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MULTI-LEADER-FOLLOWER GAMES: MODELS, METHODS AND APPLICATIONS

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Abstract The multi-leader-follower game serves as an important model in game theory with many applications in economics, operations research and other fields. In this survey paper, we first recall some background materials in game theory and optimization. In particular, we present several extensions of Nash equilibrium problems including the multi-leader-follower game. We then give some applications as well as solution methods of multi-leader-follower games.

Keywords: Game theory, Nash equilibrium, multi-leader-follower game, equilibrium problem with equilibrium constraints

1. Introduction

As a solid mathematical methodology to deal with many problems in social and natural sciences, such as economics, operations research, political science, management, computer science, biology and so on, game theory [26, 51] studies the strategic decision making, where an individual makes a choice by taking into account the others' choices. In a typical game, the following three elements should be specified: the players of the game, the strategies available to each player, and the payoffs for each outcome. Generally, there are two branches in game theory: cooperative game [18] and non-cooperative game [63].

Game theory has been widely developed since 1950 when John Nash introduced the well-known concept of Nash equilibrium [53, 54] in non-cooperative games involving two or more players. In such a game, called the Nash game or Nash equilibrium problem (NEP for short), all players are assumed to know the objective functions of other players and make decisions to choose their own strategies at the same time by taking into account the strategies of other players. When each player can obtain no more benefit by changing his/her current strategy unilaterally (i.e., all players have no incentive to change their current strategies), the strategy tuple comprised of the current strategies of all players constitutes a Nash equilibrium. By using this fundamental concept in game theory, the NEP becomes a powerful mathematical model to deal with many real-world problems, such as arms races [64] in politics, auction theory [48] and the electricity markets [34, 35, 44] in economics.

In the NEP, each player tries to observe the strategies of other players to choose his/her optimal strategy, but cannot affect the strategy sets of other players. That is, each player's strategy set is independent of the strategies of other players. However, in many real-world problems, such as those from the telecommunication field [61] and environmental pollution control [11], each player's strategy set may depend on the strategies of other players. In this case, the NEP can be extended to the generalized Nash game, or the generalized Nash equilibrium problem (GNEP for short). The early study of the GNEP started from Debreu

[15] and Arrow and Debreu [3]. In recent years, the GNEP has drawn much attention of researchers from practical and theoretical standpoints; see, e.g., Facchinei and Kanzow [21] and the references therein.

In a NEP or GNEP, all players are in a position of the same level and make their own decisions simultaneously by estimating the decisions of other players. However, in some real-world situation, e.g., in some electricity power market, a well established firm (called leader) with sound assets has the ability to decide the quantities or price of electricity by anticipating those of other more fragile firms (called followers). The followers make their decisions after observing the decision of the leader. The mathematical formulation to model such problems is the Stackelberg game [5, 68], also called the single-leader-follower game. Generally, in a Stackelberg game, there is a distinctive player called the leader, who optimizes the upper-level problem, and a number of remaining players called the followers, who optimize the lower-level problems jointly. In particular, the leader anticipates the responses of the followers, and then uses this ability to select his/her optimal strategy. At the same time, all followers select their own optimal responses by competing with each other in a NEP or GNEP parameterized by the leader's decision. Many researchers have studied the Stackelberg game extensively and have found wide applications in various areas, such as oligopolistic market analysis [56, 67], optimal product design [13], quality control in services [2], and pricing of electric transmission [36].

As a bilevel program [72], the Stackelberg game can be looked on as a special case of the mathematical program with equilibrium constraints (MPEC for short), when one replaces the followers' problems by their optimality conditions. Generally, an MPEC is an optimization problem which contains two sets of variables called decision variables and response variables. Some or all of MPEC constraints are represented by a parametric variational inequality or complementarity problem with respect to the response variables, which is parameterized by the decision variables. The MPEC has been studied extensively in the last two decades; see, e.g., [47, 58].

