less than $-\frac{g}{r}$. To rule out this possibility, we must restrict $R > -\frac{g}{r}$ or introduce an abandonment option (Capozza and Li, 1994, p.893, footnote 14). Since we are interested in development, we assume the former restriction. In other words, we assume that $R_t \in (-\frac{g}{r}, \infty)$ for all t and its process can be *approximated* by ABM in the range. In this case, it is sufficient that we show that the following condition is satisfied instead of (A4) to apply Proposition 2:

- **(A7)** We can find $y^o \in [-\frac{g}{r}, \infty)$ such that the intrinsic value function $V : \Re \to \Re$ is positive and belongs to C^2 in (y^o, ∞) and V is nonpositive and continuous in $[-\frac{g}{r}, y^o]$.

We can show that $V(R)$ satisfies (A7) (see Appendix 3(2)); therefore, we can apply Proposition 2.

Proposition 5 ***(Existence of an optimal development time: ABM case)****. Suppose that* $R_t \in (-\frac{g}{r}, \infty)$ *for all* t *and its process can be approximated by ABM. Define* $K^o \equiv Q'^{-1}\left(\frac{rC}{R^o + \frac{g}{r}}\right)$*, where* R^o *is defined as* y^o *in (A7). If* $\widetilde{\varepsilon}'(K) \leq 0$*,* $\lim_{K \to \infty} \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} < \alpha rC$*, and* $\lim_{K \downarrow K^o} \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} > \alpha rC$*, where* $\widetilde{\varepsilon}(K)$ *and* α *are defined in proposition 4 and 2, respectively, then a unique optimal development time* τ_D *exists, where* $D = \{(s, R) : s \in \Re_+, -\frac{g}{r} < R < R^*\}$*,* $R^* = \frac{rC}{Q'(K^*)} - \frac{g}{r}$*, and* K^* *is the solution of* $\frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} = \alpha rC$*. Furthermore, if we let* $w^*(s, R) \equiv$

[4] To be exact, he analyzed the stochastic cost model discussed below, but we obtained the same result in the model.

$V(R^*)e^{\alpha(R-R^*)-rs}$ for $R \in (-\frac{g}{r}, R^*)$ and $w^*(s,R) \equiv V(R)e^{-rs}$ for $R \geq R^*$, then w^* is the optimal reward function.

Proof. See Appendix 3(2). $\square$

Remarks. Here, the remarks are similar to those made in proposition 4 except for the remark (iii). If we assume $\lim_{K\to\infty} \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} = 0$ instead of the condition $\lim_{K\to\infty} \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} < \alpha rC$, then the optimal stopping time exists for any level of uncertainty. Again, if we consider the case that $Q(K) = K^\gamma$ $(0 < \gamma < 1)$, then we have $\widetilde{\varepsilon}'(K) \leq 0$ since $\varepsilon'(K) = 0$. We also gain $\lim_{K\to\infty} \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} = 0$ and $\lim_{K\downarrow K^o} \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} > \infty$; therefore, the optimal stopping time exists for any level of uncertainty. This fact is also referred to by Capozza and Li [4] (p.894, footnote 16).

3.2. Stochastic cost case

If we assume (A5) and $b = 0$, then we have the intrinsic value function

$$V(R,C) = \max_K \left[\frac{Q(K)R}{r-g} - \frac{aR}{r-g} - CK\right]. \tag{18}$$

It is a homogenous function of degree one, so we obtain

$$\widetilde{V}(Z) = \max_K \left[\frac{Q(K)Z}{r-g} - \frac{aZ}{r-g} - K\right], \tag{19}$$

where $Z \equiv \frac{R}{C}$. We can show that $\widetilde{V}(Z)$ satisfies (A6) (see Appendix 3(3)); therefore, we can apply Proposition 3.

Proposition 6 ***(Existence of an optimal development time: stochastic cost case).*** *Suppose (A5) and $b = 0$. Define $K^o \equiv Q'^{-1}\left(\frac{r-g}{Z^o}\right)$, where $K^a \equiv Q^{-1}(a)$ and Z^o is defined as z^o in (A6). If $\widetilde{\varepsilon}'(K) < 0$, $\lim_{K\to\infty} \widetilde{\varepsilon}(K) < \frac{\delta-1}{\delta}$, and $\lim_{K\downarrow K^o} \widetilde{\varepsilon}(K) > \frac{\delta-1}{\delta}$, where $\widetilde{\varepsilon}(K)$ and δ are defined in proposition 4 and 3, respectively, then a unique optimal development time τ_D exists, where $D = \{(s,R,C) : s \in \Re_+, 0 < \frac{R}{C} < Z^*\}$, $Z^* = \frac{r-g}{Q'(K^*)}$, and $K^* = \widetilde{\varepsilon}^{-1}\left(\frac{\delta-1}{\delta}\right)$. Furthermore, if we let $w^*(s,R,C) \equiv C\left(\frac{R/C}{Z^*}\right)^\delta \widetilde{V}(Z^*)e^{-rs}$ for $\frac{R}{C} \in [0, Z^*)$ and $w^*(s,R,C) \equiv V(R,C)e^{-rs}$ for $\frac{R}{C} \geq Z^*$, then w^* is the optimal reward function.*

