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AN OPTIMAL CONTROL MODEL FOR UNCERTAIN SYSTEMS WITH TIME-DELAY

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Abstract An uncertain control model with time-delay is investigated based on the concept of uncertain process. The value function of the model is infinite-dimensional in the state. Some conditions are presented to guarantee the model is equivalent to a finite-dimensional one. The latter is solved by the equation of optimality. Then the solution of an uncertain linear quadratic optimal control problem with time-delay is obtained. An example is given to show how to solve an uncertain linear quadratic optimal control model with time-delay. Finally, as an application of the result, an optimal consumption problem with delay in financial market is dealt with.

Keywords: Control, dynamic programming, uncertain process, time-delay, linear quadratic model, consumption

1. Introduction

Since stochastic optimal control theory initiated in 1970's, it has been an important branch of modern control theory. The study of stochastic optimal control greatly attracted the attention of many mathematicians. Some researches on optimal control of Brownian motion or stochastic differential equations and applications in finance can be found in some books: Fleming and Rishel [4], Harrison [5] and so on.

Moreover, the phenomenon of time-delay in stochastic optimal control is ubiquitous. It widely exists in physical, chemical, manufacturing, environmental, biological, and global economic and business systems. A set of infinite-dimensional differential equations could model the dynamic systems which do not only depend on the current state but also on the states during the last d time units, where d denotes a fixed delay. Stochastic optimal control problems with time-delay were discussed in Larssen and Risebro [7], Yong and Zhou [13], Bauer and Rieder [1] and so on. Dynamic programming principle can be used to the stochastic control problems with delay. The method of dynamic programming in optimization over Ito's process was also found in Dixit and Pindyck [3].

In the real world, however, everyday we need to face many indeterminacy but randomness, such as "high speed", "about 90km" and "roughly 65kg". This fact motivates researchers to invent a new mathematical tool. Liu [8] found uncertainty theory through introducing an uncertain measure based on normality, self-duality, countable subadditivity, and product measure axioms. Liu [10] presented uncertain differential equations based on the concepts of uncertain variable, uncertain process and canonical process. Nowadays uncertainty theory which was refined by Liu [11] has been a branch of mathematics for modeling human uncertainty.

Based on the uncertainty theory, Zhu [14] dealt with an uncertain optimal control model by using dynamic programming and presented an equation of optimality for the model.

This equation of optimality is very essential for uncertain optimal control problems. It has been applied in uncertain bang-bang control problems by Xu and Zhu [12]. A multi-stage uncertain bang-bang control problem was studied by Kang and Zhu [6].

In this paper, we will consider an uncertain optimal control problem with time-delay. Note that the value function of the problem is infinite-dimensional in the state. Stimulated by Bauer and Rieder's work [1] on stochastic control problems with delay, we will introduce a method under some conditions for transforming the uncertain optimal control problem with time-delay to a finite-dimensional one which may be solved by the equation of optimality.

The organization of the paper is as follows. In section 2, some basic concepts are reviewed. In section 3, an uncertain optimal control model with time-delay is formulated, and then a result is derived in the model. In section 4, a linear quadratic problem with time-delay and a numerical example are solved by the result obtained in the previous section. In section 5, a consumption problem is studied as an application of the results.

2. Preliminary

Some knowledge about uncertain measure and uncertain variable can be found in Liu [8]. In convenience, we give some useful concepts.

Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. The set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ is called an uncertain measure if it satisfies the three axioms : $\mathcal{M}\{\Gamma\} = 1$; $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$; $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\{\Lambda_i\} \subset \mathcal{L}$. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set R of real numbers, and an uncertain vector is a measurable function from an uncertainty space to R^n. The uncertainty distribution $\Phi : R \to [0, 1]$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x. An uncertain process is a measurable function from $V \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers where V is an index set.

Definition 2.1. *(Liu [10]) The uncertain variables $\xi_1, \xi_2, \cdots \xi_m$ are said to be independent if $\mathcal{M}\{\cap_{i=1}^{m}(\xi_i \in B_i)\} = \min_{1\leq i\leq m} \mathcal{M}\{\xi_i \in B_i\}$ for any Borel sets $B_1, B_2, \cdots B_m$ of real numbers.*

Definition 2.2. *(Liu [8])The expected value of an uncertain variable ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

For any numbers a and b, $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ if ξ and η are independent uncertain variables.

Definition 2.3. *(Liu [10]) An uncertain process C_t is said to be a canonical process if (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, denoted by $C_{s+t} - C_s \sim \mathcal{N}(0, t)$, whose distribution is*

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in R.$$

Remark 2.1. *Canonical process is different from Wiener process that almost all sample paths of Wiener process have an infinite variation and are differentiable nowhere, and almost all sample paths of canonical process have finite variation and are differentiable almost everywhere. Furthermore, the squared variation of Wiener process on $[0, t]$ is equal to t both in mean square and almost surely, while that of canonical process has the same order as t^2 for small t.*

For the sake of convenience, we review the following note. For a multi-variable and vector-value function $f : R^n \to R^m$, its Jacobi matrix is defined by

$$D_x f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in R^{m\times n}$$

for $f(x) = (f_1, f_2, \cdots, f_m)^\tau$ and $x = (x_1, x_2, \cdots, x_n)^\tau$.

Definition 2.4. *(Liu [10]) Let X_t be an uncertain process and C_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|$. Then the uncertain integral of X_t with respect to C_t is*

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

Definition 2.5. *(Liu [10]) Let C_t be a canonical process and let Z_t be an uncertain process. If there exist two uncertain processes μ_t and σ_t such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$$

for any $t \ge 0$, then we say Z_t has an uncertain differential $\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t$.

Fundamental theorems of uncertain calculus and chain rule were presented by Liu [10, 11]. If X_t is an uncertain vector, and C_t is a multi-dimensional uncertain canonical process, fundamental theorems of uncertain calculus and chain rule may be rewritten like the following theorems.

