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LEAST DISTANCE BASED INEFFICIENCY MEASURES ON THE PARETO-EFFICIENT FRONTIER IN DEA

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Abstract Since Briec developed a family of the least distance based inefficiency measures satisfying weak monotonicity over weakly efficient frontier, the existence of a least distance based efficiency measure satisfying strong monotonicity on the strongly efficient frontier is still an open problem. This paper gives a negative answer to the open problem and its relaxed open problem. Modifying Briec's inefficiency measures gives an alternative solution to the relaxed open problem, that can be used for theoretical and practical applications.

Keywords: DEA, the least distance, efficiency measure

1. Introduction

Data Envelopment Analysis (DEA) provides not only the efficiency score of each Decision Making Unit (DMU) but also a target to improve the efficiency of the DMU. The efficient frontier E is a subset of production possibility set T and any activity of the efficient frontier can not improve any input and output levels without worsening some others. Therefore, any improvement target must be located in the efficient frontier.

Identification of the improvement target in the efficient frontier depends on DEA models that are formulated as mathematical programming problems with some variables, an objective function and some constraints. Generally speaking, DEA models are classified into two groups, radial type and non-radial type. The radial type models (e.g., CCR: Charnes et al. [7], and BCC: Banker et al. [4]) originated from the Debreu-Farrell measure [10], which is an efficiency measure identifying an improvement target on the boundary E^w of T. Therefore, the Debreu-Farrell measure may identify the target on $E^w \setminus E$, while CCR and BCC adopt the two stage approach, the first stage to measure the efficiency score and the second stage to find the target on E. However, as is well known, the efficiency (inefficiency) score of the radial type measure does not satisfy the axiom of strong monotonicity which states that the efficiency (inefficiency) score strictly increases when any output (input) increases or any input (output) decreases.

Non-radial DEA models are mainly developed with the aim of satisfying the strong monotonicity axiom. Some non-radial type DEA models are one step approach that simultaneously finds the improvement target on E and measures the efficiency score. For example, a non-radial DEA model, RAM (range-adjusted measure) developed by Cooper et al. [8], defines the difference between 1 and its optimal value as its efficiency measure, which satisfies the strong monotonicity axiom under an assumption that the worst observed-data of input and output are constant. Therefore, the optimal value of RAM is an inefficiency measure satisfying the strong monotonicity axiom. In addition to strong monotonicity, RAM has various desirable properties such as unit invariance (the efficiency score is not influenced by units in which inputs and outputs are measured) and translation invariance (the efficiency score is not influenced by a change in the origin of coordinate values).

Briec [5] proposes a family of least distance based inefficiency measures under arbitrary p norm. Since the difference between 1 and the efficiency score is defined as an inefficiency measure, the whole of inefficiency measures includes any efficiency measures. In fact, Briec [5] shows that the Debreu-Farrell measure is reduced to the least distance based inefficiency measure generated by incorporating a weighted matrix into Briec's inefficiency measure specified as $p = \infty$. Therefore, his inefficiency measure satisfies weak monotonicity, that is, the inefficiency (efficiency) score does not decrease when an input (output) does not decrease or an output (input) does not increase in contrast to strong monotonicity. Moreover, his inefficiency measure may find the target on E^w. Two properties, weak monotonicity and the target existence on E^w, of his inefficiency measure can be stated as the definition of Pastor and Aparicio's [15] efficiency measure. An efficiency measure satisfying weak (strong) monotonicity and identifying the target on E is called weak (strong) monotonicity over the strongly efficient frontier.

Is there a least distance based efficiency measure satisfying strong monotonicity over the strongly efficient frontier? Recently, Baek and Lee [3] propose a least distance based efficiency measure by combining RAM and Briec's inefficiency measure for $p = 2$. Furthermore, they assert that their least distance based efficiency measure satisfies strong monotonicity over the strongly efficient frontier. Pastor and Aparicio [15] disprove this assertion by giving a counterexample and they pose a question: "it is still an open problem whether there exists an efficiency measure based on closest targets which satisfies strong monotonicity on the strong efficient frontier."

The present study gives a negative answer to the open problem and its relaxed problem "whether there exists an inefficiency measure based on closest targets which satisfies weak monotonicity on the strongly efficient frontier." Furthermore, we show that weak monotonicity on the strongly efficient frontier is satisfied by a family of inefficiency measures on arbitrary p norm by slightly modifying Briec's least distance based inefficiency measure. The slight modification is to incorporate slack variables into Briec's least distance based inefficiency measure. A priori information and additive constraints can be flexibly incorporated into our inefficiency measure. For example, by adding the convexity constraint into the inefficiency measure, we develop a least distance based inefficiency measure under the assumption of Variable Return to Scales (VRTS), which satisfies weak monotonicity over the strongly efficient frontier for arbitrary p norm. The developed inefficiency measure is applied to teamwork effectiveness ranking of undergraduate student groups, which is difficult to accept as an improvement target of decreasing inputs from the viewpoint of education. Our inefficiency measure by adding constraints of non-decreasing inputs satisfies weak monotonicity over the strongly efficient frontier for arbitrary p norm.

An improvement target attaining the least distance is referred to as a shortest path or closest target, whose calculation and application are studied in [2, 3, 12, 14]. Since our inefficiency measure satisfies weak monotonicity on the strongly efficient frontier, we can find potential rank reversal in the existing applications [3, 12]. Aparicio et al. [2] develop a mixed integer programming (MIP) approach for finding the least distance improvement target under implicit representation of efficient face. Our inefficiency measure can be incorporated into the MIP model by adding slack variables. Gonzalez et al. [14] propose enumerating efficient faces as a preprocessor for finding the shortest path, which plays the same role in our inefficiency measure.

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The remainder of this paper is organized as follows: Section 2 introduces a family of least distance based inefficiency measures developed by [5, 6] and its limitations that gives a negative answer to the open problem by [15]. Section 3 develops our inefficiency measure by modifying the least distance based inefficiency measures and it shows that our inefficiency measure satisfies axioms of Färe [11] and Russell [16]. Section 4 shows that our inefficiency measure can be answered to theoretical and practical requests. The final section summarizes our main results, and discusses possible extensions.

2. Limitation of Least Distance Based Inefficiency Measures

We assume that there are J decision-making units (DMUs), each DMU$_j$ ($j = 1, \dots, J$) of which transforms N inputs $\boldsymbol{x}_j \in R^N_{++}$ into M outputs $\boldsymbol{y}_j \in R^M_{++}$. Here, R^L_{++} denotes the positive orthant of the L dimensional space. Under several assumptions including a convex technology with freely disposable inputs and outputs, DEA considers a production possibility set

$$T \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}.$$

Following the works by Afrait [1], Charnes et al. [7] and Banker et al. [4], T is traditionally formulated as the minimal weakly monotonic convex hull (conical hull) containing all DMUs.

The *weakly efficient frontier* of T is defined by

$$E^w \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in T \mid (\bar{\boldsymbol{x}}, -\bar{\boldsymbol{y}}) < (\boldsymbol{x}, -\boldsymbol{y}) \Longrightarrow (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \notin T\}. \tag{2.1}$$

The *strongly efficient frontier* of T is defined by

$$E \equiv \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in T \;\middle|\; \begin{array}{l} (\bar{\boldsymbol{x}}, -\bar{\boldsymbol{y}}) \leq (\boldsymbol{x}, -\boldsymbol{y}) \\ (\bar{\boldsymbol{x}}, -\bar{\boldsymbol{y}}) \neq (\boldsymbol{x}, -\boldsymbol{y}) \end{array} \Longrightarrow (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \notin T \right\}. \tag{2.2}$$

The various measures of inefficiency appear in the literature in Operations Research and Economics [4, 8, 13, 18]. An *inefficiency measure* is a real-valued function f defined on T which satisfies some desirable properties such as

Axiom A: $(\boldsymbol{x}, \boldsymbol{y}) \in E$ if and only if $f(\boldsymbol{x}, \boldsymbol{y}) = 0$.
Axiom B: For any $(\boldsymbol{x}, \boldsymbol{y}) \in T$, $0 \leq f(\boldsymbol{x}, \boldsymbol{y})$.
Axiom C: For $(\boldsymbol{x}^a, \boldsymbol{y}^a) \in T$ and $(\boldsymbol{x}^b, \boldsymbol{y}^b) \in T$, $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$ implies $f(\boldsymbol{x}^a, \boldsymbol{y}^a) \geq f(\boldsymbol{x}^b, \boldsymbol{y}^b)$.

Note that these axioms for inefficiency measure f are essentially equal to that for efficiency measure given in [4, 8, 11, 16]. Axioms A and B correspond to a part of efficiency requirement of efficiency measure discussed in [4, 8, 11] and Axiom C does to monotonicity of efficiency measure [16]. If an inefficiency measure f satisfies Axioms A, B and C and it has an upper bound 1, then we have $0 \leq f(\boldsymbol{x}, \boldsymbol{y}) \leq 1$ for all $(\boldsymbol{x}, \boldsymbol{y}) \in T$ and the difference

$$1 - f(\boldsymbol{x}, \boldsymbol{y}) \tag{2.3}$$

satisfies efficiency requirement and the efficiency measure properties corresponding to Axioms A, B and C. In fact, both RAM and Debreu-Farrell measure are defined by the same manner as (2.3).