Proof. See Appendix 3(3). $\square$

Remarks. Here, the remarks are similar to those made in Proposition 4. If we assume $\lim_{K\to\infty} \widetilde{\varepsilon}(K) = 0$ instead of the condition $\lim_{K\to\infty} \widetilde{\varepsilon}(K) < \frac{\delta-1}{\delta}$, then the optimal stopping time exists for any level of uncertainty.

Proposition 4 to 6 show that the AL condition ($\varepsilon'(K) < 0$) guarantees the existence of the optimal development time in a more general setting if the uncertainty level is in an appropriate range. This fact is important in an empirical sense, since the AL condition is testable. Indeed, Arnott and Lewis [1] showed some empirical results supporting this condition. In addition, the AL condition is essential for a main result of Clarke and Reed [5], i.e., the optimal development time is delayed when uncertainty increases, because $\varepsilon'(K) < 0 \Rightarrow \widetilde{\varepsilon}'(K) < 0 \Leftrightarrow h'(R) < 0$ (see Appendix 3 and Remark (ii) of Proposition 1). Therefore, our analyses give the Clarke=Reed model rigorous foundations and generalize it.

4. Concluding Remarks

Many researchers have recently studied optimal investment problems under uncertainty using the optimal stopping theory. They often use a partial differential equation, the value-matching condition, and the smooth-pasting condition to derive the optimal solution; however, these are just necessary conditions. In this article, first, we derive sufficient conditions for the existence of the optimal solution for a type of optimal stopping problem. The conditions concern the intrinsic value function and are natural extensions of the certainty case. Additionally, they are essential for a well-known result in recent investment theory. Second, we apply the conditions to an optimal land development problem. By the analyses, we can give the Clarke=Reed model rigorous foundations and generalize it. Also, we can systematically understand some results of other existing studies assuming a Cobb-Douglas building production function (Williams [16], Capozza and Li [4]).

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Appendix 1. Sufficiency theorem of the smooth pasting condition

An optimal stopping problem is the problem of finding $\sup_{\tau} E_0[v(X_\tau)]$, where E_0 is the expectation conditional on the present (time 0) information, X_t is a stochastic process, v is a function, and τ is a F_t-stopping time. F_t is the σ-algebra generated by X_s, $s \le t$. Brekke and Øksendal [3] proved the following theorem about this problem:

1 (Assumptions for the process X_t and a region D)

(a) $X_t = (K_t, Y_t) \in U \equiv M \times N$, where $M \subset \Re^m$ and $N \subset \Re^n$, is an $(m+n)$-dimensional Ito diffusion $dX_t = g(X_t)dt + \sigma(X_t)dB_t$, where g and σ are Lipschitz continuous functions and B_t is a k-dimensional standard Brownian motion.

(b) An open set $D \subset U$ with a C^1 boundary exists and $\tau_D \equiv \inf\{t > 0 : X_t \notin D\} < \infty$ a.s.

(c) Y_t is *uniformly elliptic* in N and, for each $k \in M$, the set $\{y \in N : (k, y) \in \partial D\}$ has a zero n-dimensional Lebesgue measure, where ∂D is the boundary of D.[5]

2 (Assumptions for the reward function v and a region D)

(a) $v : U \to \Re$ is continuous and belongs to $C^1(\partial D \cap U) \cap C^2(U \backslash \overline{D})$, where $\overline{D}$ is the closure of D.[6]

(b) Lv in $U \backslash \overline{D}$, where $L = \sum_i g_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ and $a = \sigma\sigma^T$.

3 (Assumptions for a function w and a region D)

(a) A function $w : \overline{D} \to \Re$ exists and $w \in C^1(\overline{D}) \cap C^2(D)$.

(b) $w \ge v$ in D.

4 (Assumptions for a function w, the reward function v, and a region D)

(a) $Lw = 0$ for $x \in D$.

(b) $w(x) = v(x)$ for $x \in \partial D \cap U$ (the value matching condition).

(c) $\nabla_y w(x) = \nabla_y v(x)$ for $x \in \partial D \cap U$ (the smooth pasting condition).

If all of the above assumptions are satisfied, then τ_D is an optimal stopping time and w^* is the optimal reward function, where $w^*(x) \equiv w(x)$ if $x \in D$, otherwise $w^*(x) \equiv v(x)$.