In a game, when several players take the position as leaders and the rest of players take the position as followers, it becomes a multi-leader-follower game. Multi-leader-follower games arise from some oligopoly markets with two or more oligopolistic enterprises, such as deregulated electricity market [12, 37, 39, 45, 60]. One may also explain it in such an automobile manufacturing market. Several large enterprises (leaders) with adequate funding and technology have the ability to develop and produce new-fashioned cars and their quantities, thereby making their decisions first. After observing the decisions of the leaders, the other small enterprises (followers) choose their optimal strategies to decide the types of cars and their quantities they will produce. Like the leader in a Stackelberg game, the leaders in a multi-leader-follower game also have the ability to anticipate the responses of followers.

Generally, in a multi-leader-follower game, there are several players who serve as leaders and the rest of players who serve as followers. As a bilevel program, all leaders compete with each other in a non-cooperative Nash game in the upper-level and make their decisions first by anticipating the responses of followers. Upon receipt of the leaders' decisions, all followers compete with each other in a parametric non-cooperative Nash game in the lower-level with the strategies of leaders as exogenous parameters. The multi-leader-follower game may further be classified into the game which contains only one follower, called the multi-leader single-follower game, and the game which contains multiple followers, called the multi-leader multi-follower game. The leader-follower (L/F for short) Nash equilibrium, a solution concept for the multi-leader-follower game, can be defined as a set of leaders' and followers' strategies such that no player (leader or follower) can improve his/her status by changing his/her own current strategy unilaterally. Depending on whether each leader anticipates the responses of all followers optimistically or pessimistically, one can define the optimistic L/F Nash equilibrium and pessimistic L/F Nash equilibrium for the multi-leader-follower game.

A mathematical formulation to model the multi-leader-follower game is the equilibrium problem with equilibrium constraints (EPEC for short). An EPEC is an equilibrium problem consisting of several parametric MPECs which contain the strategies of other players as parameters. The equilibria of an EPEC can be achieved when all MPECs are solved simultaneously. The EPEC can also be looked on as a generalization of the NEP or GNEP, where some parametric variational inequality or complementarity problems appear in each player's constraints. The EPEC models have wide applications in different fields, such as engineering design, economics, etc.; see [20, 37, 38, 42, 43, 50, 70, 71].

The early study associated with the multi-leader-follower game and EPEC could date back at least to Sherali [66], where a multi-leader-follower game was called a multiple Stackelberg model. Sherali [66] established existence of an equilibrium by assuming that each leader can exactly anticipate the aggregate follower reaction curve. He also showed the uniqueness of equilibrium for a special case where all leaders share an identical cost function and make the identical decisions. As Ehrenmann [19, 20] pointed out, the assumption that all leaders make the identical decisions is essential for ensuring the uniqueness result. He also gave a counterexample to show that, when all leaders with identical cost functions make different decisions, the game could reach multiple equilibria. In addition, Su [71] considered a forward market equilibrium model that extended the existence result of Sherali [66] under some weaker assumptions. Pang and Fukushima [60] considered a class of remedial models for the multi-leader-follower game that can be formulated as a GNEP with convexified strategy sets. They further defined a new equilibrium concept called remedial L/F Nash equilibrium and presented an existence result with this equilibrium concept. They also proposed some examples about oligopolistic electricity market that lead to the multi-leader-follower games. Based on the strong stationarity conditions of each leader in a multi-leader-follower game, Leyffer and Munson [45] derived a family of nonlinear complementarity problem, nonlinear program, and MPEC formulations of the multi-leader-follower games. They also reformulated the game as a square nonlinear complementarity problem by imposing an additional restriction. Outrata [57] first derived two types of necessary conditions on the equilibria of the EPECs. Other optimality conditions were further studied in [33, 49]. Guo and Lin [27] presented some algorithms to compute various stationary points of the EPECs by reformulating the stationary systems of the EPEC as constrained equations. One of early methods to solve the EPEC is the diagonalization method [9, 37, 38, 59], such as Gauss-Jacobi and Gauss-Seidel methods. The main idea underlying this method is to solve all MPECs in the EPEC one by one. At each time, only one MPEC is solved. The procedure is repeated cyclically for every MPEC in the EPEC until some equilibrium is found. Su [69] presented a method called the sequential nonlinear complementarity algorithm to solve the EPECs. Its main idea is to relax the complementarity constraints in each MPEC simultaneously and solve a sequence of nonlinear complementarity problems derived from the EPECs. Hu [38] presented an approach to the EPECs by concatenating all leaders' first-order optimality conditions, where each MPEC is treated as a standard nonlinear program, and then the mixed complementarity problem comprising the first order optimality conditions of all MPECs is solved by the PATH solver [17, 24, 62]. Ehrenmann [20] also introduced a mixed complementarity formulation for the EPECs by using a big-M approach. Hu and Fukushima [40] considered a special class of EPECs with shared equilibrium constraints. They formulated it as a linear complementarity system and proposed to find an equilibrium by solving a sequence of smoothed NEPs.