Let $\|\boldsymbol{z}\|_p$ be the p-norm:

$$\|\boldsymbol{z}\|_p = \begin{cases} \left(\sum_{l=1}^{n} |z_l|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z_1|, \dots, |z_n|\} & \text{if } p = \infty, \end{cases} \tag{2.4}$$

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Table 1: Counterexample

	inputs			output		location
DMU	x_1	x_2	x_3	y_1	y_2	in (2.8)
1	1	1	1	10	1	E
2	5	5	1	10.5	1	E
3	2	4	1	10	1	$E^w \setminus E$
4	5	5	1	10	1	$E^w \setminus E$

and let

$$f^p(\boldsymbol{x}, \boldsymbol{y}) \equiv \min \{\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E^w\}, \tag{2.5}$$

which is called the *Hölder distance function* by [5]. The Hölder distance function is a natural extension of the classical profit function based efficiency measure [1, 9]. Furthermore, Debreu-Farrell measure [10], $\min\{\theta \mid (\theta \boldsymbol{x} - \boldsymbol{d}^x, \boldsymbol{y} + \boldsymbol{d}^y) \in T,\ \boldsymbol{d}^x \geq \boldsymbol{0},\ \boldsymbol{d}^y \geq \boldsymbol{0}\}$, is a special case of a weighted Hölder distance function [5]. However, a family of Hölder distance functions $\hat{f}^p$ does not satisfy Axiom A because $f^p(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = 0$ for all weakly efficient activity $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in E^w$. The Debreu-Farrell measure also violates Axiom A.

Consider variants of Axioms A and C:

Axiom A': $(\boldsymbol{x}, \boldsymbol{y}) \in E^w$ if and only if $f(\boldsymbol{x}, \boldsymbol{y}) = 0$.

Axiom C': For $(\boldsymbol{x}^a, \boldsymbol{y}^a) \in T$ and $(\boldsymbol{x}^b, \boldsymbol{y}^b) \in T$, $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$ and $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \neq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$ imply $f(\boldsymbol{x}^a, \boldsymbol{y}^a) > f(\boldsymbol{x}^b, \boldsymbol{y}^b)$.

The measure f satisfying the pair of Axioms A' and C (A and C') is called *weakly (strongly) monotonic over E^w (E)*. Briec [5] shows that a family of Hölder distance functions is an inefficient measure f^p satisfying weak monotonicity over E^w. However, the evolution of DEA from radial type to non-radial type gives a question whether there exists an inefficiency measure satisfying strong monotonicity over E.

From the viewpoint of efficiency measure, Baek and Lee [3] develop a new least distance based efficiency measure by combining RAM and a kind of inefficiency measure

$$\min \{\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}')\|_2 \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \tag{2.6}$$

and they assert that the new efficiency measure satisfies strong monotonicity over E. Pastor and Aparicio [15], however, show that their assertion is not true by giving an input-output data where weak monotonicity over E is not satisfied by the new efficiency measure. That is, the inefficiency measure of (2.6) does not satisfy Axiom C nor C'. Furthermore, Pastor and Aparicio [15] wonder if Briec's assertion, weak monotonicity over E^w of f^p, can be extended into strong one over E and they give an open problem whether there exists an efficiency measure based on closest points which satisfies strong monotonicity over E (see p. 397 in [15]).

By extending the inefficiency measure of (2.6) into arbitrary p norm, a new family of a least distance based inefficiency measures is defined as

$$\bar{f}^p(\boldsymbol{x}, \boldsymbol{y}) \equiv \min \{\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \tag{2.7}$$

for any $p \in [1, \infty]$. The inefficiency measure of (2.6) is $\bar{f}^2$. We will show a counterexample in Table 1 to the problem on strong monotonicity over E of a family of $\bar{f}^p$.

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Consider a production possibility set:

$$T_c = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \,\middle|\, \sum_{j=1}^{J} \boldsymbol{x}_j \lambda_j \leq \boldsymbol{x}, \sum_{j=1}^{J} \boldsymbol{y}_j \lambda_j \geq \boldsymbol{y}, \boldsymbol{\lambda} \geq \boldsymbol{0} \right\}, \tag{2.8}$$

where $\mathbf{0}$ is the zero vector of an appropriate dimension. The production possibility set T_c corresponding to the input-output data of Table 1 has the strongly efficient frontier

$$E = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} = \boldsymbol{x}_1\lambda_1 + \boldsymbol{x}_2\lambda_2, \boldsymbol{y} = \boldsymbol{y}_1\lambda_1 + \boldsymbol{y}_2\lambda_2, \boldsymbol{\lambda} \geq \boldsymbol{0}\}. \tag{2.9}$$

Note that $(\boldsymbol{x}_3, -\boldsymbol{y}_3) \leq (\boldsymbol{x}_4, -\boldsymbol{y}_4)$ and $(\boldsymbol{x}_3, -\boldsymbol{y}_3) \neq (\boldsymbol{x}_4, -\boldsymbol{y}_4)$. If $\bar{f}^p$ has strong monotonicity over E, then $\bar{f}^p(\boldsymbol{x}_3, \boldsymbol{y}_3) < \bar{f}^p(\boldsymbol{x}_4, \boldsymbol{y}_4)$. However, we have

$$\begin{aligned} \bar{f}^p(\boldsymbol{x}_3, \boldsymbol{y}_3) &= \min\{\|(\boldsymbol{x}_3, \boldsymbol{y}_3) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\geq \min\{\|(\boldsymbol{x}_3, \boldsymbol{y}_3) - (\boldsymbol{x}', \boldsymbol{y}')\|_\infty \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\geq \min\{\max\{|2 - (\lambda_1 + 5\lambda_2)|, |4 - (\lambda_1 + 5\lambda_2)|\} \mid \boldsymbol{\lambda} \geq \boldsymbol{0}\} \\ &= 1 \end{aligned} \tag{2.10}$$

and it follows from $(\boldsymbol{x}_2, \boldsymbol{y}_2) \in E$ and $(\boldsymbol{x}_4, \boldsymbol{y}_4) - (\boldsymbol{x}_2, \boldsymbol{y}_2) = (0, 0, 0, -0.5, 0)$ that

$$\begin{aligned} \bar{f}^p(\boldsymbol{x}_4, \boldsymbol{y}_4) &= \min\{\|(\boldsymbol{x}_4, \boldsymbol{y}_4) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\leq \|(\boldsymbol{x}_4, \boldsymbol{y}_4) - (\boldsymbol{x}_2, \boldsymbol{y}_2)\|_p \\ &= \|(0, 0, 0, -0.5, 0)\|_p = 0.5. \end{aligned} \tag{2.11}$$

Therefore, it follows from (2.10) and (2.11) that $\bar{f}^p(\boldsymbol{x}_3, \boldsymbol{y}_3) \geq 1 > 0.5 \geq \bar{f}^p(\boldsymbol{x}_4, \boldsymbol{y}_4)$ for all $p \in [1, \infty]$.

We have seen that Table 1 gives a counterexample to the problem on the production possibility set T under the assumption of constant returns to scale (CRTS). By adding a condition $\sum_{j=1}^{J} \lambda_j = 1$ into T_c defined by (2.8), DEA assumes the following production possibility set under variable returns to scale (VRTS):

$$T_v = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \,\middle|\, \boldsymbol{x} \geq \sum_{j=1}^{J} \boldsymbol{x}_j \lambda_j, \boldsymbol{y} \leq \sum_{j=1}^{J} \boldsymbol{y}_j \lambda_j, \sum_{j=1}^{J} \lambda_j = 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0} \right\}. \tag{2.12}$$

Table 1 dropping the third input x_3 and the second output y_2 will be a counterexample of the problem under VRTS. In fact, 4 DMUs with two inputs and one output, $(\boldsymbol{x}_1, y_1) = (1, 1, 10)$, $(\boldsymbol{x}_2, y_2) = (5, 5, 10.5)$, $(\boldsymbol{x}_3, y_3) = (2, 4, 10)$ and $(\boldsymbol{x}_4, y_4) = (5, 5, 10)$ form a production possibility set and the efficient frontier under VRTS as follows:

$$T_v = \left\{ (x_1, x_2, y) \,\middle|\, \sum_{j=1}^{4} (x_{1j}, x_{2j})\lambda_j \leq (x_1, x_2), \sum_{j=1}^{4} y_{1j}\lambda_j \geq y_1, \sum_{j=1}^{4} \lambda_j = 1, \boldsymbol{\lambda} \geq \boldsymbol{0} \right\}$$

and

$$E = \left\{ (x_1, x_2, y) \,\middle|\, \sum_{j=1}^{2} (x_{1j}, x_{2j})\lambda_j = (x_1, x_2), \sum_{j=1}^{2} y_{1j}\lambda_j = y_1, \sum_{j=1}^{2} \lambda_j = 1, \boldsymbol{\lambda} \geq \boldsymbol{0} \right\},$$

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respectively. Since $(\boldsymbol{x}_4, -y_4) \geq (\boldsymbol{x}_3, -y_3)$, the least distance from $(\boldsymbol{x}_3, y_3)$ to E must not be greater than that from $(\boldsymbol{x}_4, y_4)$ to E. However, we have

$$\begin{aligned} &\min\{\|(\boldsymbol{x}_3, y_3) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\qquad \geq \min\{\|(\boldsymbol{x}_3, y_3) - (\boldsymbol{x}', \boldsymbol{y}')\|_\infty \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\qquad \geq \min\left\{\max\left\{\begin{array}{c} |2-(\lambda_1+5\lambda_2)| \\ |4-(\lambda_1+5\lambda_2)| \end{array}\right\} \middle| \boldsymbol{\lambda} \geq \mathbf{0}, \lambda_1+\lambda_2=1\right\} \\ &\qquad = 1 \end{aligned} \tag{2.13}$$

and it follows from $(\boldsymbol{x}_2, y_2) \in E$ and $(\boldsymbol{x}_4, y_4) - (\boldsymbol{x}_2, y_2) = (0, 0, -0.5)$ that

$$\begin{aligned} \min\{\|(\boldsymbol{x}_4, y_4) - (\boldsymbol{x}', y')\|_p \mid (\boldsymbol{x}', y') \in E\} &\leq \|(\boldsymbol{x}_4, y_4) - (\boldsymbol{x}_2, y_2)\|_p \\ &= \|(0, 0, -0.5)\|_p = 0.5. \end{aligned} \tag{2.14}$$

Due to the counterexample, a family of least distance based inefficiency measures $\bar{f}^p$ does not satisfy weak monotonicity over E, and hence, the data of 4 DMUs with two inputs and one output is a negative answer to the problem under VRTS. For decreasing or increasing returns to scale ($\sum \lambda_j \geq 1$ or $\sum \lambda_j \leq 1$), we can show that Table 1 plays a role of a negative answer to the problem as well.