Our assumptions (A1), (A3), and (A5) are special cases of their assumption 1(a) and guarantee an assumption in 1(c), i.e., Y_t is uniformly elliptic in the range. Additionally, a

[5] We say Y_t is *uniformly elliptic* in N, if and only if there exists $\lambda > 0$ such that $z^T\sigma_y(y)\sigma_y(y)^T z \ge \lambda |z|^2$ for all $y \in N$, $z \in \Re^n$, where $dY_t = g_y(Y_t)dt + \sigma_y(Y_t)dB_t$. If Y_t is uniformly elliptic in N, then the expected length of time Y_t stays in any area with a zero n-dimensional Lebesgue measure is zero.

[6] Brekke and Øksendal [3] suppose $v \in C^1(U) \cap C^2(U \backslash \overline{D})$. Clearly, it is not necessary.

candidate of the continuation region D must contain the set $\{(s,y) : s \in \Re_+, y < y^o\}$ in the constant cost case (Section 2.1), and the set $\{(s,y) : s \in \Re_+, \frac{y}{c} < z^o\}$ in the stochastic cost case (Section 2.2). This implies that our assumptions (A2), (A4), and (A6) guarantee an assumption in 2(a), i.e., $v \in C^2(U \backslash \overline{D})$. Thus, the remaining assumptions are additionally required. We restate their assumptions 1(b)(c), 3(a), 2, 3(b), and 4 as (C1), (C2), (C3), (C4), and (C5), respectively in Section 2.

Appendix 2. Proofs of propositions in Section 2

(1) Proposition 1

A candidate of continuation region D must be invariant w.r.t. time t (Øksendal, 1998, p.210), so we estimate that D has the form $\{(s,y) : s \in \Re_+, 0 < y < y^*\}$, where y^* is a positive number.

It is reasonable to assume that $w(s,y) = W(y)e^{-rs}$, where W is a function of y. By (C5)(a) and the characteristic operator (4), we get the following differential equation of W:

$$\frac{1}{2}\sigma^2 y^2 W''(y) + gyW'(y) - rW(y) = 0. \tag{20}$$

The general solution of (20) is $W(y) = B_1 y^{\beta_1} + B_2 y^{\beta_2}$, where B_1 and B_2 are arbitrary constants and β_1 and β_2 are roots of the equation

$$\frac{1}{2}\sigma^2\beta^2 + (g - \frac{1}{2}\sigma^2)\beta - r = 0. \tag{21}$$

When we assume $\beta_1 > \beta_2$, we get $\beta_1 > 1$ and $\beta_2 < 0$. $W(y)$ must be bounded as $y \to 0$, so we must have $B_2 = 0$. If we restate B_1 and β_1 as B and β, respectively, the solution is $W(y) = By^\beta$.

From (5) and (6), we have $By^{*\beta} = V(y^*)$ and $\beta B y^{*\beta-1} = V'(y^*)$, respectively. When we use the two equations and notice $V(y^*) > 0$, we obtain $h(y^*) \equiv \frac{y^*V'(y^*)}{V(y^*)} = \beta$. From the assumptions $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \beta$, and $\lim_{y \downarrow y^o} h(y) > \beta$, $y^* \in (y^o, \infty)$ and B must exist uniquely. Therefore, we can find D and w satisfying (C5).

We show these D and w satisfy (C4). If we define $Z(y) \equiv W(y) - V(y)$, then we get

$$\begin{aligned} Z''(y^*) &= \beta(\beta-1)By^{*\beta-2} - V''(y^*) \\ &= (\beta-1)y^{*-1}V'(y^*) - V''(y^*) \qquad \text{(from (6))} \\ &= \frac{y^*V'(y^*)^2 - \{V'(y^*) + y^*V''(y^*)\}V(y^*)}{y^*V(y^*)}. \quad \text{(from } h(y^*) = \beta) \end{aligned}$$

We also obtain

$$h'(y) = \frac{\{V'(y) + yV''(y)\}V(y) - yV'(y)^2}{V(y)^2}. \tag{22}$$

Therefore, we have $Z''(y^*) = -\frac{h'(y^*)V(y^*)}{y^*} > 0$ by $h'(y) < 0$ and $V(y^*) > 0$. Equations (5) and (6) and this inequality show that $W(y)$ is tangent to $V(y)$ at y^* and $W(y) \geq V(y)$ in the neighborhood of y^*. Next, we show that $W(y)$ and $V(y)$ do not intersect in (y^o, y^*). Since $h(y) > \beta$ in (y^o, y^*), we have

$$\frac{V(y)}{y^\beta} - \frac{V'(y)}{\beta y^{\beta-1}} = \frac{V(y)}{y^\beta}\left[1 - \frac{h(y)}{\beta}\right] < 0 \ \text{ in } (y^o, y^*). \tag{23}$$