In the above mentioned two equilibrium concepts, Nash equilibrium and L/F Nash equilibrium, each player is assumed to have complete information about the game. This means that, in a NEP, each player can observe other players' strategies and choose his/her own strategy exactly, while in a multi-leader-follower game, each leader can anticipate each follower's response to the leaders' strategies exactly. However, in many real-world problems, such strong assumptions are not always satisfied. Another kind of game with uncertain data and the corresponding concept of equilibria need to be considered.

There has been important work about games with uncertain data. Under the assumption on probability distributions called the Bayesian hypothesis, Harsanyi [29–31] considered a game with incomplete information, where the players have no complete information about some important parameters of the game. Further assuming all players share some common knowledge about those probability distributions, the game was finally reformulated as a game with complete information, called the Bayes equivalent of the original game. Stochastic optimization technique [10, 14] can also be used to deal with the Stackelberg game and the MPEC with uncertain data. One may see the details about the stochastic Stackelberg game and the stochastic MPEC in the survey paper [46] and the references therein. DeMiguel and Xu [16] considered stochastic multi-leader-follower game applied in the telecommunications industry and established the existence and uniqueness of the equilibrium. Shanbhag, Infanger and Glynn [65] considered a class of stochastic multi-leader-follower games and established the existence of a local equilibrium by a related simultaneous stochastic Nash equilibrium problems.

Besides the probability distribution models, the distribution-free models based on the worst-case scenario have received attention in recent years; see [1, 32, 55]. In the latter models, each player makes a decision according to the concept of robust optimization [6–8]. Basically, in robust optimization, uncertain data are assumed to belong to some set called an uncertainty set, and then a solution is sought by taking into account the worst case in terms of the objective function value and/or the constraint violation. In a NEP containing some uncertain parameters, we may also define an equilibrium called a robust Nash equilibrium. Namely, if each player has chosen a strategy pessimistically and no player can obtain more benefit by changing his/her own current strategy unilaterally (i.e., the other players hold their current strategies), then the tuple of current strategies of all players is defined as a robust Nash equilibrium and the problem of finding a robust Nash equilibrium is called the robust Nash equilibrium problem. Such a problem was studied by Hayashi, Yamashita and Fukushima [32] for the bimatrix game with uncertain data. Under some assumptions on the uncertainty sets, they presented some existence results about robust Nash equilibria. Aghassi and Bertsimas [1] considered a robust Nash equilibrium in an N-person NEP with bounded polyhedral uncertainty sets, where each player solves a linear programming problem. They also proposed a method for computing robust Nash equilibria. Note that both of these models in [1, 32] particularly deal with linear objective functions in each player's problem. More recently, Nishimura, Hayashi and Fukushima [55] considered a more general NEP with uncertain data, where each player solves an optimization problem with a nonlinear objective function. Under some mild assumptions on the uncertainty sets, the authors presented some results about the existence and uniqueness, as well as the computation, of a robust Nash equilibria.

In the field of multi-leader-follower games, Hu and Fukushima [41] further extended their work in [39] under the uncertainty assumption by the robust optimization technique.

A new concept called the robust L/F Nash equilibrium was introduced and its existence and uniqueness results were established for a class of multi-leader-follower games with some special structure.