Proposition 2.1. There exists an input-output data where neither strong nor weak monotonicity over E is satisfied by a family of $\bar{f}^p$.

Both additive model and RAM model are reduced to an inefficiency measure under VRTS and additive constraints $\boldsymbol{x}' \leq \boldsymbol{x}$ and $\boldsymbol{y}' \geq \boldsymbol{y}$ for $(\boldsymbol{x}', \boldsymbol{y}') \in E$. By adding the constraints $\boldsymbol{x}' \leq \boldsymbol{x}$ and $\boldsymbol{y}' \geq \boldsymbol{y}$ into $\bar{f}^p(\boldsymbol{x}, \boldsymbol{y})$, we have a side constrained $\bar{f}^p(\boldsymbol{x}, \boldsymbol{y})$ that is

$$\bar{f}^p_{xy}(\boldsymbol{x}, \boldsymbol{y}) \equiv \min\{\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E,\ \boldsymbol{x}' \leq \boldsymbol{x} \text{ and } \boldsymbol{y}' \geq \boldsymbol{y}\}. \tag{2.15}$$

It follows from (2.13), (2.14), $(\boldsymbol{x}_4, -y_4) \geq (\boldsymbol{x}_2, -y_2)$ and $(\boldsymbol{x}_2, y_2) \in E$ that

$$\begin{aligned} \bar{f}^p_{xy}(\boldsymbol{x}_3, y_3) &= \min\{\|(\boldsymbol{x}_3, y_3) - (\boldsymbol{x}', y')\|_p \mid (\boldsymbol{x}', y') \in E, \boldsymbol{x}' \leq \boldsymbol{x}, y' \geq y\} \\ &\geq \min\{\|(\boldsymbol{x}_3, y_3) - (\boldsymbol{x}', y')\|_p \mid (\boldsymbol{x}', y') \in E\} \\ &= 1 > 0.5 = \|(\boldsymbol{x}_4, y_4) - (\boldsymbol{x}_2, y_2)\|_p \\ &\geq \min\{\|(\boldsymbol{x}_4, y_4) - (\boldsymbol{x}', y')\|_p \mid (\boldsymbol{x}', y') \in E, \boldsymbol{x}' \leq \boldsymbol{x}_4, y' \geq y_4\} \\ &= \bar{f}^p_{xy}(\boldsymbol{x}_4, y_4). \end{aligned}$$

This means $\bar{f}^p_{xy}(\boldsymbol{x}_3, y_3) > \bar{f}^p_{xy}(\boldsymbol{x}_4, y_4)$ while $(\boldsymbol{x}_3, -y_3) \leq (\boldsymbol{x}_4, -y_4)$.

Adding constraints of $\sum \lambda_j = 1$, $\boldsymbol{x}' \leq \boldsymbol{x}$ and $\boldsymbol{y}' \geq \boldsymbol{y}$ into $\bar{f}^p$ is not an effective way to have weak monotonicity over E. When $\bar{f}^p_{xy}$ is defined under any assumption of constant, decreasing or increasing returns to scale, it follows from (2.10) and (2.11) that Table 1 gives a negative answer to the open problem.

Proposition 2.2. There exists an input-output data where neither strong nor weak monotonicity over E is satisfied by a family of $\bar{f}^p_{xy}$.

3. An Axiomatic Revision of Least Distance Based Inefficiency Measures

Let $D(\boldsymbol{x}, \boldsymbol{y}) \equiv \{(\boldsymbol{x} + \boldsymbol{d}^x, \boldsymbol{y} - \boldsymbol{d}^y) \mid \boldsymbol{d}^x \geq \mathbf{0}, \boldsymbol{d}^y \geq \mathbf{0}\}$. For $(\boldsymbol{x}, \boldsymbol{y}) \in T$, $D(\boldsymbol{x}, \boldsymbol{y})$ is a subset of T whose activity is dominated by $(\boldsymbol{x}, \boldsymbol{y})$. For any activity $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in D(\boldsymbol{x}, \boldsymbol{y})$, $\bar{\boldsymbol{x}}$ can produce $\bar{\boldsymbol{y}}$ by the same technology as that of $(\boldsymbol{x}, \boldsymbol{y})$. The inefficiency measure $\bar{f}^p(\boldsymbol{x}, \boldsymbol{y})$ is the least

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Table 2: An example of input-output data

DMU	x_1	x_2	y_1
DMU$_a$	1	10	3
DMU$_b$	1/2	17/2	3
DMU$_c$	1	37/4	9/2
DMU$_d$	3/2	10	19/4

Table 3: Efficiency scores of DMU$_a$

	$f^1(\boldsymbol{x}_a, \boldsymbol{y}_a)$	$\bar{f}^1(\boldsymbol{x}_a, \boldsymbol{y}_a)$	$\hat{f}^1(\boldsymbol{x}_a, \boldsymbol{y}_a)$	$\hat{f}^{w1}(\boldsymbol{x}_a, \boldsymbol{y}_a)$
distance	1/2	2	7/4	23/40
$(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$	$(1/2, 10, 3)$	$(1/2, 17/2, 3)$	$(3/2, 10, 19/4)$	$(1, 37/4, 9/2)$
$(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$	—	—	$(3/2, 10, 3)$	$(1, 10, 3)$

distance from $(\boldsymbol{x}, \boldsymbol{y})$ to E. We modify $\bar{f}^p(\boldsymbol{x}, \boldsymbol{y})$ into the least distance between $D(\boldsymbol{x}, \boldsymbol{y})$ and E, that is,

$$\hat{f}^p(\boldsymbol{x}, \boldsymbol{y}) \equiv \min \left\{ \|(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) - (\boldsymbol{x}', \boldsymbol{y}')\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E, (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in D(\boldsymbol{x}, \boldsymbol{y}) \right\}. \tag{3.1}$$

By the definition of the norm $\|\cdot\|_p$, a family of the least distance based inefficiency measures $\hat{f}^p$ satisfies Axioms A and B. Axiom C is also valid for $\hat{f}^p$: If $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$, then we have $D(\boldsymbol{x}^a, \boldsymbol{y}^a) \subseteq D(\boldsymbol{x}^b, \boldsymbol{y}^b)$, and hence, $\hat{f}^p(\boldsymbol{x}^b, \boldsymbol{y}^b) \leq \hat{f}^p(\boldsymbol{x}^a, \boldsymbol{y}^a)$.

Proposition 3.1. A family of least distance based inefficiency measures $\hat{f}^p$ satisfies weak monotonicity over E.

Russell [17] suggests that inefficiency and efficiency measures should be independent of the units in which the input and output variables are measured. We refer to this property as *the units invariance*. All of f^p, $\bar{f}^p$ and $\hat{f}^p$ are not unit invariance. The unit invariance is satisfied by a class of directional distance based inefficiency measures including the Debreu-Farrell measure [10]. The inefficiency measure class narrows down the production possibility set T to R^{N+M}_{++}. This section considers $T_+ = T \cap R^{N+M}_{++}$ as the domain of least distance inefficiency measures and discusses weak monotonicity and unit invariance over T_+.

For any $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$, we extend an inefficiency measure $\hat{f}^p$ into its weighted version as follows:

$$\hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) \equiv \min \left\{ \| ((\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) - (\boldsymbol{x}', \boldsymbol{y}'))\, Z(\boldsymbol{x}, \boldsymbol{y}) \|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E, (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in D(\boldsymbol{x}, \boldsymbol{y}) \right\}, \tag{3.2}$$

where $Z(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} \mathrm{diag}(\boldsymbol{x})^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{diag}(\boldsymbol{y})^{-1} \end{bmatrix}$.

By using an illustrative example, we show the difference among f^p,$\bar{f}^p$,$\hat{f}^p$ and $\hat{f}^{wp}$. The example consists of 4 DMUs with 2 inputs and 1 output whose data is listed in Table 2. Under the assumption of variable returns to scale, we draw the production possibility set in the 3 dimensional space which is given in Figure 1. We choose $p = 1$ and calculate the efficiency scores of DMU$_a$ by using four types of measures, f^1, $\bar{f}^1$, $\hat{f}^1$ and $\hat{f}^{w1}$. Table 3 reports these scores and corresponding optimal solutions $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ and $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$. Their optimal targets $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ are different each other as seen in Table 3. Figure 1 also illustrates the difference among optimal targets and $(3/2, 10, 3) \in D(\boldsymbol{x}_a, \boldsymbol{y}_a)$.