Suppose $\overline{y}$ is an intersection of $W(y)$ and $V(y)$ and $y^o < \overline{y} < y^*$. From (23), we get

$$\frac{V(\overline{y})}{B\overline{y}^\beta} - \frac{V'(y)}{B\beta\overline{y}^{\beta-1}} < 0.$$

Since $V(\overline{y}) = B\overline{y}^\beta$, we have $V'(\overline{y}) > B\beta\overline{y}^{\beta-1} = W'(\overline{y})$. However, we must have another intersection $\widetilde{y}$ such as $V'(\widetilde{y}) \le W'(\widetilde{y})$ and $\overline{y} < \widetilde{y} < y^*$ because $W(y) \ge V(y)$ in the neighborhood of y^* (Figure 2). This contradicts (23), so intersections never exist in (y^o, y^*). In addition, $W(y)$ and $V(y)$ do not intersect in $(0, y^o)$, since $V(y) \le 0$ in the area. We conclude that $W(y) \ge V(y)$ in $(0, y^*)$ and (C4) is satisfied.

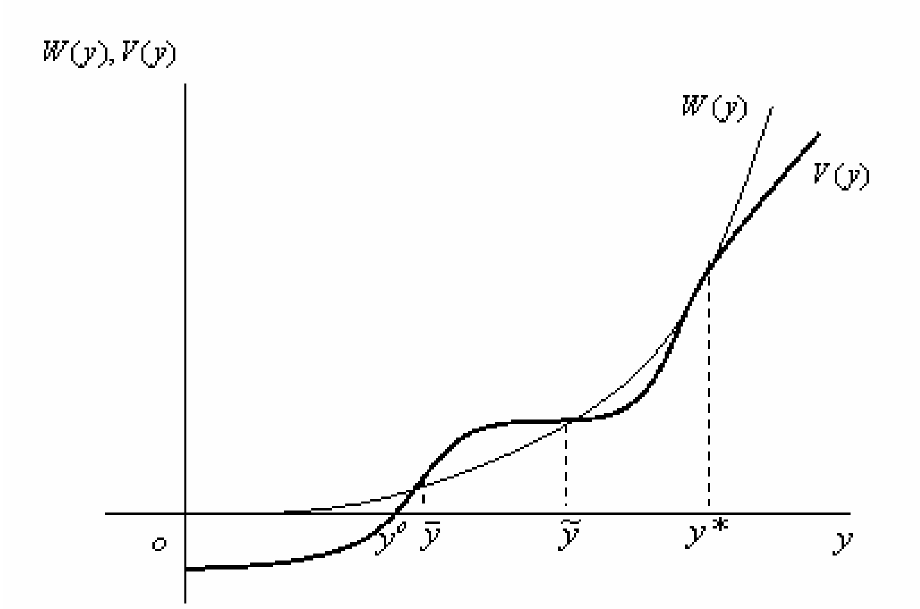

Figure 2: Contradiction in existence of intersections

Next, we show that (C3) is also satisfied. First, from (A2), we have that $V \in C^2$ at y^*, since $y^* \in (y^o, \infty)$; in other words, it is sufficient for the condition $v \in C^1(\partial D \cap U)$. Second, the condition $Lv \le 0$ outside $\overline{D}$ is expressed as

$$\frac{1}{2}\sigma^2 y^2 V''(y) + gyV'(y) - rV(y) \le 0 \ \text{ in } (y^*, \infty). \tag{24}$$

Since $0 < h(y) \le \beta$ in $[y^*, \infty)$, where the first inequality is Equation (10), we have $h(y)^2 - h(y) \le \beta^2 - \beta$ in $[y^*, \infty)$. Since (22) can be restated as

$$h'(y) = \frac{1}{y}\left[\frac{y^2V''(y)}{V(y)} + h(y) - h(y)^2\right],$$

and we have $\frac{y^2V''(y)}{V(y)} < h(y)^2 - h(y)$, we obtain $\frac{y^2V''(y)}{V(y)} < \beta^2 - \beta = \frac{r-g\beta}{\frac{1}{2}\sigma^2}$ in $[y^*, \infty)$, where the last equality follows from (21). This inequality can be restated as

$$\beta < \frac{r}{g} - \frac{\sigma^2}{2g}\frac{y^2V''(y)}{V(y)} \ \text{ in } [y^*, \infty).$$

Using $h(y) \le \beta$ in $[y^*, \infty)$, we obtain (24). (C5) guarantees that D must have the form $\{(s, y) : s \in \Re_+, 0 < y < y^*\}$ and could not have any other components (Øksendal [13], p.210).