The organization of this paper is as follows. In the next section, we collect some basic definitions and present some basic models and formulations related to the multi-leader-follower games. In Section 3, we introduce some applications to motivate the multi-leader-follower games. In Section 4, we discuss some reformulations of the multi-leader-follower games and the corresponding existence and uniqueness results. Finally, we conclude the paper in Section 5.

Throughout this paper, we use the following notations. The gradient $\nabla f(x)$ of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is regarded as a column vector. The nonnegative orthant in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$. For any vector $x \in \mathbb{R}^n$, its Euclidean norm is defined by $\|x\| := \sqrt{x^\top x}$, where $\top$ denotes transposition. If a vector x consists of several subvectors $x^1, \dots, x^N$, it is denoted for simplicity of notation as $(x^1, \dots, x^N)$ instead of $((x^1)^\top, \dots, (x^N)^\top)^\top$. For any set X, $\mathcal{P}(X)$ denotes the set comprising all the subsets of X.

2. Basic Models and Formulations

2.1. Variational inequality and complementarity problem

In this subsection, we introduce some basic concepts and properties of the variational inequality (VI for short), its special case called the complementarity problem (CP for short), and its generalization called the generalized variational inequality (GVI for short).

Definition 2.1. *The* variational inequality *denoted by VI(K, F) is a problem of finding a vector $x \in K$ such that*

$$F(x)^\top (y - x) \geq 0, \quad \forall y \in K, \tag{2.1}$$

where K is a nonempty closed convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping.

As to the existence and uniqueness of a solution in the VI, a number of results are known. One of the most fundamental results relies on the compactness of set K. Other existence results can be obtained by imposing another condition, such as coerciveness of F, instead of the compactness of K. On the other hand, under some monotonicity assumptions on F, we have the following two results on the uniqueness of a solution:

Proposition 2.1. *If F is strictly monotone on K, i.e., $(F(x) - F(y))^\top (x - y) > 0, \forall x, y \in K, x \neq y$, and the VI$(K, F)$ has at least one solution, then the solution is unique.*

Proposition 2.2. *If F is strongly monotone on K, i.e., there exists $\mu > 0$ such that $(F(x) - F(y))^\top (x - y) \geq \mu \|x - y\|^2, \forall x, y \in K$, then there exists a unique solution to the VI(K, F).*

The VI is a very large class of problems, which contains many problems as its special cases, such as the system of equations, the convex programming problem, and the CP. For example, when $K = \mathbb{R}^n_+$, the VI(K, F) (2.1) is equivalent to the complementarity problem denoted by CP(F), which is to find a vector $x \in \mathbb{R}^n$ such that

$$F(x) \geq 0, \quad x \geq 0, \quad F(x)^\top x = 0. \tag{2.2}$$

When F is an affine function given by $F(x) = Mx + q$ with a square matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, CP(F) (2.2) becomes the linear complementarity problem (LCP) denoted by LCP(M, q), which is to find a vector $x \in \mathbb{R}^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad (Mx + q)^\top x = 0.$$

Applications of the VI and CP can be found in various areas, such as transportation systems, mechanics, and economics; see [22, 25, 28, 52] and the references therein.

For the VI, there exist several important generalizations, one of which is the generalized variational inequality (GVI for short) [23] defined as follows.

Definition 2.2. *The* generalized variational inequality*, denoted by GVI$(K, \mathcal{F})$, is a problem of finding a vector $x \in K$ such that*

$$\exists\, \xi \in \mathcal{F}(x), \quad \xi^\top (y - x) \geq 0, \quad \forall x \in K, \tag{2.3}$$

where $K \subseteq \mathbb{R}^n$ is a nonempty closed convex set and $\mathcal{F} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued mapping.

It is easy to see that if the set-valued mapping $\mathcal{F}$ happens to be a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\mathcal{F}(x) = \{F(x)\}$, then GVI (2.3) reduces to the VI(K, F) (2.1). The GVI also shares some similar properties with the VI.