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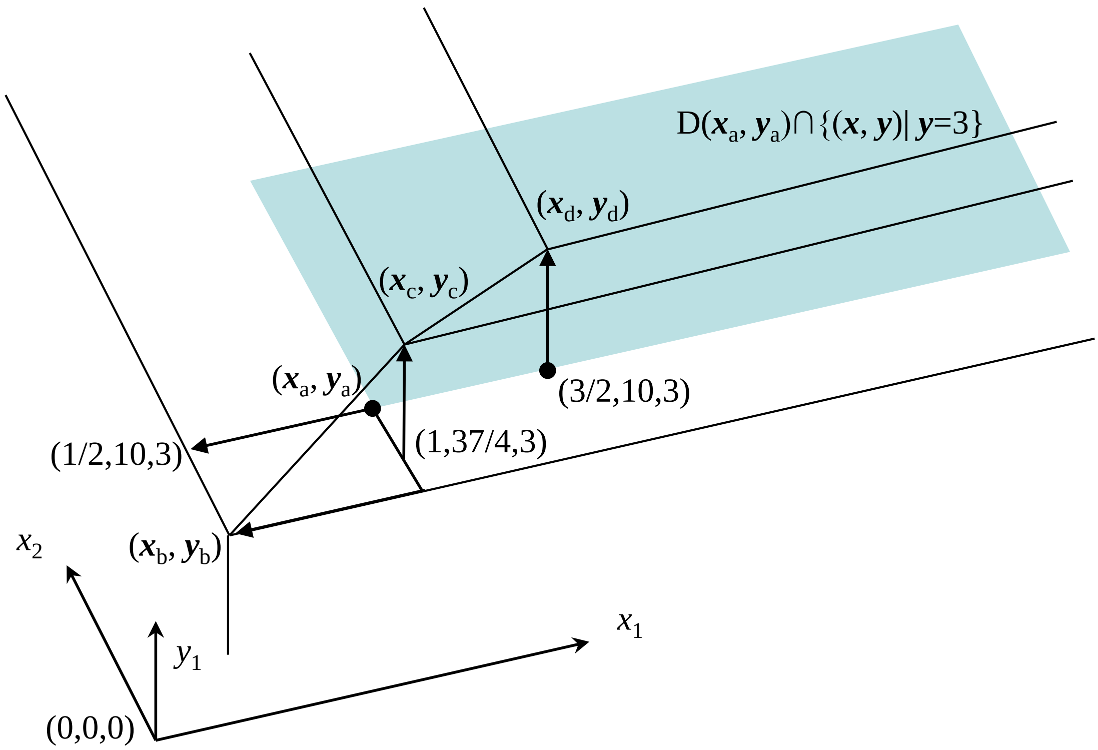

Figure 1: Optimal targets by $f^1, \bar{f}^1, \hat{f}^1$ and $\hat{f}^{w1}$

For all $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in D(\boldsymbol{x}, \boldsymbol{y})$ there exist $\boldsymbol{d}^x \geq \boldsymbol{0}$ and $\boldsymbol{d}^y \geq \boldsymbol{0}$ such that $\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{d}^x$ and $\hat{\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{d}^y$, and hence, we have

$$\begin{aligned} \|((\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) - (\boldsymbol{x}', \boldsymbol{y}'))Z(\boldsymbol{x}, \boldsymbol{y})\|_p &= \|(\boldsymbol{x} - \boldsymbol{x}' + \boldsymbol{d}^x, \boldsymbol{y} - \boldsymbol{y}' - \boldsymbol{d}^y)Z(\boldsymbol{x}, \boldsymbol{y})\|_p \\ &= \|(\boldsymbol{x}' - \boldsymbol{d}^x, \boldsymbol{y}' + \boldsymbol{d}^y)Z(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{e}, \boldsymbol{e})\|_p \end{aligned}$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$. Let $G(\boldsymbol{x}, \boldsymbol{y}) \equiv \{(\boldsymbol{x}' - \boldsymbol{d}^x, \boldsymbol{y}' + \boldsymbol{d}^y)Z(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{d}^x \geq \boldsymbol{0}, \boldsymbol{d}^y \geq \boldsymbol{0}, (\boldsymbol{x}', \boldsymbol{y}') \in E\}$. Then, the following property of $G(\boldsymbol{x}, \boldsymbol{y})$ holds.

Proposition 3.2. Choose $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$ arbitrarily. If $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in G(\boldsymbol{x}, \boldsymbol{y})$, then

$$(\bar{\boldsymbol{x}} - \boldsymbol{d}^x, \bar{\boldsymbol{y}} + \boldsymbol{d}^y) \in G(\boldsymbol{x}, \boldsymbol{y}) \text{ for any } (\boldsymbol{d}^x, \boldsymbol{d}^y) \in R_+^{N+M}.$$

Proof. Since $(\boldsymbol{d}^x, \boldsymbol{d}^y)Z(\boldsymbol{x}, \boldsymbol{y})^{-1} \in R_+^{N+M}$, we have $(\bar{\boldsymbol{x}} - \boldsymbol{d}^x, \bar{\boldsymbol{y}} + \boldsymbol{d}^y) \in G(\boldsymbol{x}, \boldsymbol{y})$. □

From the definition of $G(\boldsymbol{x}, \boldsymbol{y})$, we have

$$\hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) = \min\{\|(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) - (\boldsymbol{e}, \boldsymbol{e})\|_p \mid (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in G(\boldsymbol{x}, \boldsymbol{y})\}, \quad (3.3)$$

where $\boldsymbol{e}$ is the vector of all ones of an appropriate dimension. A family of weighted versions $\hat{f}^{wp}$ satisfies both Axioms A and B. Moreover, it is unit invariant.

Proposition 3.3. A family of weighted distance based inefficiency measures $\hat{f}^{wp}$ satisfies weak monotonicity over E. Moreover, a family of $\hat{f}^{wp}$ is unit invariant.

Proof. Consider $(\boldsymbol{x}^a, \boldsymbol{y}^a) \in T_+$ and $(\boldsymbol{x}^b, \boldsymbol{y}^b) \in T_+$ such that $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$. If $G(\boldsymbol{x}^a, \boldsymbol{y}^a) \subseteq G(\boldsymbol{x}^b, \boldsymbol{y}^b)$, then it follows from (3.3) that $\hat{f}^{wp}(\boldsymbol{x}^a, \boldsymbol{y}^a) \geq \hat{f}^{wp}(\boldsymbol{x}^b, \boldsymbol{y}^b)$, which means Axiom C. Therefore, it suffices to show that $G(\boldsymbol{x}^a, \boldsymbol{y}^a) \subseteq G(\boldsymbol{x}^b, \boldsymbol{y}^b)$.

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Table 4: Inefficiency scores for the counterexample in Table 1

	inputs			output		inefficiency scores		
DMU	x_1	x_2	x_3	y_1	y_2	DF	DF^E	$\hat{f}^{w\infty}$
1	1	1	1	10	1	0	0	0
2	5	5	1	10.5	1	0	0	0
3	2	4	1	10	1	0	0.333	0.018
4	5	5	1	10	1	0	0.024	0.024

Choose any $(\boldsymbol{x}' - \boldsymbol{d}^x, \boldsymbol{y}' + \boldsymbol{d}^y)Z(\boldsymbol{x}^a, \boldsymbol{y}^a) \in G(\boldsymbol{x}^a, \boldsymbol{y}^a)$ and let $Z_x^c = \text{diag}(\boldsymbol{x}^c)^{-1}$ and $Z_y^c = \text{diag}(\boldsymbol{y}^c)^{-1}$ for $c = a, b$. We have $(\boldsymbol{x}'-\boldsymbol{d}^x, \boldsymbol{y}'+\boldsymbol{d}^y)Z(\boldsymbol{x}^a, \boldsymbol{y}^a) = ((\boldsymbol{x}'-\boldsymbol{d}^x)Z_x^a, (\boldsymbol{y}'+\boldsymbol{d}^y)Z_y^a)$ and $(\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^a \geq (\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^b$. Therefore, we have

$$(\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^a = (\boldsymbol{y}' + \boldsymbol{d}^y + (\boldsymbol{y}' + \boldsymbol{d}^y)(Z_y^b)^{-1}Z_y^a - (\boldsymbol{y}' + \boldsymbol{d}^y))Z_y^b \quad \text{and} \tag{3.4}$$

$$\boldsymbol{d}^y + (\boldsymbol{y}' + \boldsymbol{d}^y)(Z_y^b)^{-1}Z_y^a - (\boldsymbol{y}' + \boldsymbol{d}^y) \geq \boldsymbol{0}. \tag{3.5}$$

First, we consider the case when $\boldsymbol{x}' - \boldsymbol{d}^x \geq \boldsymbol{0}$. We have $(\boldsymbol{x}' - \boldsymbol{d}^x)Z_x^a \leq (\boldsymbol{x}' - \boldsymbol{d}^x)Z_x^b$. This means that $(\boldsymbol{x}' - \boldsymbol{d}^x)Z_x^a = (\boldsymbol{x}' - \boldsymbol{d}^x + (\boldsymbol{x}' - \boldsymbol{d}^x)(Z_x^b)^{-1}Z_x^a - (\boldsymbol{x}' - \boldsymbol{d}^x))Z_x^b$ and $\boldsymbol{d}^x - (\boldsymbol{x}' - \boldsymbol{d}^x)(Z_x^b)^{-1}Z_x^a + (\boldsymbol{x}' - \boldsymbol{d}^x) \geq \boldsymbol{0}$. Therefore, it follows from (3.4) and (3.5) that

$$(\boldsymbol{x}' - \boldsymbol{d}^x, \boldsymbol{y}' + \boldsymbol{d}^y)Z(\boldsymbol{x}^a, \boldsymbol{y}^a) \in G(\boldsymbol{x}^b, \boldsymbol{y}^b) \text{ if } \boldsymbol{x}' - \boldsymbol{d}^x \geq \boldsymbol{0}. \tag{3.6}$$