Theorem 2.1. *Let X_t be an n-dimensional uncertain process, and $h(t, x) : R \times R^n \to R^n$ be a continuously differentiable vector-value function. Then the uncertain process $Y_t = h(t, X_t)$ is differentiable and has an uncertain differential*

$$\mathrm{d}Y_t = h_t(t, X_t)\mathrm{d}t + D_x h(t, X_t)\mathrm{d}X_t, \tag{2.1}$$

where $h_t(t, x)$ is the partial derivative of the function $h(t, x)$ in t, and $D_x h(t, x)$ is the Jacobi matrix of $h(t, x)$ in x.

Theorem 2.2. *Let $f : R^n \to R^n$ and $g : R^n \to R^n$ be continuously differentiable vector-value functions. Then the uncertain process $f(g(X_t))$ has an uncertain differential*

$$\mathrm{d}f(g(X_t)) = D_g f(g(X_t)) D_x g(X_t)\mathrm{d}X_t. \tag{2.2}$$

Uncertain differential equation was introduced by Liu [9]. Chen and Liu [2] proved an existence and uniqueness theorem of solution of uncertain differential equation.

Definition 2.6. *(Liu [9])Suppose C_t is a canonical process, and f_1 and f_2 are some given functions. Then*

$$\mathrm{d}X_t = f_1(t, X_t)\mathrm{d}t + f_2(t, X_t)\mathrm{d}C_t \tag{2.3}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (2.3).

If f_1 is a vector-value function, f_2 is a matrix-value function, X_t is an uncertain vector, and C_t is a multi-dimensional uncertain canonical process, then (2.3) is a system of uncertain differential equations.

Theorem 2.3. *Suppose C_t is an l-dimensional canonical process, $f : [0, +\infty) \times R^n \to R^n$ is a vector-value function , $g : [0, +\infty) \times R^n \to R^{n\times l}$ is a matrix-value functions. Let X_t be an uncertain process satisfying the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \tag{2.4}$$

If $y = Y(t, x) : [0, +\infty) \times R^n \to R^n$ is continuously differentiable, then

$$\mathrm{d}Y(t, X_t) = [Y_t(t, X_t) + D_xY(t, X_t)f(t, X_t)]\mathrm{d}t + D_xY(t, X_t)g(t, X_t)\mathrm{d}C_t. \tag{2.5}$$

Proof. By using Theorem 2.1 and Theorem 2.2, we have

$$\begin{aligned}\mathrm{d}Y(t, X_t) &= Y_t(t, X_t)\mathrm{d}t + D_xY(t, X_t)\mathrm{d}X_t \\ &= Y_t(t, X_t)\mathrm{d}t + D_xY(t, X_t)(f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t) \\ &= [Y_t(t, X_t) + D_xY(t, X_t)f(t, X_t)]\mathrm{d}t + D_xY(t, X_t)g(t, X_t)\mathrm{d}C_t.\end{aligned}$$

□

Remark 2.2. *Note that probability theory is a branch of mathematics for studying the behavior of random phenomena, but uncertainty theory is a branch of mathematics for modeling human uncertainty. The main difference is that the product probability measure is the product of probability measures of individual events, i.e., $\Pr\{A \times B\} = \Pr\{A\} \times \Pr\{B\}$, and the product uncertain measure is the minimum of uncertain measures of individual events, i.e., $\mathcal{M}\{A \times B\} = \mathcal{M}\{A\} \wedge \mathcal{M}\{B\}$.*

In practice, when the sample points of an indeterminant event are many enough, we may obtain a probability distribution to describe the event and employ the probability theory to study it. Otherwise, when we are lack of observed data for an indeterminant event, judgements of some specialists may be used to describe the event and uncertainty theory may be employed to deal with it provided that these judgement data could be quantified by an uncertain measure.

3. Uncertain Optimal Control With Time-delay

Let $C = \{C_t, t \geq 0\}$ denote an l-dimensional uncertain canonical process. Assume that an uncertain process $X = \{X_t, t \geq -d\}$ taking values in a closed set $\mathbf{A} \subset R^n$, which describes the state of a system at time t that started at time $-d < 0$. Here, d describes a constant delay inherent to the system. Let $C_{\mathbf{A}}[-d, 0]$ denote the space of all continuous functions on $[-d, 0]$ taking values in $\mathbf{A}$. For $t \in [-d, 0]$, the process X_t is consistent with a function $\varphi_0 \in C_{\mathbf{A}}[-d, 0]$. For $t \geq 0$, X_{t+s} $(s \in [-d, 0])$ describes the associated segment process of X_t, denoted by

$$\varphi_t(s) = X_{t+s}, \quad s \in [-d, 0].$$

In this paper, we consider a system whose dynamics may not only depend on the current state but also depend on the segment process through the processes

$$Y_t = \int_{-d}^{0} e^{\lambda s} f(X_{t+s})\mathrm{d}s, \quad \zeta_t = f(X_{t-d}), \quad t \geq 0$$

where $f : R^n \to R^k$ is a differentiable function and $\lambda \in R$ is a constant. The system can be controlled by $u = \{u_t, t \geq 0\}$ taking values in a closed subset U of R^m.

At every time $t \geq 0$, an immediate reward $F(t, X_t, Y_t, u_t)$ is accrued and the terminal state of the system earns a reward $h(X_T, Y_T)$. Then we are looking for a control process u that maximizes the overall expected reward over the horizon $[0, T]$. That is, we consider the following uncertain optimal control problem with time-delay:

$$\begin{cases} J(0, \varphi_0) = \sup_{u \in U} E\left[\int_0^T F(s, X_s, Y_s, u_s)\mathrm{d}s + h(X_T, Y_T)\right] \\ \text{subject to} \\ \quad \mathrm{d}X_s = \mu_1(s, X_s, Y_s, u_s)\mathrm{d}s + \mu_2(X_s, Y_s)\zeta_s \mathrm{d}s \\ \qquad\quad + \sigma(s, X_s, Y_s, u_s)\mathrm{d}C_s, \quad s \in [0, T] \\ \quad X_s = \varphi_0(s), \quad -d \leq s \leq 0. \end{cases} \tag{3.1}$$