Proposition 2.3. *Suppose that the set-valued mapping $\mathcal{F} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is strictly monotone on K, i.e., $(\xi - \eta)^\top (x - y) > 0, \forall x, y \in K, \xi \in \mathcal{F}(x), \eta \in \mathcal{F}(y), x \neq y$, and the GVI* (2.3) *has at least one solution. Then the solution is unique.*

2.2. Nash equilibrium problem and generalized Nash equilibrium problem

In this subsection, we describe the Nash equilibrium problem (NEP for short) and its generalization, the generalized Nash equilibrium problem (GNEP for short).

Formally, in a NEP, there are N players labelled by integers $\nu = 1, \ldots, N$. Player ν's strategy is denoted by vector $x^\nu \in \mathbb{R}^{n_\nu}$ and his/her cost function $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ depends on all players' strategies, which are collectively denoted by the vector $x \in \mathbb{R}^n$ consisting of subvectors $x^\nu \in \mathbb{R}^{n_\nu}$, $\nu = 1, \ldots, N$, and $n := n_1 + \cdots + n_N$. Player ν's strategy set $X^\nu \subseteq \mathbb{R}^{n_\nu}$ is independent of the other players' strategies, which are denoted collectively as $x^{-\nu} := (x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^N) \in \mathbb{R}^{n_{-\nu}}$, where $n_{-\nu} := n - n_\nu$. For every fixed but arbitrary vector $x^{-\nu} \in \mathbb{R}^{n_{-\nu}}$, player ν solves the following optimization problem for his/her own variable x^ν:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}) \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned} \tag{2.4}$$

where we denote $\theta_\nu(x) := \theta_\nu(x^\nu, x^{-\nu})$ to emphasize the particular role of x^ν in this problem. Let X denote the Cartesian product of all players' strategy sets X^ν, i.e.,

$$X := X^1 \times \cdots \times X^N. \tag{2.5}$$

Then an equilibrium concept for the NEP is defined as follows.

Definition 2.3. *A tuple of strategies $x^* := (x^{*,\nu})_{\nu=1}^N \in X$ is called a* Nash equilibrium *if*

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^\nu, x^{*,-\nu}), \quad \forall x^\nu \in X^\nu$$

hold simultaneously for all players $\nu = 1, \ldots, N$.

Definition 2.4. *A tuple of strategies $x^* := (x^{*,\nu})_{\nu=1}^N \in X$ is called a* stationary Nash equilibrium *if for each $\nu = 1, \ldots, N$, $x^{*,\nu}$ is a stationary point of the optimization problem* (2.4) *with $x^{-\nu} = x^{*,-\nu}$, where a stationary point means that it satisfies the first-order optimality condition for the problem.*

Under the assumption of the differentiability of the cost functions θ_ν and the convexity of the strategy sets X^ν, a stationary Nash equilibrium is characterized as a tuple $x^* = (x^{*,\nu})_{\nu=1}^N \in X$ that satisfies the following conditions for all $\nu = 1, \dots, N$:

$$\nabla_{x^\nu} \theta_\nu(x^{*,\nu}, x^{*,-\nu})^\top (x^\nu - x^{*,-\nu}) \geq 0, \quad \forall x^\nu \in X^\nu.$$

If, in addition, θ_ν is convex with respect to x^ν for each ν, then a stationary Nash equilibrium reduces to a Nash equilibrium. When θ_ν is non-differentiable, one needs to introduce a more general notion of stationarity; see [40] for more details.

For an N-person NEP, we have the following well-known result on the existence of a Nash equilibrium [4].

Lemma 2.1. *Suppose that for each player $\nu = 1, \dots, N$,*
(a) *the strategy set X^ν is nonempty, convex and compact;*
(b) *the objective function θ_ν is continuous;*
(c) *the objective function θ_ν is convex with respect to x^ν.*
Then, the NEP comprised of N players' problems (2.4) *has at least one Nash equilibrium.*

The following proposition shows a basic relation between the NEP and the VI [22].