Next, we consider the case when $\boldsymbol{x}' - \boldsymbol{d}^x \not\geq \boldsymbol{0}$. Let

$$\bar{d}_i^x \equiv \begin{cases} x_i' & \text{if } x_i' < d_i^x, \\ d_i^x & \text{otherwise.} \end{cases}$$

It follows from $\boldsymbol{x}' \in R_+^N$ that $\bar{\boldsymbol{d}}^x \geq \boldsymbol{0}$, and hence, $((\boldsymbol{x}' - \bar{\boldsymbol{d}}^x)Z_x^a, (\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^a) \in G(\boldsymbol{x}^a, \boldsymbol{y}^a)$. Since $\boldsymbol{x}' - \bar{\boldsymbol{d}}^x \geq \boldsymbol{0}$, it follows from (3.6) that $((\boldsymbol{x}' - \bar{\boldsymbol{d}}^x)Z_x^a, (\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^a) \in G(\boldsymbol{x}^b, \boldsymbol{y}^b)$. Since $\big((\boldsymbol{x}' - \boldsymbol{d}^x)Z_x^a, -(\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^a\big) \leq \big((\boldsymbol{x}' - \bar{\boldsymbol{d}}^x)Z_x^a, -(\boldsymbol{y}' + \boldsymbol{d}^y)Z_y^a\big)$, it follows from Proposition 3.2 that $(\boldsymbol{x}' - \boldsymbol{d}^x, \boldsymbol{y}' + \boldsymbol{d}^y)Z(\boldsymbol{x}^a, \boldsymbol{y}^a) \in G(\boldsymbol{x}^b, \boldsymbol{y}^b)$. □

Briec [5] shows that the Debreu-Farrell measure can be seen as the weighted distance based inefficiency measure

$$f^{w\infty}(\boldsymbol{x}, \boldsymbol{y}) = \min \{\| ((\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}'))\, Z(\boldsymbol{x}, \boldsymbol{y})\|_\infty \mid (\boldsymbol{x}', \boldsymbol{y}') \in E^w\}. \tag{3.7}$$

Let us denote $f^{w\infty}(\boldsymbol{x}, \boldsymbol{y})$ by $\text{DF}(\boldsymbol{x}, \boldsymbol{y})$. Replacing E^w of (3.7) with E, we consider

$$\min \{\| ((\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}'))\, Z(\boldsymbol{x}, \boldsymbol{y})\|_\infty \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \tag{3.8}$$

and denote it by $\text{DF}^E(\boldsymbol{x}, \boldsymbol{y})$. As proved in [5], DF satisfies weak monotonicity over E^w. Table 4 shows that DF^E does not satisfy weak monotonicity over E while $\hat{f}^{w\infty}$ does. Note that $(\boldsymbol{x}_1, \boldsymbol{y}_1) \in E$, $(\boldsymbol{x}_2, \boldsymbol{y}_2) \in E$, $(\boldsymbol{x}_3, \boldsymbol{y}_3) \in E^w \setminus E$, $(\boldsymbol{x}_4, \boldsymbol{y}_4) \in E^w \setminus E$ and $(\boldsymbol{x}_3, -\boldsymbol{y}_3) \leq (\boldsymbol{x}_4, -\boldsymbol{y}_4)$. The Debreu-Farrell measure DF provides all same inefficiency scores 0, however, both DF^E and $\hat{f}^{w\infty}$ provide positive ones for DMUs 3 and 4. Since $\text{DF}^E(\boldsymbol{x}_3, \boldsymbol{y}_3) > \text{DF}^E(\boldsymbol{x}_4, \boldsymbol{y}_4)$, DF^E is not weakly monotonic.

Consider a generalized version $\bar{f}^{wp}$ including (3.8) as follows:

$$\bar{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) \equiv \min \{\| ((\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}'))\, Z(\boldsymbol{x}, \boldsymbol{y})\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\}. \tag{3.9}$$

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Table 1 is also a counterexample to a weak monotonicity over E of a family of $\bar{f}^{wp}$. In fact, we have

$$\begin{aligned}\bar{f}^{wp}(\boldsymbol{x}_3, \boldsymbol{y}_3) &= \min\{\|((\boldsymbol{x}_3, \boldsymbol{y}_3) - (\boldsymbol{x}', \boldsymbol{y}'))Z(\boldsymbol{x}_3, \boldsymbol{y}_3)\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\geq \min\{\|((\boldsymbol{x}_3, \boldsymbol{y}_3) - (\boldsymbol{x}', \boldsymbol{y}'))Z(\boldsymbol{x}_3, \boldsymbol{y}_3)\|_\infty \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\geq \min\{\max\{|2 - (\lambda_1 + 5\lambda_2)|/2, |4 - (\lambda_1 + 5\lambda_2)|/4\} \mid \lambda_1 \geq 0, \lambda_2 \geq 0\} \\ &= 1/2 \end{aligned} \tag{3.10}$$

and it follows that

$$\begin{aligned}\bar{f}^{wp}(\boldsymbol{x}_4, \boldsymbol{y}_4) &= \min\{\|((\boldsymbol{x}_4, \boldsymbol{y}_4) - (\boldsymbol{x}', \boldsymbol{y}'))Z(\boldsymbol{x}_4, \boldsymbol{y}_4)\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E\} \\ &\leq \|((\boldsymbol{x}_4, \boldsymbol{y}_4) - (\boldsymbol{x}_2, \boldsymbol{y}_2))Z(\boldsymbol{x}_4, \boldsymbol{y}_4)\|_p \\ &= \|(0, 0, 0, -0.5/10, 0)\|_p = 1/20. \end{aligned} \tag{3.11}$$

Therefore, the following proposition holds from (3.10) and (3.11):

Proposition 3.4. There exists an input-output data where neither weak nor strong monotonicity over E is satisfied by a family of $\bar{f}^{wp}$.

The inefficiency measure $\hat{f}^{wp}$ is bounded as follows:

Proposition 3.5. Consider T_+ under CRTS and choose any $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$ arbitrarily. We have

$$0 \leq \hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) \leq \|(\boldsymbol{e}, \boldsymbol{e})\|_p. \tag{3.12}$$

Furthermore, $\hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) = 0$ if and only if $(\boldsymbol{x}, \boldsymbol{y}) \in E$.

Proof. It follows from the definition (3.2) of $\hat{f}^{wp}$ that $\hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ for all $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$ and $\hat{f}^{wp}(\boldsymbol{x}', \boldsymbol{y}') = 0$ if and only if $(\boldsymbol{x}', \boldsymbol{y}') \in E \cap T_+$.

Choose $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$ arbitrarily. Suppose that $(\boldsymbol{x}', \boldsymbol{y}') \in E$. Then, $\alpha(\boldsymbol{x}', \boldsymbol{y}') \in E$ holds for all $\alpha > 0$. Therefore, there exists $(\boldsymbol{x}', \boldsymbol{y}') \in E \cap R^{N+M}_{++}$ such that

$$(\boldsymbol{x}', \boldsymbol{y}') \leq (\boldsymbol{x}, \boldsymbol{y}). \tag{3.13}$$

It follows from (3.13) that

$$\begin{aligned}\hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y}) &= \min\{\|((\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) - (\boldsymbol{x} + \boldsymbol{d}^x, \boldsymbol{y} - \boldsymbol{d}^y))Z(\boldsymbol{x}, \boldsymbol{y})\|_p \mid (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in E, \boldsymbol{d}^x \geq \boldsymbol{0}, \boldsymbol{d}^y \geq \boldsymbol{0}\} \\ &\leq \min\{\|((\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) - (\boldsymbol{x}, \boldsymbol{y}))Z(\boldsymbol{x}, \boldsymbol{y})\|_p \mid (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in E\} \\ &= \min\{\|(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})Z(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{e}, \boldsymbol{e})\|_p \mid (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in E\} \\ &\leq \|(\boldsymbol{x}', \boldsymbol{y}')Z(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{e}, \boldsymbol{e})\|_p = \left(\sum_{i=1}^N \left|1 - \frac{x_i'}{x_i}\right|^p + \sum_{r=1}^M \left|1 - \frac{y_r'}{y_r}\right|^p\right)^{1/p} \\ &= \left(\sum_{i=1}^N \left(1 - \frac{x_i'}{x_i}\right)^p + \sum_{r=1}^M \left(1 - \frac{y_r'}{y_r}\right)^p\right)^{1/p} \\ &< \left(\sum_{i=1}^N (1)^p + \sum_{r=1}^M (1)^p\right)^{1/p} = \|(\boldsymbol{e}, \boldsymbol{e})\|_p. \end{aligned}$$

□

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Proposition 3.5 gives a least distance based efficiency measure $\left(1 - \hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y})/\|(\boldsymbol{e}, \boldsymbol{e})\|_p\right)$ satisfying the following properties:

Proposition 3.6. Consider T_+ under CRTS and let

$$g^p(\boldsymbol{x}, \boldsymbol{y}) = 1 - \frac{\hat{f}^{wp}(\boldsymbol{x}, \boldsymbol{y})}{\|(\boldsymbol{e}, \boldsymbol{e})\|_p}, \tag{3.14}$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$. A family of g^p satisfies unit invariance property and the following three ones:

Property A': $(\boldsymbol{x}, \boldsymbol{y}) \in E \cap T_+$ if and only if $g^p(\boldsymbol{x}, \boldsymbol{y}) = 1$.
Property B': For any $(\boldsymbol{x}, \boldsymbol{y}) \in T_+$, $0 \leq g^p(\boldsymbol{x}, \boldsymbol{y}) \leq 1$.
Property C': For $(\boldsymbol{x}^a, \boldsymbol{y}^a) \in T_+$ and $(\boldsymbol{x}^b, \boldsymbol{y}^b) \in T_+$, $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$ implies $g^p(\boldsymbol{x}^a, \boldsymbol{y}^a) \leq g^p(\boldsymbol{x}^b, \boldsymbol{y}^b)$.