In the above model, X_s is the state vector of n dimension, u_s takes values in a closed subset U of R^m, $F : [0, +\infty) \times R^n \times R^k \times U \to R$ the objective function, and $h : R^n \times R^k \to R$ the function of terminal reward. In addition, $\mu_1 : [0, +\infty) \times R^n \times R^k \times U \to R^n$ is a column-vector function, $\mu_2 : R^n \times R^k \to R^{n\times k}$ a matrix function, $\sigma : [0, +\infty) \times R^n \times R^k \times U \to R^{n\times l}$ a matrix function, and $C_s = (C_{s_1}, C_{s_2}, \cdots C_{s_l})^\tau$, where $C_{s_1}, C_{s_2}, \cdots C_{s_l}$ are independent canonical process. The function $J(0, \varphi_0)$ is the expected optimal reward obtainable in $[0, T]$ with the initial condition that at time 0 we have the state $\varphi_0(s)$ between $-d$ and 0, where $\varphi_0 \in C_{\mathbf{A}}[-d, 0]$ is a given function. The final time $T > 0$ is fixed or free. A feasible control process means that it takes values in the set U.

For any $0 < t < T$, $J(t, \varphi_t)$ is the expected optimal reward obtainable in $[t, T]$ with the condition that at time t we have the state $\varphi_t(s)$ between $t - d$ and t. That is, consider the following problem (P):

$$(P)\begin{cases} J(t, \varphi_t) = \sup_{u \in U} E\left[\int_t^T F(s, X_s, Y_s, u_s)\mathrm{d}s + h(X_T, Y_T)\right] \\ \text{subject to} \\ \quad \mathrm{d}X_s = \mu_1(s, X_s, Y_s, u_s)\mathrm{d}s + \mu_2(X_s, Y_s)\zeta_s \mathrm{d}s \\ \qquad\quad + \sigma(s, X_s, Y_s, u_s)\mathrm{d}C_s, \quad s \in [t, T] \\ \quad X_s = \varphi_t(s), \quad s \in [-d, 0]. \end{cases} \tag{3.2}$$

Note that the value function J is defined on the infinite-dimensional space $[0, T] \times C_{\mathbf{A}}[-d, 0]$ so that the equation of optimality in Zhu [14] is not directly applicable. We will formulate an uncertain control problem $(\overline{P})$ with finite-dimensional state space such that an optimal control process for (P) can be constructed from an optimal solution of the problem $(\overline{P})$. In order to transform the uncertain control problem (P) we introduce the following

Assumption 1. *There exists an operator* $Z : R^n \times R^k \to R^n$ *such that*

$$e^{\lambda d} D_x Z(x, y)\mu_2(x, y) - D_y Z(x, y) = 0, \quad \forall (x, y) \in R^n \times R^k, \tag{3.3}$$

where $D_x Z(x, y)$ *and* $D_y Z(x, y)$ *denote the Jacobi matrices of* Z *in* x *and in* y*, respectively.*

This transformation yields a new state process $Z_t = Z(X_t, Y_t)$. Let $S = \mathbf{A} \times y(C_{\mathbf{A}}[-d, 0])$. For $\psi \in C_{\mathbf{A}}[-d, 0]$, we denote $x(\psi) = \psi(0)$, $y(\psi) = \int_{-d}^0 e^{\lambda s} f(\psi(s))\mathrm{d}s$, $\zeta(\psi) = f(\psi(-d))$. Then Z_t take values in $Z(S)$. In order to derive the dynamics of the transformed process Z we need the following lemma.

Lemma 3.1. *Let $G(t,x,y):[0,+\infty)\times R^n\times R^k\to R^n$ be continuously differentiable function and consider a feasible control process $u_t\in U$. Then the uncertain process $G(t,X_t,Y_t)$ satisfies*

$$\begin{aligned}\mathrm{d}G(t,X_t,Y_t)=&\{G_t(t,X_t,Y_t)+D_xG(t,X_t,Y_t)(\mu_1(t,X_t,Y_t,u_t)+\mu_2(X_t,Y_t)\zeta_t)\}\mathrm{d}t\\&+D_xG(t,X_t,Y_t)\sigma(t,X_t,Y_t,u_t)\mathrm{d}C_t+D_yG(t,X_t,Y_t)(f(X_t)\\&-e^{-\lambda d}\zeta_t-\lambda Y_t)\mathrm{d}t.\end{aligned}\tag{3.4}$$

Proof. For a given feasible control process u_t with state process X_t, define a process $\tilde{F}_t$ by

$$\tilde{F}_t=\int_0^t f(X_s)ds.$$

Then the process Y_t has the representation

$$\begin{aligned}Y_t&=\int_{-d}^0 e^{\lambda s}f(X_{t+s})\mathrm{d}s=\int_{-d}^0 e^{\lambda s}\mathrm{d}\tilde{F}_{t+s}=e^{\lambda s}\tilde{F}_{t+s}\mid_{-d}^0-\int_{-d}^0\tilde{F}_{t+s}\mathrm{d}e^{\lambda s}\\&=\tilde{F}_t-e^{-\lambda d}\tilde{F}_{t-d}-\int_{-d}^0\lambda e^{\lambda s}\tilde{F}_{t+s}\mathrm{d}s\\&=\int_0^t\left(f(X_s)-e^{-\lambda d}f(X_{s-d})-\lambda\int_{-d}^0 e^{\lambda r}f(X_{s+r})\mathrm{d}r\right)\mathrm{d}s.\end{aligned}$$

Thus

$$\mathrm{d}Y_t=\big(f(X_t)-e^{-\lambda d}f(X_{t-d})-\lambda Y_t\big)\mathrm{d}t=\big(f(X_t)-e^{-\lambda d}\zeta_t-\lambda Y_t\big)\mathrm{d}t.$$