Proposition 2.4. *Consider the NEP comprised of N players' problems* (2.4)*. If each strategy set X^ν is a nonempty, closed and convex subset of $\mathbb{R}^{n_\nu}$ and, for each fixed $x^{-\nu}$, the objective function $\theta_\nu(x^\nu, x^{-\nu})$ is convex and continuously differentiable with respect to x^ν, then a strategy tuple x is a Nash equilibrium if and only if x solves the VI(X, F), where X is given by* (2.5) *and $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined by*

$$F(x) := (\nabla_{x^\nu} \theta_\nu(x))_{\nu=1}^N.$$

In a NEP, if the strategy set of each player depends upon the strategies of his/her rivals, that is to say, for each player $\nu = 1, \dots, N$, his/her strategy set X^ν is replaced by $X^\nu(x^{-\nu})$, then the NEP is generalized as a GNEP, where each player $\nu = 1, \dots, N$ solves the following optimization problem for his/her own variable x^ν:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} && \theta_\nu(x^\nu, x^{-\nu}) \\ &\text{subject to} && x^\nu \in X^\nu(x^{-\nu}). \end{aligned}$$

Let $X(x) := X^1(x^{-1}) \times \cdots \times X^N(x^{-N})$ denote the Cartesian product of the strategy sets of all players. The corresponding equilibrium concept for the GNEP can be defined as follows.

Definition 2.5. *A tuple of strategies $x^* := (x^{*,\nu})_{\nu=1}^N \in X(x^*)$ is called a* generalized Nash equilibrium *of the GNEP if*

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^\nu, x^{*,-\nu}), \quad \forall x^\nu \in X^\nu(x^{-\nu})$$

hold simultaneously for all players $\nu = 1, \dots, N$.

2.3. Multi-leader-follower game and equilibrium problem with equilibrium constraints

In this subsection, we introduce the main topics of this paper, the multi-leader-follower game and the EPEC. Before doing so, we first introduce the Stackelberg game and the MPEC.

In a Stackelberg game, also called the single-leader-follower game, there are a distinctive player called the leader, who optimizes the upper-level problem, and several remaining players called followers, who optimize the lower-level problems jointly in a Nash noncooperative way. In particular, the leader makes the decision first by anticipating the response of the followers. At the same time, for the given leader's strategy, all followers select their own optimal responses while competing with each other. More precisely, in a general Stackelberg game with one leader and M followers, each follower $\omega = 1, \ldots, M$ solves the following optimization problem:

$$\begin{aligned} \underset{y^\omega}{\text{minimize}} \quad & \gamma_\omega(x, y^\omega, y^{-\omega}) \\ \text{subject to} \quad & y^\omega \in Y^\omega(x). \end{aligned} \tag{2.6}$$

Notice that the followers' objective functions and strategy sets depend on the leader's decision x. Moreover, we assume that each follower's strategy set $Y^\omega(x) \subseteq \mathbb{R}^{m_\omega}$ is closed and convex, and for any given x^* and $y^{*,-\omega}$, the objective function $\gamma_\omega(x^*, \cdot, y^{*,-\omega})$ is convex and continuously differentiable. Let $m := \sum_{\omega=1}^{M} m_\omega$ and denote $y := (y^\omega)_{\omega=1}^M \in \mathbb{R}^m$. By anticipating the optimal response vector $y(x)$ which comprises a Nash equilibrium in the lower-level, the leader solves the following optimization problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \theta(x, y) \\ \text{subject to} \quad & x \in X. \end{aligned} \tag{2.7}$$

One may define an equilibrium in the Stackelberg game as follows: Suppose that all players (the leader and the followers) have chosen their own strategies. There is no player who can reduce his/her cost by changing his/her current strategy unilaterally.

In a Stackelberg game, the leader chooses his/her strategy from the strategy set $X \subseteq \mathbb{R}^n$. For each one of the leader's strategy $x \in X$, the followers compete in the Nash noncooperative way. Then, by the convexity assumption on the followers' problems (2.6), the above Stackelberg game can be equivalently reformulated as the following MPEC:

$$\begin{aligned} \text{minimize} \quad & \theta(x, y) \\ \text{subject to} \quad & x \in X, \\ & y \text{ solves } \mathrm{VI}(Y(x), F(x, \cdot)), \end{aligned} \tag{2.8}$$

where for $y \in \mathbb{R}^m$ and $x \in X$,

$$F(x, y) := (\nabla_{y^\omega} \gamma_\omega(x, y^\omega, y^{-\omega}))_{\omega=1}^M,$$

and

$$Y(x) := \prod_{\omega=1}^{M} Y^\omega(x).$$

Generally, the MPEC is an optimization problem with two types of variables, called decision variables $x \in \mathbb{R}^n$ and response variables $y \in \mathbb{R}^m$, in which some or all of its constraints with respect to the response variables are expressed by a VI or CP parameterized by the decision variables. More precisely, this problem is stated as follows.