Clearly, (C2) is satisfied. Since $D = \{(s, y) : s \in \Re_+, 0 < y < y^*\}$ and $\tau_D < \infty$ a.s. from (A1), (C1) is also satisfied. Therefore, we find a unique optimal stopping time τ_D and the optimal reward function w^*, where $w^*(s, y) \equiv V(y^*)\left(\frac{y}{y^*}\right)^\beta e^{-rs}$ for $y \in [0, y^*)$ and $w^* \equiv v$ for $y \geq y^*$. The proof is complete. □

(2) Proposition 2

A candidate of continuation region D must be invariant w.r.t. time t, so we guess that D has the form $\{(s, y) : s \in \Re_+, -\infty < y < y^*\}$, where y^* is a positive number.

It is reasonable to assume that $w(s, y) = W(y)e^{-rs}$, where W is a function of y. By (C5)(a) and the characteristic operator (11), we get the following differential equation of W

$$\frac{1}{2}\sigma^2 W''(y) + gW'(y) - rW(y) = 0. \tag{25}$$

The general solution of (25) is $W(y) = A_1 e^{\alpha_1 y} + A_2 e^{\alpha_2 y}$, where A_1 and A_2 are arbitrary constants and α_1 and α_2 are roots of the equation

$$\frac{1}{2}\sigma^2\alpha^2 + g\alpha - r = 0. \tag{26}$$

When we assume $\alpha_1 > \alpha_2$, we get $\alpha_1 > 0$ and $\alpha_2 < 0$. $W(y)$ must be bounded as $y \to -\infty$, so we must have $A_2 = 0$. If we restate A_1 and α_1 as A and α respectively, the solution is $W(y) = Ae^{\alpha y}$. It is easy to show that $\frac{\partial \alpha}{\partial \sigma^2} < 0$, $\lim_{\sigma^2 \to 0} \alpha = \frac{r}{g} > 1$, and $\lim_{\sigma^2 \to \infty} \alpha = 0$.

If we use α, A, $e^{\alpha y}$, and $h_A(y)$, (11) instead of β, B, $e^{\beta y}$, and $h(y)$, (4), respectively, we can show that (C2) - (C5) are all satisfied in the same manner as in Proposition 1, except that, for (C3), we use $h_A(y)^2 \leq \alpha^2$ in $[y^*, \infty)$ instead of $h(y)^2 - h(y) \leq \beta^2 - \beta$ in $[y^*, \infty)$. Since $D = \{(s, y) : s \in \Re_+, -\infty < y < y^*\}$ and $\tau_D < \infty$ a.s. from a characteristic of the standard Brownian motion (e.g. Øksendal, 1998, p.119), (C1) is also satisfied. Therefore, we find a unique optimal stopping time τ_D and the optimal reward function w^*, where $w^*(s, y) \equiv V(y^*)e^{\alpha(y-y^*)-rs}$ for $y \in (-\infty, y^*)$ and $w^* \equiv v$ for $y \geq y^*$. The proof is complete. □

(3) Proposition 3

A candidate of continuation region D must be invariant w.r.t. time t, and the agent takes notice of the ratio $\frac{Y}{C}$ since the intrinsic value function V is a homogenous function of degree one. Thus, we estimate that D has the form $\{(s, y, c) : s \in \Re_+, 0 < \frac{y}{c} < z^*\}$, where z^* is a positive number.

It is reasonable to assume that $w(s, y, c) = W(y, c)e^{-rs}$, where W is also a homogenous function of degree one. If we define $\widetilde{W}(z) \equiv \frac{1}{c}W(y, c) = W(\frac{y}{c}, 1)$, where $z \equiv \frac{y}{c}$, we get the following differential equation of $\widetilde{W}$

$$\frac{1}{2}(\sigma_y^2 - 2\rho\sigma_y\sigma_c + \sigma_c^2)z^2\widetilde{W}''(z) + (g_y - g_c)z\widetilde{W}'(z) - (r - g_c)\widetilde{W}(z) = 0, \tag{27}$$

by (C5)(a) and the characteristic operator (13). The general solution of (27) is $\widetilde{W}(z) = \Delta_1 e^{\delta_1 z} + \Delta_2 e^{\delta_2 z}$, where Δ_1 and Δ_2 are arbitrary constants and δ_1 and δ_2 are the roots of the equation

$$\frac{1}{2}(\sigma_y^2 - 2\rho\sigma_y\sigma_c + \sigma_c^2)\delta(\delta - 1) + (g_y - g_c)\delta - (r - g_c) = 0. \tag{28}$$

When we assume $\delta_1 > \delta_2$, we get $\delta_1 > 1$ and $\delta_2 < 0$. $\widetilde{W}(z)$ must be bounded as $z \to 0$, so we must have $\Delta_2 = 0$. If we restate Δ_1 and δ_1 as Δ and δ, respectively, the solution is $\widetilde{W}(z) = \Delta e^{\delta z}$.