Applying Theorem 2.3 to $G(t,X_t,Y_t)$, the equation (3.4) follows. □

Now we are able to present the dynamics for $Z_t=Z(X_t,Y_t)$ by using (3.3) and (3.4):

$$\begin{aligned}\mathrm{d}Z_t&=\mathrm{d}Z(X_t,Y_t)\\&=D_xZ(X_t,Y_t)(\mu_1(t,X_t,Y_t,u_t)+\mu_2(X_t,Y_t)\zeta_t)\mathrm{d}t\\&\quad+D_xZ(X_t,Y_t)\sigma(t,X_t,Y_t,u_t)\mathrm{d}C_t+D_yZ(X_t,Y_t)(f(X_t)-e^{-\lambda d}\zeta_t-\lambda Y_t)\mathrm{d}t\\&=D_xZ(X_t,Y_t)\mu_1(t,X_t,Y_t,u_t)\mathrm{d}t+D_yZ(X_t,Y_t)(f(X_t)-\lambda Y_t)\mathrm{d}t\\&\quad+D_xZ(X_t,Y_t)\sigma(t,X_t,Y_t,u_t)\mathrm{d}C_t.\end{aligned}$$

Define $\widetilde{\mu}:[0,+\infty)\times R^n\times R^k\times U\to R^n$ by

$$\widetilde{\mu}(t,x,y,u)=D_xZ(x,y)\mu_1(t,x,y,u)+D_yZ(x,y)(f(x)-\lambda y),$$

and $\widetilde{\sigma}:[0,+\infty\times R^n\times R^k\times U\to R^{n\times l}$ by

$$\widetilde{\sigma}(t,x,y,u)=D_xZ(x,y)\sigma(t,x,y,u).$$

If the functions $\widetilde{\mu}$ and $\widetilde{\sigma}$ as well as h would depend on (x,y) through $Z(x,y)$ only, then the problem (P) could be reduced to a finite-dimensional problem.

Assumption 2. *There are functions*

$$\begin{aligned}&\overline{\mu}:[0,+\infty)\times R^n\times U\to R^n, &&\overline{\sigma}:[0,+\infty)\times R^n\times U\to R^{n\times l},\\&\overline{F}:[0,+\infty)\times R^n\times U\to R, &&\overline{h}:R^n\to R\end{aligned}$$

such that for all $t\in[0,T],u\in U,(x,y)\in R^n\times R^k$, we have

$$\begin{aligned}&\overline{\mu}(t,Z(x,y),u)=\widetilde{\mu}(t,x,y,u), &&\overline{\sigma}(t,Z(x,y),u)=\widetilde{\sigma}(t,x,y,u),\\&\overline{F}(t,Z(x,y),u)=F(t,x,y,u), &&\overline{h}(Z(x,y))=h(x,y).\end{aligned}$$

Now we can introduce a finite-dimensional control problem $(\overline{P})$ associated to (P) via the transformation. For $\varphi_t \in C_{\mathbf{A}}[-d,0]$, define $z = Z(x(\varphi_t), y(\varphi_t)) \in Z(S)$. Then for $t \in [0,T]$, the problem (P) can be transformed to the problem $(\overline{P})$

$$(\overline{P}) \begin{cases} \overline{J}(t,z) = \sup\limits_{u_t \in U} E\left[\int_t^T \overline{F}(s, Z_s, u_s)\mathrm{d}s + \overline{h}(Z_T)\right] \\ \text{subject to} \\ \quad dZ_s = \overline{\mu}(s, Z_s, u_s)\mathrm{d}s + \overline{\sigma}(s, Z_s, u_s)\mathrm{d}C_s, \quad s \in [t,T] \\ \quad Z_t = z, \\ \quad u_s \in U, \quad s \in [t,T]. \end{cases} \tag{3.5}$$

The value function $\overline{J}$ of the uncertain optimal control problem $(\overline{P})$ has a finite-dimensional state space. So we can directly use the equation of optimality in Zhu [14] for $(\overline{P})$ and have the main result of this paper.

Theorem 3.1. *Suppose that Assumptions 1 and 2 hold and $\overline{J}_t(t,z)$ is twice differentiable on $[0,T] \times R^n$. Then we have*

$$\begin{cases} -\overline{J}_t(t,z) = \sup\limits_{u \in U}\{\overline{F}(t,z,u_t) + \nabla_z \overline{J}(t,z)^{\tau}\overline{\mu}(t,z,u_t)\} \\ \overline{J}(T, Z_T) = \overline{h}(Z_T), \end{cases} \tag{3.6}$$

and $\overline{J}(t,z) = J(t,\varphi_t)$, where $\overline{J}_t(t,z)$ is the partial derivative of the function $\overline{J}(t,z)$ in t, and $\nabla_z \overline{J}(t,z)$ is the gradient of $\overline{J}(t,x)$ in z.

Proof. The equation (3.6) directly follows from the equation of optimality [12]. For any $u_t \in U$, we have

$$\begin{aligned} \overline{J}(t,z) &\geq E\left[\int_t^T \overline{F}(s, Z_s, u_s)\mathrm{d}s + \overline{h}(Z_T)\right] \\ &= E\left[\int_t^T F(s, X_s, Y_s, u_s)\mathrm{d}s + h(X_T, Y_T)\right]. \end{aligned}$$

Thus,

$$\overline{J}(t,z) \geq \sup_{u_t \in U} E\left[\int_t^T F(s, X_s, Y_s, u_s)\mathrm{d}s + h(X_T, Y_T)\right] = J(t, \varphi_t).$$

Similarly we can get $J(t,\varphi_t) \geq \overline{J}(t,z)$. Therefore, the theorem is proved. □

Remark 3.1. *The optimal decision and optimal expected value of problem (P) are determined if the equation (3.6) has solutions.*