$$\begin{aligned} \text{minimize} \quad & \theta(x, y) \\ \text{subject to} \quad & (x, y) \in Z, \\ & y \text{ solves } \mathrm{VI}(Y(x), F(x, \cdot)), \end{aligned} \tag{2.9}$$

where $\theta : \mathbb{R}^{n+m} \to \mathbb{R}$ and $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are given functions, $Z \subseteq \mathbb{R}^{n+m}$ is a closed subset, and $Y : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ is a set-valued mapping from $\mathbb{R}^n$ to the set of non-empty closed convex subsets of $\mathbb{R}^m$.

As a generalization of the Stackelberg game, the multi-leader-follower game has several leaders in the upper-level. Similarly to the Stackelberg game, each leader can also anticipate the response of the followers, and uses this ability to select his/her strategy to compete with the other leaders in the Nash noncooperative way. At the same time, for a given strategy tuple of all leaders, each follower selects his/her own optimal response by competing with the other followers in the Nash noncooperative way, too. Formally, the multi-leader-follower game consists of N leaders and M followers, where each leader $\nu = 1, \ldots, N$ determines his/her decision variable $x^\nu \in \mathbb{R}^{n_\nu}$ and each follower $\omega = 1, \ldots, M$ determines his/her response variable $y^\omega \in \mathbb{R}^{m_\omega}$ to respond to the vector tuple $x := (x^1, \ldots, x^N) \in \mathbb{R}^n, n := \sum_{\nu=1}^N n_\nu$, which is formed by all leaders' decision variables. Sometimes we write $(x^\nu, x^{-\nu}) \in \mathbb{R}^{n_\nu + n_{-\nu}}, n_{-\nu} := n - n_\nu$ instead of $x \in \mathbb{R}^n$ in order to emphasize leader ν's decision variable $x^\nu \in \mathbb{R}^{n_\nu}$ in $x \in \mathbb{R}^n$. Similarly, we can also denote the vector tuple $y := (y^1, \ldots, y^M) \in \mathbb{R}^m, m := \sum_{\omega=1}^M m_\omega$, which is formed by all followers' response variables, and write $(y^\omega, y^{-\omega}) \in \mathbb{R}^{m_\omega + m_{-\omega}}, m_{-\omega} := m - m_\omega$ instead of $y \in \mathbb{R}^m$ in order to emphasize follower ω's response variable $y^\omega \in \mathbb{R}^{m_\omega}$ in $y \in \mathbb{R}^m$.

Depending on particular applications, the objective function of a leader or a follower is often called the utility function, payoff function, cost function or loss function. Each leader ν's objective function $\theta_\nu : \mathbb{R}^{n+m} \to \mathbb{R}$ is dependent upon his/her own decision variable x^ν and those of other leaders $x^{-\nu}$, as well as the response variables of all followers y. Similarly, each follower ω's objective function $\gamma_\omega : \mathbb{R}^{n+m} \to \mathbb{R}$ is also dependent upon his/her own response variable y^ω and those of other followers $y^{-\omega}$, as well as the decision variables of all leaders x.