If we use z, δ, Δ, $\widetilde{h}$, $\widetilde{W}$, and $\widetilde{V}$, (13), (14), (15) instead of y, β, B, h, W, and V, (4), (5), (6), respectively, we can show that (C2) - (C5) are all satisfied in the same manner as in Proposition 1. Since $D = \{(s, y, c) : s \in \Re_+, 0 < \frac{y}{c} < z^*\}$ and $\tau_D < \infty$ a.s. from (A5), (C1) is also satisfied. Therefore, we find a unique optimal stopping time τ_D and the optimal reward function w^*, where $w^*(s, y, c) \equiv c\left(\frac{y/c}{z^*}\right)^\delta \widetilde{V}(z^*)e^{-rs}$ for $\frac{y}{c} \in [0, z^*)$ and $w^* \equiv v$ for $\frac{y}{c} \geq z^*$. The proof is complete. $\square$

Appendix 3. Proofs of propositions in Section 3

Before proofs, we confirm some facts concerning output elasticity capital $\varepsilon(K)$ and $\widetilde{\varepsilon}(K)$, which is defined in Proposition 4. First, we clearly have $\widetilde{\varepsilon}(K) \geq \varepsilon(K)$, where the equality is supported if and only if $a = b = 0$. Next, differentiating them, we obtain

$$\varepsilon'(K) = \frac{Q'(K)}{Q(k)}\left[1 + \frac{Q''(K)K}{Q'(K)} - \varepsilon(K)\right], \tag{29}$$

$$\widetilde{\varepsilon}'(K) = \frac{Q'(K)}{Q(k) - a}\left[1 + \frac{Q''(K)(\frac{b}{rC} + K)}{Q'(K)} - \widetilde{\varepsilon}(K)\right]. \tag{30}$$

(29) implies that $\varepsilon'(K) \leq 0 \Leftrightarrow \varepsilon(K) \geq 1 + \frac{Q''(K)K}{Q'(K)}$, and (30) implies that $\widetilde{\varepsilon}'(K) < 0 \Leftrightarrow \widetilde{\varepsilon}(K) > 1 + \frac{Q''(K)(\frac{b}{rC}+K)}{Q'(K)}$, where we should note that $\widetilde{\varepsilon}(K)$ is defined only in (K^a, ∞). Hence, if $a > 0$ or $b > 0$, we have that $\varepsilon'(K) \leq 0 \Rightarrow \widetilde{\varepsilon}'(K) < 0$, since $\widetilde{\varepsilon}(K) > \varepsilon(K) \geq 1 + \frac{Q''(K)K}{Q'(K)} > 1 + \frac{Q''(K)(\frac{b}{rC}+K)}{Q'(K)}$. Therefore, we conclude that $\varepsilon'(K) \leq 0 \Rightarrow \widetilde{\varepsilon}'(K) \leq 0$, considering the case that $a = b = 0$.

(1) Proposition 4

To prove this proposition from Proposition 1, we show that (i) (A2) is satisfied, (ii) $h'(R) < 0 \Leftrightarrow \widetilde{\varepsilon}'(K) < 0$ for $R \in (R^o, \infty)$, and (iii) $h(R) \lesseqgtr \beta \Leftrightarrow \widetilde{\varepsilon}(K) \lesseqgtr \frac{\beta-1}{\beta}$ for $R \in (R^o, \infty)$.

The first-order condition for RHS of (16) is $R = \frac{(r-g)C}{Q'(K)}$, which is defined by $R(K)$. We have $R' > 0$ since $Q'' < 0$. If $R \leq R(0) = \frac{(r-g)C}{Q'(0)}$, then the optimal capital stock K^* is 0 and the intrinsic value $V(R) = -\frac{aR}{r-g} - \frac{b}{r} \leq 0$. Otherwise, K^* is a solution of $R(K) = R$ and by substituting $R(K)$ into (16) and using the envelope theorem, we get

$$V(R(K)) = [Q(K) - a]\frac{C}{Q'(K)} - \left(\frac{b}{r} + CK\right), \tag{31}$$

$$V'(R(K)) = \frac{C}{R(K)}\frac{Q(K) - a}{Q'(K)}. \tag{32}$$

From these equations, we have $V(R(K^a)) \leq 0$ and $V'(R(K^a)) = 0$, where we should note that $R(K^a) \geq R(0)$ and the equality is gained if and only if $a = 0$.

Thus, if $a \neq 0$ (i.e., $K^a > 0$), we have $V'(R) < 0$ for $R < R(K^a)$ and $V'(R) > 0$ for $R > R(K^a)$. Otherwise, when $a = 0$ (i.e., $K^a = 0$), we have $V'(R) = 0$ for $R < R(K^a)$ and $V'(R) > 0$ for $R > R(K^a)$ (Figure 3). Thus, we can find nonnegative R^o such as V is positive and belongs to C^2 in (R^o, ∞) and $V \leq 0$ in $[0, R^o]$, so (i) is verified.