4. Uncertain Linear Quadratic Problem With Time-delay

In this section, we apply the result obtained in the previous section to study an uncertain LQ problem with time-delay. Let $A_1(t)$, $A_2(t)$, $A_4(t)$, $A_5(t)$, $A_6(t)$, $A_7(t)$, $B(t)$, $H(t)$, $I(t)$, $L(t)$, $M(t)$, $N(t)$, $R(t)$ be continuously differentiable functions of t. What's more, let $A_3 \neq 0$ and G be constants, and $I(t) \leq 0$, $R(t) < 0$. For $\psi \in C_R[-d,0]$, denote $x(\psi) = \psi(0)$, $y(\psi) = \int_{-d}^{0} e^{\lambda s}\psi(s)ds$, $\zeta(\psi) = \psi(-d)$. Then an uncertain LQ problem with time-delay is stated as

$$(LQ)\begin{cases} J(t,\varphi_t) = \sup\limits_{u\in U} E\Big[\int_t^T \{I(s)(e^{-\lambda d}X_s + A_3Y_s)^2 + R(s)u_s^2 + H(s)(e^{-\lambda d}X_s + A_3Y_s)u_s \\ \qquad +L(s)(e^{-\lambda d}X_s + A_3Y_s) + M(s)u_s + N(s)\}\,\mathrm{d}s + G(e^{-\lambda d}X_T + A_3Y_T)^2\Big] \\ \text{subject to} \\ \quad \mathrm{d}X_s = \{A_1(s)X_s + A_2(s)Y_s + A_3\zeta_s + B(s)u_s + A_4(s)\}\mathrm{d}s + \{A_5(s)X_s \\ \qquad +A_6(s)Y_s + A_7(s)\}\mathrm{d}C_s, \quad s\in[t,T] \\ \quad Y_s = \int_{-d}^0 e^{\lambda r}X_{s+r}\mathrm{d}r, \quad \zeta_s = X_{s-d}, \quad s\in[t,T] \\ \quad X_s = \varphi_t(s), \quad -d\le s\le 0 \\ \quad u_s\in U, \quad s\in[t,T] \end{cases}$$

where $\varphi_0 \in C_R[-d,0]$ is a given initial function and $\varphi_t \in C_R[-d,0]$ is the segment of X_t for $t>0$, and U is the set of feasible controls. In addition, we are in state $X_t = x$ at time t.

Theorem 4.1. *If $A_2(t) = e^{\lambda d}A_3(A_1(t) + e^{\lambda d}A_3 + \lambda)$ and $A_6(t) = e^{\lambda d}A_3A_5(t)$ hold in the (LQ) model, then the optimal control u_t^* of (LQ) is*

$$u_t^* = -\frac{(H(t) + e^{-\lambda d}B(t)P(t))z + e^{-\lambda d}B(t)Q(t) + M(t)}{2R(t)}, \tag{4.1}$$

where $P(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = \dfrac{e^{-2\lambda d}B(t)^2}{2R(t)}P(t)^2 + \left(\dfrac{e^{-\lambda d}H(t)B(t)}{R(t)} - 2A_1(t) - 2A_3e^{\lambda d}\right)P(t) \\ \qquad\qquad +\dfrac{H(t)^2}{2R(t)} - 2I(t) \\ P(T) = 2G, \end{cases} \tag{4.2}$$

and $Q(t)$ is a solution of the following differential equation

$$\begin{cases} \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t} = \left(\dfrac{e^{-\lambda d}H(t)B(t) + e^{-2\lambda d}B(t)^2P(t)}{2R(t)} - A_1(t) - A_3e^{\lambda d}\right)Q(t) \\ \qquad\qquad -e^{-\lambda d}P(t)A_4(t) - L(t) + \dfrac{e^{-\lambda d}M(t)B(t)P(t) + H(t)M(t)}{2R(t)} \\ Q(T) = 0. \end{cases} \tag{4.3}$$

The optimal value of (LQ) is

$$J(t,\varphi_t) = \frac{1}{2}P(t)z^2 + Q(t)z + K(t), \tag{4.4}$$

where $z = e^{-\lambda d}x + A_3\int_{-d}^0 e^{\lambda s}X_{t+s}\mathrm{d}s$, and

$$\begin{aligned} K(t) &= \int_t^T \left\{\frac{M(s)^2}{4R(s)} + \frac{e^{-2\lambda d}B(t)^2Q(t)^2}{4R(s)} + \frac{e^{-\lambda d}B(t)M(s)Q(t)}{2R(s)} - N(s)\right. \\ &\qquad -e^{-\lambda d}Q(s)A_4(s)\Big\}\mathrm{d}s. \end{aligned} \tag{4.5}$$

Proof. The problem (LQ) is a special case of (P). In order to solve (LQ) by employing Theorem 3.1, we need to check Assumptions 1 and 2 for the (LQ) model. Note that

$$\begin{aligned}\mu_1(t,x,y,u) &= A_1(t)x + A_2(t)y + B(t)u + A_4(t), \quad \mu_2(x,y) = A_3,\\ F(t,x,y,u) &= I(t)(e^{-\lambda d}x + A_3 y)^2 + R(t)u^2 + H(t)(e^{-\lambda d}x + A_3 y)u\\ &\quad + L(t)(e^{-\lambda d}x + A_3 y) + M(t)u + N(t),\\ h(x,y) &= G(e^{-\lambda d}x + A_3 y)^2, \quad \sigma(t,x,y,u) = A_5(t)x + A_6(t)y + A_7(t).\end{aligned}$$

We set $Z(x,y) = e^{-\lambda d}x + A_3 y$ so that Assumption 1 is supported in this (LQ) problem. Furthermore, we have

$$\begin{aligned}\widetilde{\mu}(t,x,y,u) &= Z_x(x,y)\mu_1(t,x,y,u) + Z_y(x,y)(f(x) - \lambda y)\\ &= e^{-\lambda d}(A_1(t)x + A_2(t)y + B(t)u + A_4(t)) + A_3(x - \lambda y)\\ &= (A_1(t) + e^{\lambda d}A_3)Z(x,y) + (e^{-\lambda d}A_2(t) - A_3A_1(t) - e^{\lambda d}A_3^2 - \lambda A_3)y\\ &\quad + e^{-\lambda d}(B(t)u + A_4(t)),\\ \overline{F}(t,x,y,u) &= I(t)Z(x,y)^2 + R(t)u^2 + H(t)Z(x,y)u + L(t)Z(x,y) + M(t)u + N(t),\\ \overline{h}(x,y) &= GZ(x,y)^2,\\ \widetilde{\sigma}(t,x,y,u) &= Z_x(x,y)\sigma(t,x,y,u)\\ &= e^{-\lambda d}(A_5(t)x + A_6(t)y + A_7(t))\\ &= A_5(t)Z(x,y) - (A_3A_5(t) - A_6(t)e^{-\lambda d})y + e^{-\lambda d}A_7(t).\end{aligned}$$