Proof. The assertions hold from Propositions 3.3 and 3.5. □

4. Some Variants from Theoretical and Practical Applications

This section shows theoretical and practical applications of $\hat{f}^p$ and $\hat{f}^{wp}$ given in Section 3. The first part of the section develops a variant of $\hat{f}^p$ satisfying the same properties as RAM [8]. The last part applies $\hat{f}^{wp}$ into ranking the team effectiveness in an educational program.

4.1. RAM type variant of efficiency measure

Consider a production possibility set T_v under VRTS and let

$$\begin{aligned}
\boldsymbol{\gamma}^{\max} &\equiv \left(\max_{j=1,\ldots,J} x_{1j}, \ldots, \max_{j=1,\ldots,J} x_{Nj}, \max_{j=1,\ldots,J} y_{1j}, \ldots, \max_{j=1,\ldots,J} y_{Mj}\right), \\
\boldsymbol{\gamma}^{\min} &\equiv \left(\min_{j=1,\ldots,J} x_{1j}, \ldots, \min_{j=1,\ldots,J} x_{Nj}, \min_{j=1,\ldots,J} y_{1j}, \ldots, \min_{j=1,\ldots,J} y_{Mj}\right), \\
\Gamma &\equiv \operatorname{diag}\left(\boldsymbol{\gamma}^{\max} - \boldsymbol{\gamma}^{\min}\right)^{-1}, \\
\Omega(X, Y) &\equiv \left\{(\boldsymbol{x}, \boldsymbol{y}) \in T_v | \boldsymbol{\gamma}^{\min} \leq (\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{\gamma}^{\max}\right\}.
\end{aligned}$$

The efficiency measure RAM is defined by

$$\mathrm{RAM}(\boldsymbol{x}, \boldsymbol{y}) \equiv 1 - \frac{1}{N+M} \max\left\{\|((\boldsymbol{x}', \boldsymbol{y}') - (\boldsymbol{x}, \boldsymbol{y}))\Gamma\|_1 \mid (\boldsymbol{x}', \boldsymbol{y}') \in T_v, \boldsymbol{x}' \leq \boldsymbol{x},\ \boldsymbol{y}' \geq \boldsymbol{y}\right\}$$

for $(\boldsymbol{x}, \boldsymbol{y}) \in T_v$ and $f(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{RAM}(\boldsymbol{x}, \boldsymbol{y})$ satisfies the following three properties:

Property A": $(\boldsymbol{x}, \boldsymbol{y}) \in E$ if and only if $f(\boldsymbol{x}, \boldsymbol{y}) = 1$.
Property B": For any $(\boldsymbol{x}, \boldsymbol{y}) \in \Omega(X, Y)$, $0 \leq f(\boldsymbol{x}, \boldsymbol{y}) \leq 1$.
Property C": For $(\boldsymbol{x}^a, \boldsymbol{y}^a) \in \Omega(X, Y)$ and $(\boldsymbol{x}^b, \boldsymbol{y}^b) \in \Omega(X, Y)$, $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$ and $(\boldsymbol{x}^a, \boldsymbol{y}^a) \neq (\boldsymbol{x}^b, \boldsymbol{y}^b)$ imply $f(\boldsymbol{x}^a, \boldsymbol{y}^a) > f(\boldsymbol{x}^b, \boldsymbol{y}^b)$.

Properties B" and C" would not be satisfied by RAM if we replaced $\Omega(X, Y)$ with T_v (see details in p. 778 of [20]). The weight matrix Γ makes RAM unit invariant, furthermore, VRTS type of the production possibility set T_v makes it translation invariant. Since an optimal solution of $\max\left\{\|((\boldsymbol{x}', \boldsymbol{y}') - (\boldsymbol{x}, \boldsymbol{y}))\Gamma\|_1 \mid (\boldsymbol{x}', \boldsymbol{y}') \in T_v, \boldsymbol{x}' \leq \boldsymbol{x},\ \boldsymbol{y}' \geq \boldsymbol{y}\right\}$ belongs to E, $\mathrm{RAM}(\boldsymbol{x}, \boldsymbol{y})$ is reduced to

$$1 - \frac{1}{N+M} \min\left\{\|((\boldsymbol{x}', \boldsymbol{y}') - (\boldsymbol{x}, \boldsymbol{y}))\Gamma\|_1 \mid (\boldsymbol{x}', \boldsymbol{y}') \in E, \boldsymbol{x}' \leq \boldsymbol{x},\ \boldsymbol{y}' \geq \boldsymbol{y}\right\}. \tag{4.1}$$

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Baek and Lee [3] modify (4.1) into a least distance based measure

$$\mathrm{BL}(\boldsymbol{x},\boldsymbol{y}) \equiv 1 - \frac{1}{N+M} \min\left\{ \|((\boldsymbol{x}',\boldsymbol{y}') - (\boldsymbol{x},\boldsymbol{y}))\Gamma\|_2 \mid (\boldsymbol{x}',\boldsymbol{y}') \in E \right\}. \tag{4.2}$$

However, BL satisfies neither Property C" nor its weakened property:

Property D: For $(\boldsymbol{x}',\boldsymbol{y}') \in \Omega(X,Y)$ and $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) \in \Omega(X,Y)$, $(\boldsymbol{x}',-\boldsymbol{y}') \geq (\hat{\boldsymbol{x}},-\hat{\boldsymbol{y}})$ implies $f(\boldsymbol{x}',\boldsymbol{y}') \geq f(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})$.

Let $(X^\gamma, Y^\gamma) \equiv \begin{bmatrix} (\boldsymbol{x}_1,\boldsymbol{y}_1)\Gamma \\ \vdots \\ (\boldsymbol{x}_J,\boldsymbol{y}_J)\Gamma \end{bmatrix}$. The VRTS type production possibility set generated by (X^γ, Y^γ) is $\{(\boldsymbol{x},\boldsymbol{y})\Gamma \mid (\boldsymbol{x},\boldsymbol{y}) \in T_v\}$ and its efficient frontier is $E^\gamma = \{(\boldsymbol{x},\boldsymbol{y})\Gamma \mid (\boldsymbol{x},\boldsymbol{y}) \in E\}$. Let $(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) = (\boldsymbol{x},\boldsymbol{y})\Gamma$. Then, the efficiency measure $\mathrm{BL}(\boldsymbol{x},\boldsymbol{y})$ is

$$1 - \frac{1}{N+M} \min\left\{ \|(\boldsymbol{x}',\boldsymbol{y}') - (\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)\|_2 \mid (\boldsymbol{x}',\boldsymbol{y}') \in E^\gamma \right\} = 1 - \frac{1}{N+M}\bar{f}^2(\boldsymbol{x},\boldsymbol{y}). \tag{4.3}$$

The efficiency measure BL is essentially equal to $\bar{f}^p(\boldsymbol{x},\boldsymbol{y})$ with $p=2$ and $E = E^\gamma$. Therefore, Proposition 2.1 means that BL does not satisfy Property D. On the other hand, consider an efficiency measure

$$h^p(\boldsymbol{x},\boldsymbol{y}) \equiv 1 - \frac{1}{N+M} \min\left\{ \|(\boldsymbol{x}',\boldsymbol{y}') - (\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\|_p \mid (\boldsymbol{x}',\boldsymbol{y}') \in E^\gamma,\ (\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) \in D(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \right\}. \tag{4.4}$$

over $\Omega(X,Y)$. Then, we have $h^p(\boldsymbol{x},\boldsymbol{y}) = 1 - \frac{1}{N+M}\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$. It follows from Proposition 3.1 that a family of $\hat{f}^p$ satisfies Property D, and hence, a family of h^p also satisfies Property D. The efficiency measure h^p has the almost same properties as RAM.

Proposition 4.1. A family of efficiency measures h^p satisfies unit invariance, translation invariance and Properties A", B", D.

Proof. Let $\Omega^\gamma(X,Y) = \{(\boldsymbol{x},\boldsymbol{y})\Gamma \,|(\boldsymbol{x},\boldsymbol{y}) \in \Omega(X,Y)\}$. Then, it follows from $E \subseteq \Omega(X,Y)$ that $E^\gamma \subseteq \Omega^\gamma(X,Y)$. Choose $p = 1, 2, \ldots, \infty$, arbitrarily. Since $h^p(\boldsymbol{x},\boldsymbol{y}) = 1 - \frac{1}{N+M}\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$, we will show that $\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$ on $\Omega^\gamma(X,Y)$ is unit invariant and translation invariant and $\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$ satisfies the following three conditions:

Pa: $\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) = 0$ if and only if $(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \in E^\gamma$.

Pb: $0 \leq \hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \leq N+M$ for all $(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \in \Omega^\gamma(X,Y)$.

Pc: For $(\boldsymbol{x}^{\gamma a},\boldsymbol{y}^{\gamma a}) \in \Omega^\gamma(X,Y)$ and, $(\boldsymbol{x}^{\gamma b},\boldsymbol{y}^{\gamma b}) \in \Omega^\gamma(X,Y)$, $(\boldsymbol{x}^{\gamma a},-\boldsymbol{y}^{\gamma a}) \geq (\boldsymbol{x}^{\gamma b},-\boldsymbol{y}^{\gamma b})$ imply $\hat{f}^p(\boldsymbol{x}^{\gamma a},\boldsymbol{y}^{\gamma a}) \geq \hat{f}^p(\boldsymbol{x}^{\gamma b},\boldsymbol{y}^{\gamma b})$.