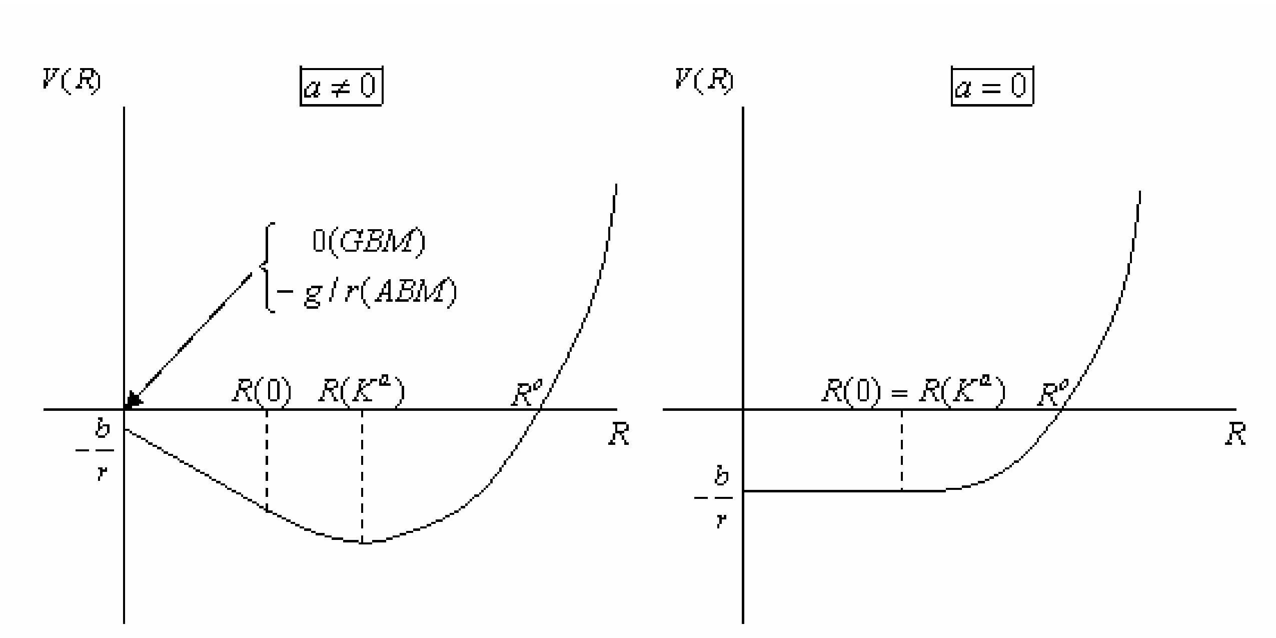

Figure 3: The graph of $V(R)$

Using the definition of h, (31), and (32), we obtain

$$h(R(K)) = [1 - \widetilde{\varepsilon}(K)]^{-1}. \tag{33}$$

If we define $K^o \equiv R^{-1}(R^o)$, then we have $K^o \geq K^a$. Since $V(R(K)) > 0$ for $K > K^o$, we have

$$[Q(K) - a]\frac{C}{Q'(K)} > \frac{b}{r} + CK \Leftrightarrow \widetilde{\varepsilon}(K) \equiv \frac{Q'(K)(\frac{b}{rC} + K)}{Q(K) - a} < 1 \text{ for } K > K^o,$$

from (31). Therefore, from (33), we conclude that $h(R(K)) \lesseqgtr \beta \Leftrightarrow \widetilde{\varepsilon}(K) \lesseqgtr \frac{\beta-1}{\beta}$ for $K > K^o$, which implies (iii). Differentiating (33), we gain $h'(R(K))R'(K) = [1-\widetilde{\varepsilon}(K)]^{-2}\widetilde{\varepsilon}'(K)$. Since $R' > 0$ and $\widetilde{\varepsilon}(K) < 1$ for $K > K^o$, h' and $\widetilde{\varepsilon}'$ have the same sign. Hence, (ii) is verified. It is easy to obtain R^*, K^*, and w^* in the proposition from the above description. The proof is complete.