Therefore, Assumption 2 hold if only if

$$A_2(t) = e^{\lambda d}A_3(A_1(t) + e^{\lambda d}A_3 + \lambda), \; A_6(t) = e^{\lambda d}A_3A_5(t).$$

The reduced finite-dimensional uncertain control problem becomes

$$(\overline{LQ})\begin{cases}\overline{J}(t,z) = \sup\limits_{u\in U} E\left[\int_t^T \{I(s)Z_s^2 + R(s)u_s^2 + H(s)Z_su_s + L(s)Z_s + M(s)u_s\right.\\ \qquad\qquad\qquad \left. +N(s)\}\,\mathrm{d}s + GZ_T^2\right]\\ \text{subject to}\\ \quad \mathrm{d}Z_s = \left\{(A_1(s) + e^{\lambda d}A_3)Z_s + e^{-\lambda d}(B(s)u_s + A_4(s))\right\}\mathrm{d}s\\ \qquad\qquad +\{A_5(s)Z_s + e^{-\lambda d}A_7(s)\}\mathrm{d}C_s, \quad s\in[t,T]\\ \quad Z_t = z,\\ \quad u_s \in U, \quad s\in[t,T]\end{cases} \tag{4.6}$$

where $z = Z(x(\varphi_t), y(\varphi_t))$. By using Theorem 3.1, we know that $\overline{J}(t,z)$ satisfies

$$-\overline{J}_t(t,z) = \sup_{u_t\in U}\{\overline{F}(t,z,u_t) + \overline{J}_z(t,z)\overline{\mu}(t,z,u_t)\},$$

that is,

$$\begin{aligned}-\overline{J}_t(t,z) &= \sup_{u\in U}\{I(t)z^2 + R(t)u_t^2 + H(t)zu_t + L(t)z + M(t)u_t + N(t)\\ &\qquad +[(A_1(t) + e^{\lambda d}A_3)z + e^{-\lambda d}(B(t)u_t + A_4(t))]\overline{J}_z\}.\end{aligned} \tag{4.7}$$

Let

$$\begin{aligned}g(u_t) &= I(t)z^2 + R(t)u_t^2 + H(t)zu_t + L(t)z + M(t)u_t + N(t)\\ &\quad +[(A_1(t) + e^{\lambda d}A_3)z + e^{-\lambda d}(B(t)u_t + A_4(t))]\overline{J}_z.\end{aligned}$$

Setting $\dfrac{\partial g(u_t)}{\partial u_t}=0$ yields

$$2R(t)u_t+H(t)z+M(t)+e^{-\lambda d}B(t)\overline{J}_z=0.$$

Hence

$$u_t^*=-\frac{H(t)z+M(t)+e^{-\lambda d}B(t)\overline{J}_z}{2R(t)}. \tag{4.8}$$

By equation (4.7) we have

$$\begin{aligned}-\overline{J}_t(t,z)=&\,I(t)z^2+R(t)u_t^{*2}+H(t)zu_t^*+L(t)z+M(t)u_t^*+N(t)\\&+[(A_1(t)+e^{\lambda d}A_3)z+e^{-\lambda d}(B(t)u_t^*+A_4(t))]\overline{J}_z.\end{aligned} \tag{4.9}$$

Since $\overline{J}(T,Z_T)=GZ_T^2$, we guess

$$\overline{J}(t,z)=\frac{1}{2}P(t)z^2+Q(t)z+K(t). \tag{4.10}$$

Thus

$$\overline{J}_t(t,z)=\frac{1}{2}\frac{dP(t)}{dt}z^2+\frac{dQ(t)}{dt}z+\frac{dK(t)}{dt} \tag{4.11}$$

and

$$\overline{J}_z(t,z)=P(t)z+Q(t). \tag{4.12}$$

Substituting (4.8) and (4.12) into (4.9) yields

$$\begin{aligned}\overline{J}_t(t,z)=&\left[\frac{H(t)^2}{4R(t)}+\frac{e^{-\lambda d}H(t)B(t)P(t)}{2R(t)}+\frac{e^{-2\lambda d}B(t)^2P(t)^2}{4R(t)}-P(t)A_1(t)\right.\\&\left.-P(t)A_3e^{\lambda d}-I(t)\right]z^2+\left[\frac{e^{-\lambda d}H(t)B(t)+e^{-2\lambda d}B(t)^2P(t)}{2R(t)}Q(t)\right.\\&-A_3e^{\lambda d}Q(t)-A_1(t)Q(t)-L(t)+\frac{e^{-\lambda d}M(t)B(t)P(t)+H(t)M(t)}{2R(t)}\\&\left.-e^{-\lambda d}P(t)A_4(t)\right]z+\frac{M(t)^2}{4R(t)}+\frac{e^{-2\lambda d}B(t)^2Q(t)^2}{4R(t)}+\frac{e^{-\lambda d}B(t)M(t)Q(t)}{2R(t)}\\&-N(t)-e^{-\lambda d}Q(t)A_4(t).\end{aligned} \tag{4.13}$$