Properties A", B" and D correspond to Pa, Pb and Pc, respectively. From the definition of Γ and VRTS type production possibility set $\{(\boldsymbol{x},\boldsymbol{y})\Gamma \mid (\boldsymbol{x},\boldsymbol{y}) \in T_v\}$, $\hat{f}^p$ is unit invariant and translation invariant.

Choose any $(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \in \Omega^\gamma(X,Y)$, arbitrarily. It follows from the definition of (3.1) that $\hat{f}^p$ satisfies Pa and $0 \leq \hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$. Proposition 3.1 and $\Omega^\gamma(X,Y) \subseteq \{(\boldsymbol{x},\boldsymbol{y})\Gamma \mid (\boldsymbol{x},\boldsymbol{y}) \in T_v\}$ imply that $\hat{f}^p$ satisfies Pc. It follows from the definition (2.7) of $\bar{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$ that

$$\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \leq \bar{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma). \tag{4.5}$$

Let $(\bar{\boldsymbol{x}}^\gamma,\bar{\boldsymbol{y}}^\gamma)$ be an optimal solution of $\min\{\|(\boldsymbol{x},\boldsymbol{y}) - (\boldsymbol{x}',\boldsymbol{y}')\|_p \mid (\boldsymbol{x}',\boldsymbol{y}') \in E^\gamma\}$. Then, it follows from (2.7) that $\|(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) - (\bar{\boldsymbol{x}}^\gamma,\bar{\boldsymbol{y}}^\gamma)\|_p = \bar{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma)$ and that

$$0 \leq |\bar{x}_i^\gamma - x_i^\gamma| \leq 1 \quad i = 1,\ldots,N \quad \text{and} \quad 0 \leq |\bar{y}_i^\gamma - y_r^\gamma| \leq 1 \quad r = 1,\ldots,M.$$

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Therefore, it follows from (4.5) that

$$\begin{aligned}\hat{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) \le \bar{f}^p(\boldsymbol{x}^\gamma,\boldsymbol{y}^\gamma) &= \left(\sum_{i=1}^N |\bar{x}_i^\gamma - x_i^\gamma|^p + \sum_{r=1}^M |\bar{y}_r^\gamma - y_r^\gamma|^p\right)^{1/p} \le \|(\boldsymbol{e},\boldsymbol{e})\|_p \\ &\le \|(\boldsymbol{e},\boldsymbol{e})\|_1 = N+M.\end{aligned}$$

□

4.2. A side-constrained inefficiency measure for target identification and efficiency ranking of group activity in an education program

Department of Systems Engineering in Shizuoka University has a compulsory subject, called "Program Contest," where all the second year undergraduate students in the department are divided into groups consisting of four or five members and each group develops a solver for the traveling salesman problem by using programming language C and web-application technique. Each group competes in the computational performance race and presentation contest for his/her algorithmic strategy. Group study outcomes are ranked in the race and the presentation contest.

It is very hard to fairly divide all students into groups by programming achievement and skill level of web application. A group whose members have high levels of the achievements and the skills is easy to get top rankings in the race and the contest. On the other hand, a group consisting of members with low level of the achievements or the skills is hard to get noticeable results in either the race or the contest. The group consisting of members with low level of the achievements or the skills is difficult to stay motivated to compete with other groups. A way to let every student in the class stay motivated is to evaluate teamwork effectiveness of each group, that is, to rank groups in the viewpoint of two outcomes based on the achievements and the skills.

To do so, we regard each group in the class as a decision making unit that transforms the two inputs: the total programming achievement level and the total web-application skill into the two outputs: the score of the race and the score of the presentation. Then, we evaluate the teamwork effectiveness of each group by the least distance based inefficiency measure so that we can provide each group with an improvement target, which is also useful for the teaching staffs to give the group members some comments or suggestions.

The programming achievement level of a student is measured by the score of a prerequisite subject for the program contest "Fundamentals of Computer Programming" and the skill level of web application of a student is measured by the score of the programming homework assignments given in the first half of the program contest, each of which a student should complete individually. The programming achievement score of a group is the sum of all the prerequisite subject scores of its group members. The web application skill's score of a group is obtained by summing all the homework scores of its group members. The race outcome of a group is obtained by averaging T-scores of the tour lengths that are provided by the group. The presentation contest outcome of a group is estimated by the average of T-scores of points marked by all the members of the other groups. Here, the T-score used in the case study is defined as $\frac{\text{score}-\text{average}}{\text{standard deviation}} + 50$. The class of "Program Contest" in 2010 consists of 12 groups whose scores are listed in Table 5.

Let a pair of $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$ and $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ be an optimal solution of (3.2). Then, we have $(\boldsymbol{x}'^*, \boldsymbol{y}'^*) \in E$ and we define $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ as an improvement target of $(\boldsymbol{x}, \boldsymbol{y})$. Some type of inputs are difficult to be reduced from the current input activity $\boldsymbol{x}$. In this case study, the input reduction is undesirable from the educational point of view. Reduction of inputs of

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Table 5: Input-output data of 12 groups

No. group	input1 (x_1) prerequisite subject	input2 (x_2) homework	output1 (y_1) race	output2 (y_2) presentation
1	297	381.25	52.470	59.502
2	296	350.00	50.631	57.014
3	297	332.50	45.766	48.057
4	301	368.75	70.803	68.459
5	298	335.00	53.834	52.535
6	294	375.00	48.620	38.602
7	294	400.00	47.291	43.000
8	297	350.00	46.520	35.616
9	303	350.00	47.218	41.512
10	297	375.00	47.218	45.071
11	304	381.25	44.524	46.564
12	302	362.50	70.873	64.478

Table 5 requires the group to decrease the programming achievement level or the skill level of web application. Requiring members of the group to reduce their individual achievement and skill levels is inconsistent with educational goals of the prerequisite subject, "Fundamentals of Computer Programming", and the first half in "Program Contest". Therefore, reduction of inputs means purging some members from the group, which is against a goal of group study in "Program Contest". In such case, improvement target of an activity $(\boldsymbol{x}, \boldsymbol{y})$ needs to prohibit any decrease of input-levels from the current ones. Consider that any input level of improvement target $\boldsymbol{x}'^*$ must not be less than that of the current level $\boldsymbol{x}$. By adding a side constraint $\boldsymbol{x}' \geq \boldsymbol{x}$ into (3.2), we formulate

$$\hat{f}_x^{wp}(\boldsymbol{x}, \boldsymbol{y}) \equiv \min \{ \|((\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) - (\boldsymbol{x}', \boldsymbol{y}'))Z(\boldsymbol{x}, \boldsymbol{y})\|_p \mid (\boldsymbol{x}', \boldsymbol{y}') \in E, (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in D(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}' \geq \boldsymbol{x} \} \tag{4.6}$$

which is called *a side-constrained weighted distance measure*. The inefficiency measure $\hat{f}_x^{wp}$ satisfies both Axiom A and B on T_+. Furthermore, Axiom C is satisfied by $\hat{f}_x^{wp}$ as follows.

Proposition 4.2. Consider the production possibility set T_+. A family of side-constrained weighted distance measures $\hat{f}_x^{wp}$ satisfies weak monotonicity over E.

Proof. Let $G_x(\boldsymbol{x}, \boldsymbol{y}) \equiv \{(\boldsymbol{x}' - \boldsymbol{d}^x, \boldsymbol{y}' + \boldsymbol{d}^y)Z(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{d}^x \geq \boldsymbol{0}, \boldsymbol{d}^y \geq \boldsymbol{0}, (\boldsymbol{x}', \boldsymbol{y}') \in E, \boldsymbol{x}' \geq \boldsymbol{x}\}$. Then, we have

$$\hat{f}_x^{wp}(\boldsymbol{x}, \boldsymbol{y}) = \min \{ \|(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) - (\boldsymbol{e}, \boldsymbol{e})\|_p \mid (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in G_x(\boldsymbol{x}, \boldsymbol{y}) \}.$$

Consider $(\boldsymbol{x}^a, \boldsymbol{y}^a) \in T_+$ and $(\boldsymbol{x}^b, \boldsymbol{y}^b) \in T_+$ such that $(\boldsymbol{x}^a, -\boldsymbol{y}^a) \geq (\boldsymbol{x}^b, -\boldsymbol{y}^b)$. We have $\{\boldsymbol{x}' \mid \boldsymbol{x}' \geq \boldsymbol{x}^a\} \subseteq \{\boldsymbol{x}' \mid \boldsymbol{x}' \geq \boldsymbol{x}^b\}$ and we can show $G_x(\boldsymbol{x}^a, \boldsymbol{y}^a) \subseteq G_x(\boldsymbol{x}^b, \boldsymbol{y}^b)$ by the same manner as the proof of Proposition 3.3. Therefore, we have $\hat{f}_x^{wp}(\boldsymbol{x}^a, \boldsymbol{y}^a) \geq \hat{f}_x^{wp}(\boldsymbol{x}^b, \boldsymbol{y}^b)$. □

Proposition 4.3. For all $(\boldsymbol{x}, \boldsymbol{y}) \in T_+ \setminus E$, any improvement target $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ of $\hat{f}_x^{wp}(\boldsymbol{x}, \boldsymbol{y})$ satisfies $y_r'^* > y_r$ for some $r \in \{1, \ldots, M\}$.