For remark (i), we consider $\lim_{R\downarrow R^o} h(R)$. If $a > 0$ or $b > 0$, then we clearly have $\lim_{R\downarrow R^o} h(R) = \infty$, since $V(R^o) = 0$ and $V'(R^o) > 0$. If $a = b = 0$, from (31) and (32), we have $h(R(K)) = \frac{Q(K)}{Q(K)-KQ'(K)}$ and

$$\lim_{R\downarrow R^o} h(R) = \lim_{K\downarrow 0} h(R(K)) = \lim_{K\downarrow 0} \frac{Q(K)}{Q(K) - KQ'(K)} = \lim_{K\downarrow 0} \frac{Q'(K)}{-KQ''(K)}.$$

Therefore, if $a = b = 0$ and $Q'' > -\infty$, then $\lim_{R\downarrow R^o} h(R) = \infty$. Otherwise, when $a = b = 0$ and $Q''(0) = -\infty$, the condition $\lim_{R\downarrow R^o} h(R) > \beta$ is necessary. It can be restated as $\lim_{K\downarrow K^o} \widetilde{\varepsilon}(K) > \frac{\beta-1}{\beta}$, so remark (i) is verified. □

(2) Proposition 5

To prove this proposition from Proposition 2, we show that (i) (A7) is satisfied, (ii) $\widetilde{\varepsilon}'(K) \leq 0 \Rightarrow h'_A(R) < 0$ for $R \in (R^o, \infty)$, and (iii) $h_A(R) \lesseqgtr \alpha \Leftrightarrow \frac{Q'(K)}{1-\widetilde{\varepsilon}(K)} \lesseqgtr \alpha rC$ for $R \in (R^o, \infty)$.

The first-order condition for RHS of (17) is $R = \frac{rC}{Q'(K)} - \frac{g}{r}$, which is defined by $R(K)$. We have $R' > 0$ since $Q'' < 0$. If $R \le R(0) = \frac{rC}{Q'(0)} - \frac{g}{r}(> -\frac{g}{r})$, then the optimal capital stock K^* is 0 and the intrinsic value $V(R) = -\frac{aR}{r} - \frac{ag}{r^2} - \frac{b}{r} \le 0$. Otherwise, K^* is a solution of $R(K) = R$, and, by substituting $R(K)$ into (17) and using the envelope theorem, we get the same equation (31) and

$$V'(R(K)) = \frac{1}{r}[Q(K) - a], \quad (34)$$

$$h_A(R(K)) = \frac{1}{rC}\frac{Q'(K)}{1 - \widetilde{\varepsilon}(K)}. \quad (35)$$

From these equations, we can show (i) and (iii) in the same manner as in Proposition 4. If we define $K^o \equiv R^{-1}(R^o)$, then we have $K^o \ge K^a$ and $\widetilde{\varepsilon}(K) < 1$ for $K > K^o$ from (31). Differentiating (35), we gain $h'_A(R(K))R'(K) = \frac{1}{rC}\frac{Q''(K)[1-\widetilde{\varepsilon}(K)]+Q'(K)\widetilde{\varepsilon}'(K)}{[1-\widetilde{\varepsilon}(K)]^2}$. Since $R' > 0$, $Q' > 0$, $Q'' < 0$, and $\widetilde{\varepsilon}(K) < 1$ for $K > K^o$, $h'_A(R) < 0$ if $\widetilde{\varepsilon}'(K) \le 0$. Hence, (ii) is verified.

It is easy to obtain R^*, K^*, and w^* in the proposition from the above description. The proof is complete. □

(3) Proposition 6

To prove this proposition from Proposition 3, we show that (i) (A6) is satisfied, (ii) $\widetilde{h}'(Z) < 0 \Leftrightarrow \widetilde{\varepsilon}'(K) < 0$ for $Z \in (Z^o, \infty)$, and (iii) $\widetilde{h}(Z) \lesseqgtr \delta \Leftrightarrow \widetilde{\varepsilon}(K) \lesseqgtr \frac{\delta-1}{\delta}$ for $Z \in (Z^o, \infty)$.

The first-order condition for RHS of (19) is $Z = \frac{r-g}{Q'(K)}$, which is defined by $Z(K)$. We have $Z' > 0$ since $Q'' < 0$. If $Z \le Z(0) = \frac{r-g}{Q'(0)}$, then the optimal capital stock K^* is 0 and $\widetilde{V}(Z) = -\frac{aZ}{r-g} \le 0$. Otherwise, K^* is a solution of $Z(K) = Z$, and, by substituting $Z(K)$ into (19) and using the envelope theorem, we get

$$\widetilde{V}(Z(K)) = \frac{Q(K) - a}{Q'(K)} - K, \quad (36)$$

$$\widetilde{V}'(Z(K)) = \frac{Q(K) - a}{Z(K)Q'(K)}, \quad (37)$$

$$\widetilde{h}(Z(K)) = [1 - \widetilde{\varepsilon}(K)]^{-1}. \quad (38)$$

From these equations, we can show (i) - (iii) in the same manner as in Proposition 4.

It is easy to obtain Z^*, K^*, and w^* in the proposition from the above description. The proof is complete. □

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