By equation (4.11) and (4.13) we get

$$\begin{cases}\dfrac{\mathrm{d}P(t)}{\mathrm{d}t}=-2I(t)+\dfrac{H(t)^2}{2R(t)}+\dfrac{e^{-\lambda d}H(t)B(t)P(t)}{R(t)}+\dfrac{e^{-2\lambda d}B(t)^2P(t)^2}{2R(t)}\\ \qquad\qquad -2P(t)(A_1(t)+A_3e^{\lambda d})\\ \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t}=\left(\dfrac{e^{-\lambda d}H(t)B(t)+e^{-2\lambda d}B(t)^2P(t)}{2R(t)}-A_1(t)-A_3e^{\lambda d}\right)Q(t)\\ \qquad\qquad -e^{-\lambda d}P(t)A_4(t)+\dfrac{e^{-\lambda d}M(t)B(t)P(t)+H(t)M(t)}{2R(t)}-L(t),\end{cases} \tag{4.14}$$

and

$$\begin{aligned}\frac{\mathrm{d}K(t)}{\mathrm{d}t}=&\frac{M(t)^2}{4R(t)}+\frac{e^{-2\lambda d}B(t)^2Q(t)^2}{4R(t)}+\frac{e^{-\lambda d}B(t)M(t)Q(t)}{2R(t)}-N(t)\\&-e^{-\lambda d}Q(t)A_4(t).\end{aligned} \tag{4.15}$$

Since $\overline{J}(T,z) = \frac{1}{2}P(T)z^2 + Q(T)z + K(T) = Gz^2$, we have $P(T) = 2G$, $Q(T) = 0$, and $K(T) = 0$. By equation(4.14), we obtain (4.2) and (4.3). By equation(4.15), the equation (4.5) holds. Therefore,

$$J(t,\varphi_t) = \overline{J}(t,z) = \frac{1}{2}P(t)(z)^2 + Q(t)z + K(t)$$

is the optimal value of (LQ), and

$$u_t^* = -\frac{(H(t) + e^{-\lambda d}B(t)P(t))z + e^{-\lambda d}B(t)Q(t) + M(t)}{2R(t)}$$

is the optimal control, where

$$\begin{aligned} z &= e^{-\lambda d}x(\varphi_t) + A_3 y(\varphi_t) = e^{-\lambda d}\varphi_t(0) + A_3\int_{-d}^{0} e^{\lambda s}\varphi_t(s)\mathrm{d}s \\ &= e^{-\lambda d}X_t + A_3\int_{-d}^{0} e^{\lambda s}X_{t+s}\mathrm{d}s = e^{-\lambda d}x + A_3\int_{-d}^{0} e^{\lambda s}X_{t+s}\mathrm{d}s. \end{aligned}$$

□

An example

We consider the following example of uncertain optimal control model with time-delay

$$\begin{cases} J(0,\varphi_0) = \sup\limits_{u\in U} E\left[\displaystyle\int_0^2 \left\{-(e^{-1}X_s + Y_s)^2 - u_s^2\right\}\mathrm{d}s + (e^{-1}X_T + Y_T)^2\right] \\ \text{subject to} \\ \quad \mathrm{d}X_t = \{(-e-5)X_t + X_{t-0.2} + u_t\}\mathrm{d}t + \mathrm{d}C_t, \quad t\in[0,2] \\ \quad X_t = \varphi_0(t) = \cos\pi t, \quad -0.2\le t\le 0 \\ \quad Y_t = \displaystyle\int_{-0.2}^{0} e^{5s}X_{t+s}\mathrm{d}s, \quad t\in[0,2] \\ \quad u_t\in R, \quad t\in[0,2]. \end{cases} \tag{4.16}$$

We have $A_1(s) = -(e+5)$, $A_2(s) = 0$, $A_3 = 1$, $A_4(s) = 0$, $B(s) = 1$, $A_5(s) = A_6(s) = 0$, $A_7(s) = 1$, $I(s) = -1$, $R(s) = -1$, $H(s) = L(s) = M(s) = N(s) = 0$, $G = 1$, $\lambda = 5$, $d = 0.2$. Hence $A_2(t) = eA_3(A_1(t) + eA_3 + 5)$ and $A_6(t) = eA_3A_5(t)$ hold in this model. By Theorem 4.1, the function $Q(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t} = \left(-\dfrac{1}{2e^2}P(t) + 5\right)Q(t), \quad t\in[0,2] \\ Q(2) = 0. \end{cases}$$

Thus $Q(t) = 0$ for $t\in[0,2]$, and then $K(t) = 0$ for $t\in[0,2]$. Therefore, the optimal control u_t^* is $u_t^* = \dfrac{e^{-1}P(t)z_t}{2}$, where $z_t = e^{-1}x_t + y_t$, and the optimal value is $J(0,\varphi_0) = \frac{1}{2}P(0)z_0^2$, where $z_0 = e^{-1}x_0 + y_0$, and $P(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = -\dfrac{1}{2e^2}P(t)^2 + 10P(t) + 2 \\ P(2) \;= 2, \end{cases} \tag{4.17}$$

and

$$x_0 = X_0 = 1, \quad y_t = Y_t = \int_{-0.2}^{0} e^{5s} X_{t+s} \mathrm{d}s,$$

$$y_0 = Y_0 = \int_{-0.2}^{0} e^{5s} X_s ds = \int_{-0.2}^{0} e^{5s} \cos \pi s \, \mathrm{d}s = \frac{\pi \sin(0.2\pi) - 5\cos(0.2\pi) + 5e}{e(\pi^2 + 25)}.$$

Since the value of y_t is derived from the value of X_s between $t - 0.2$ and t, the analytical expression of y_t can not be obtained and so is that of u_t^*.

Now we consider the numerical solutions of the model. Let $\Pi_1 = s_0, s_1, \cdots s_{20}$ be an average partition of $[-0.2, 0]$ (i.e., $-0.2 = s_0 < s_1 < \cdots < s_{20} = 0$), and $\triangle s = 0.01$. Thus,

$$y_t = Y_t = \sum_{i=0}^{20} e^{5s_i} X_{t+s_i} \triangle s.$$

Let $\Pi_2 = t_0, t_1, \cdots t_{200}$ be an average partition of $[0, 2]$ (i.e., $0 = t_0 < t_1 < \cdots < t_{200} = 2$), and $\triangle t = 0.01$. Thus,

$$\triangle X_t = (-(e+5)X_t + X_{t-0.2} + u_t^*)\triangle t + \triangle C_t.$$

Since $\triangle C_t$ is a normal uncertain variable with expected value 0 and variance $\triangle t^2$, the distribution function of $\triangle C_t$ is $\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}\triangle t}\right)\right)^{-1}$, $x \in R$. We may get a sample point $\widetilde{c}_t$ of $\triangle C_t$ from $\widetilde{c}_t = \Phi^{-1}(rand(0,1))$ that $\widetilde{c}_t = \frac{\sqrt{3}\triangle t}{-\pi} \ln\left(\frac{1}{rand(0,1)} - 1\right)$. Thus, x_t, y_t and u_t may be given by the following iterative equations