Proof. It follows from $\boldsymbol{x}' \geq \boldsymbol{x}$ and $(\boldsymbol{x}, \boldsymbol{y}) \notin E$ that any $(\boldsymbol{x}', \boldsymbol{y}') \in E$ satisfies $\boldsymbol{y}' \not\leq \boldsymbol{y}$. Hence, it follows from $(\boldsymbol{x}'^*, \boldsymbol{y}'^*) \in E$ that $y_r'^* > y_r$ for some $r \in \{1, \ldots, M\}$. □

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Table 6: Inefficiency measure by $\hat{f}^{w\infty}$

	inefficiency score	x_1	x_2	y_1	y_2
	$\hat{f}^{w\infty}$	upper:$(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$, lower:$(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$			
1	0.165	259.862	318.352	61.127	59.102
		297.000	381.250	52.470	59.502
2	0.157	249.580	299.579	58.572	53.287
		296.000	350.000	50.631	57.014
3	0.207	235.438	282.604	55.253	50.267
		297.000	332.500	45.766	48.057
4	0.000	301.000	368.750	70.803	68.459
		301.000	368.750	70.803	68.459
5	0.130	259.234	311.167	60.838	55.348
		298.000	335.000	53.833	52.535
6	0.267	229.049	274.935	53.754	48.903
		294.000	375.000	48.620	38.602
7	0.247	251.065	301.361	58.920	53.604
		294.000	400.000	47.291	43.000
8	0.281	213.641	256.440	50.138	45.614
		297.000	350.000	46.520	35.616
9	0.218	236.869	284.321	55.589	50.573
		303.000	350.000	47.218	41.512
10	0.210	242.962	296.345	57.123	54.525
		297.000	375.000	47.218	45.071
11	0.244	235.392	288.375	55.371	53.538
		304.000	381.250	44.524	46.564
12	0.000	302.000	362.500	70.873	64.478
		302.000	362.500	70.873	64.478

By applying the inefficiency measure $\hat{f}^{wp}_x$ having constraints of nondecreasing inputs into ranking groups whose input-output data are in Table 5, we compare the inefficiency scores and improvement targets with that by applying $\hat{f}^{wp}$. Calculation of both two inefficiency measures $\hat{f}^{wp}$ and $\hat{f}^{wp}_x$ needs to identify the efficient frontier E. Since both two groups 4 and 12 are efficient and the activity $(\boldsymbol{x}_4, \boldsymbol{y}_4) + (\boldsymbol{x}_{12}, \boldsymbol{y}_{12})$ is also efficient, we have

$$E = \{(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) = \lambda_4(\boldsymbol{x}_4, \boldsymbol{y}_4) + \lambda_{12}(\boldsymbol{x}_{12}, \boldsymbol{y}_{12}), \lambda_4 \geq 0, \lambda_{12} \geq 0\}\,. \tag{4.7}$$

Inefficiency measures $\hat{f}^{wp}$ and $\hat{f}^{wp}_x$ are given in Table 6 and Table 7, respectively. It follows from two tables 6 and 7 that two inefficiency group rankings based on $\hat{f}^{wp}$ and $\hat{f}^{wp}_x$ coincide. In the tables, the third column to the last one show an optimal solution $(\boldsymbol{x}'^*, \boldsymbol{y}'^*, \hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$ for each group that separates $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ and $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$ in the upper row and in the lower row, respectively. All the lower rows $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$ of two tables coincide with the input-output data of Table 5. Therefore, by comparison between the lower and the upper rows we check the difference between current input-output activity and its improvement target. The inefficiency measure $\hat{f}^{wp}$ shows in Table 6 that two groups 1 and 2 have an output whose improvement target level is less than the current level. For example, group 1 has improvement target level (59.103) of output 2 is less than current one (59.503). Furthermore, for all groups other than efficient groups 4 and 12, the improvement target levels of the

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Table 7: Inefficiency measure by $\hat{f}_x^{w\infty}$

	inefficiency score	x_1	x_2	y_1	y_2
	$\hat{f}_x^{w\infty}$	upper:$(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$, lower: $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$			
1	0.395	311.203	381.250	73.204	70.780
		297.000	381.250	52.470	59.502
2	0.372	296.000	355.298	69.466	63.198
		296.000	350.000	50.631	57.014
3	0.523	297.000	356.498	69.701	63.411
		297.000	332.500	45.767	48.056
4	0.000	301.000	368.750	70.803	68.459
		301.000	368.750	70.803	68.459
5	0.299	298.000	357.699	69.935	63.625
		298.000	335.000	53.834	52.535
6	0.728	312.414	375.000	73.318	66.702
		294.000	375.000	48.620	38.602
7	0.655	333.241	400.000	78.206	71.149
		294.000	400.000	47.291	43.000
8	0.780	297.000	356.498	69.701	63.411
		297.000	350.000	46.520	35.616
9	0.558	303.000	363.700	71.109	64.692
		303.000	350.000	47.218	41.512
10	0.531	307.449	375.000	72.284	68.997
		297.000	375.000	47.218	45.071
11	0.644	311.203	381.250	73.204	70.780
		304.000	381.250	44.524	46.564
12	0.000	302.000	362.500	70.873	64.478
		302.000	362.500	70.873	64.478

two inputs are less than the current levels. The input reduction of improvement target is inappropriate as an educational application to "Program Contest".

The inefficiency measure $\hat{f}_x^{wp}(\boldsymbol{x}, \boldsymbol{y})$ having constraints of nondecreasing inputs $\boldsymbol{x}' \geq \boldsymbol{x}$ provides all improvement target levels $\boldsymbol{x}'^*$ of two inputs that are not less than the current ones $\boldsymbol{x}$. For all inefficient groups, the improvement target $\boldsymbol{x}'^*$ has exactly one input whose level is greater than the current one of $\boldsymbol{x}$ and the improvement target $\boldsymbol{y}'^*$ has two outputs whose levels are greater than the current levels $\boldsymbol{y}$. The improvement target output levels $\boldsymbol{y}'^*$ of an inefficient group are feasible output levels if the inefficient group sufficiently exploits improvement target input levels $\boldsymbol{x}'^* \geq \boldsymbol{x}$. The following two points can be included as the final comment to all inefficient groups:

- To what extent programming achievements or web application skills must be improved,
- To what extent the two outcomes can be simultaneously increased.

5. Conclusions and Future Research

This study has shown a counterexample such that a family of $\bar{f}^p$ gives a negative answer to the open problem raised by Pastor and Aparicio [15]. Adding constraints of an improvement target that dominates the assessed activity into f^p was not effective to solve the open

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problem positively. A weighted distance model with the incorporation of a weight matrix $Z(\boldsymbol{x}, \boldsymbol{y})$ into $\bar{f}^p(\boldsymbol{x}, \boldsymbol{y})$ also gives the negative answer to the open problem, regardless of the choice of $p \in [1, \infty]$. Furthermore, the counterexample indicates least distance models are inconsistent with strong monotonicity over E and weak monotonicity over $\bar{E}$. That is, we showed non-existence of a family of least distance based inefficiency measures $\bar{f}^p$ satisfying weak monotonicity over $\bar{E}$, which provides a negative answer to an extension of the open problem by Pastor and Aparicio [15].

This study proposes a variation of $\bar{f}^p(\boldsymbol{x}, \boldsymbol{y})$ by replacing the distance from the efficient frontier E to $(\boldsymbol{x}, \boldsymbol{y})$ with that from E to the region $D(\boldsymbol{x}, \boldsymbol{y})$ dominated by $(\boldsymbol{x}, \boldsymbol{y})$. Theoretical and practical modification of $\hat{f}^p$ satisfies weak monotonicity over $\bar{E}$. Moreover, $\hat{f}^p$ is to add linear inequality constraints into $\bar{f}^p$ and it is simply and slightly modified from $\bar{f}^p$. The simple and slight modification makes $\hat{f}^p$ satisfy all axioms of RAM except strong monotonicity. Furthermore, $\hat{f}^p$ is applied to a group working application. The application requires non-deceasing input constraints for $\hat{f}^p$, that is consistent with weak monotonicity over $\bar{E}$. The measurement with $p = \infty$ results in Table 7, whose improvement target $(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$ is unique. Table 7 shows $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*) = (\boldsymbol{x}, \boldsymbol{y})$ for all groups, however, groups, 2, 3, 5, 6, 7, 8 have an alternative result $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*) \neq (\boldsymbol{x}, \boldsymbol{y})$, where $\hat{x}_2^* \neq x_2$ or $\hat{y}_1^* \neq y_1$. The uniqueness of an optimal solution $(\boldsymbol{x}'^*, \boldsymbol{y}'^*, \hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*)$ of the right-hand side of (4.6) can be easily verified by a numerical computation. For example, see [19] and [22] for a linear programming for checking multiple optimal solutions in DEA. An efficient algorithm for enumerating all optimal solutions to (4.6) is left for one further research.

It is important to examine how often real life input-output data violate weak monotonicity in terms of $\bar{f}^p$ as in the counterexample by [15] and Table 4. In cases of $p = 1, 2, \infty$, $\bar{f}^p$ is reduced to linear or convex quadratic programming problems, which can be exactly solved by optimization softwares. It is possible to verify the different rankings by the choice of p on real life input-output data of group working applications. Numerical experiments for frequency of not only weak monotonicity violation but also the different rankings and improvement target by the choice of p are left for another future research.

This study focused on the distance $\|(\boldsymbol{x}', \boldsymbol{y}') - (\boldsymbol{x}, \boldsymbol{y})\|_p$ from $(\boldsymbol{x}, \boldsymbol{y})$ to $(\boldsymbol{x}', \boldsymbol{y}') \in E$, however, it did not investigate SBM [21] type distance, $\|\boldsymbol{x}' - \boldsymbol{x}\|_p / \|\boldsymbol{y}' - \boldsymbol{y}\|_p$, whose least distance based inefficiency measure is the other direction for future research.

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