$$\begin{aligned} y_{t_j} &= \sum_{i=0}^{20} e^{5s_i} x_{t_j+s_i} \triangle s, \quad u_{t_j} = \frac{e^{-1}}{2} P(t_j)(e^{-1}x_{t_j} + y_{t_j}), \\ x_{t_{j+1}} &= x_{t_j} + \triangle X_t \\ &= x_{t_j} + (-(e+5)x_{t_j} + x_{t_j-0.2} + u_{t_j})\triangle t + \frac{\sqrt{3}\triangle t}{-\pi} \ln\left(\frac{1}{rand(0,1)} - 1\right) \end{aligned}$$

for $j = 0, 1, 2, \cdots, 200$, and $x_{s_i} = \cos \pi s_i$ for $i = 0, 1, \cdots, 20$, where the numerical solution $P(t_j)$ of (4.17) is provided by

$$P(t_{j-1}) = P(t_j) - \left(-\frac{1}{2e^2} P(t_j)^2 + 10P(t_j) + 2\right) \Delta t$$

for $j = 200, 199, \cdots, 2, 1$ with $P(t_{200}) = 2$.

Therefore, the optimal value of the example is $J(0, \varphi_0) = -0.024429$, and the optimal controls and corresponding states are obtained in the Table 1 for part data.

5. A Consumption Problem

Consider a consumption problem in financial market. Let the wealth of an inventor be an uncertain process following an uncertain differential system with delay. The inventor consumes part of his wealth based on a consumption process ω_t. Thus his current wealth X_t may be described by

$$\begin{cases} \mathrm{d}X_t = (X_t + ae^{\lambda d}(ae^{\lambda d} + \lambda + 1)Y_t + a\zeta_t - \omega_t)\mathrm{d}t + (X_t + ae^{\lambda d}Y_t)\mathrm{d}C_t, & t \geq 0 \\ X_t = \varphi(t), \quad -d \leq t \leq 0, \end{cases}$$

Table 1: Numerical solutions

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	1.000000	0.472623	0.268005	0.163757	0.103532	0.042242	0.010214
y	0.126709	0.103694	0.063890	0.035297	0.020538	0.012068	0.005922
u	−0.018170	−0.010197	−0.005969	−0.003510	−0.002154	−0.001014	−0.000356
t	0.7	0.8	0.9	1.0	1.1	1.2	1.3
x	−0.008133	−0.028942	−0.002142	−0.014657	−0.002138	−0.046814	−0.030036
y	0.001701	−0.001587	0.000481	0.000381	−0.000242	−0.001997	−0.004128
u	0.000047	0.000449	0.000011	0.000184	0.000038	0.000704	0.000554
t	1.4	1.5	1.6	1.7	1.8	1.9	2.0
x	−0.023194	−0.022888	−0.050667	−0.013823	−0.003976	−0.012452	−0.040871
y	−0.004421	−0.003591	−0.003553	−0.003156	−0.002194	−0.001305	−0.002409
u	0.000466	0.000416	0.000682	0.000160	−0.000047	−0.000621	−0.006417

where $a > 0$. Choose the utility function as $e^{-\beta t}\frac{1}{\alpha}\omega_t^\alpha$ where $\alpha \in (0,1)$ and β is a discount factor with $(ae^{\lambda d}+1)\alpha < \beta < ae^{\lambda d}\alpha + 1$. Therefore an optimal consumption problem for uncertain system with delay may be considered:

$$\begin{cases} J(0,\varphi) = \max_{\omega} E\left[\int_0^{+\infty} e^{-\beta s}\frac{1}{\alpha}\omega_s^\alpha \mathrm{d}s\right] \\ \text{subject to} \\ \quad \mathrm{d}X_t = (X_t + ae^{\lambda d}(ae^{\lambda d}+\lambda+1)Y_t + a\zeta_t - \omega_t)\mathrm{d}t + (X_t + ae^{\lambda d}Y_t)\mathrm{d}C_t, \quad t \ge 0 \\ \quad X_t = \varphi(t), \quad -d \le t \le 0. \end{cases}$$

Let $Z(x,y) = x + ae^{\lambda d}y$. Then Assumption 1 holds. Based on the Assumption 2, above problem with delay can be transformed into the following problem without delay

$$\begin{cases} \overline{J}(0,Z_0) = \max_{\omega} E\left[\int_0^{+\infty} e^{-\beta s}\frac{1}{\alpha}\omega_s^\alpha \mathrm{d}s\right] \\ \text{subject to} \\ \quad \mathrm{d}Z_t = ((ae^{\lambda d}+1)Z_t - \omega_t)\mathrm{d}t + Z_t\mathrm{d}C_t, \quad t \ge 0 \\ \quad Z_0 = \varphi(0) + ae^{\lambda d}\int_{-d}^0 e^{\lambda s}\varphi(s)\mathrm{d}s. \end{cases}$$

By the similar technique to one in [14], we can obtain the optimal consumption

$$\omega_t = \frac{\beta - (ae^{\lambda d}+1)\alpha}{1-\alpha} z_t$$

where $z_t = x_t + ae^{\lambda d}y_t$.

6. Conclusion

In this paper, an uncertain optimal control problem with time-delay was investigated. By using two assumptions, the problem was transformed to an uncertain optimal control problem without time-delay which may be solved by the equation of optimality presented in the literature. As a special case, an LQ model was considered. The optimal control of the LQ model is a feedback of state of the uncertain linear system. Due to the time-delay of the system, the state is hardly expressed as an analytical form of time. For an example, the numerical solution of an LQ model was given to show the useableness of the result obtained in the paper. An optimal consumption problem with delay in financial market was presented to show all assumptions are easily satisfied under some practice.

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