Furthermore, each leader ν's strategy set, also called the feasible set or strategy space, denoted by $X^\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$, is dependent upon the strategies of other leaders $x^{-\nu}$, but independent of the strategies of all followers y. Each follower ω's strategy set, denoted by $Y^\omega(y^{-\omega}, x)$, depends upon the strategies of all leaders x. Under the above assumptions, for a given decision variable tuple x of N leaders, M followers compete in a parameterized Nash noncooperative way, where each follower ω solves the following optimization problem:

$$\begin{aligned} \underset{y^\omega}{\text{minimize}} \quad & \gamma_\omega(x, y^\omega, y^{-\omega}) \\ \text{subject to} \quad & y^\omega \in Y^\omega(x). \end{aligned} \tag{2.10}$$

By anticipating the optimal response vector $y(x)$, M leaders compete in a Nash noncooperative way, where each leader ν solves the following optimization problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} \quad & x^\nu \in X^\nu. \end{aligned} \tag{2.11}$$

For a given decision variable tuple x of leaders, we denote the set of Nash equilibria for the parameterized NEP consisting of M followers by $K(x)$. Then we can define a concept to describe a solution of the multi-leader-follower game.

Definition 2.6. *A strategy tuple* $(x^*, y^*) := (x^{*,1}, \ldots, x^{*,N}, y^{*,1}, \ldots, y^{*,M}) \in X \times Y(x^*) \subseteq \mathbb{R}^{n+m}$ *is called an* optimistic leader-follower Nash equilibrium *(optimistic L/F Nash equilibrium) or, more simply, an optimistic solution of the multi-leader-follower game if, for each leader* $\nu = 1, \ldots, N$, $x^{*,\nu}$ *is an optimal solution of the following optimization problem and* $y^* \in K(x^*)$*:*

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \underset{y \in K(x^\nu, x^{*,-\nu})}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{*,-\nu}, y) \\ \text{subject to} \quad & x^\nu \in X^\nu. \end{aligned} \tag{2.12}$$

The strategy tuple $(x^*, y^*) = (x^{*,1}, \dots, x^{*,N}, y^{*,1}, \dots, y^{*,M}) \in X \times Y(x^*) \subseteq \mathbb{R}^{n+m}$ *is called a* pessimistic leader-follower Nash equilibrium *(pessimistic L/F Nash equilibrium) or, more simply, a pessimistic solution of the multi-leader-follower game if, for each leader* $\nu = 1, \dots, N$, $x^{*,\nu}$ *is an optimal solution of the following optimization problem and* $y^* \in K(x^*)$*:*

$$\begin{array}{lll} \underset{x^\nu}{\text{minimize}} & \underset{y \in K(x^\nu, x^{*,-\nu})}{\text{maximize}} & \theta_\nu(x^\nu, x^{*,-\nu}, y) \\ \text{subject to} & x^\nu \in X^\nu. & \end{array} \tag{2.13}$$

From the above definition, we mention that the concept of optimistic (pessimistic) L/F Nash equilibrium is based on the assumption that each leader ν chooses his/her optimal strategy by anticipating the Nash equilibrium of the parameterized NEP consisting of the followers' problems optimistically (pessimistically). Therefore, each leader may choose a different Nash equilibrium $y^* \in K(x^*)$ as a response from the followers. However, such complication can be completely avoided in the case that the set $K(x)$ of Nash equilibria in the lower-level is a singleton for any x.

In a multi-leader-follower game comprised of (2.10) and (2.11), if each follower ω's problem (2.10) is smooth and convex with respect to his/her own variable y^ω for all feasible strategies of all leaders and the remaining followers, then the multi-leader-follower game can be reformulated as an EPEC as follows, by combining the first-order optimality conditions of the followers' problems:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} & x^\nu \in X^\nu, \\ & y \text{ solves } \mathrm{VI}(Y(x), F(x, \cdot)), \end{array} \tag{2.14}$$

where $Y(x) := \prod_{\omega=1}^M Y^\omega(x)$, and $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is defined by

$$F(x, y) := \begin{pmatrix} \nabla_{y^1} \gamma_1(x, y^1, y^{-1}) \\ \vdots \\ \nabla_{y^M} \gamma_M(x, y^M, y^{-M}) \end{pmatrix}. \tag{2.15}$$

For each feasible point $x \in \mathbb{R}^n$, y solves $\mathrm{VI}(Y(x), F(x, \cdot))$ if and only if $y \in Y(x)$ and the following inequalities hold:

$$(z - y)^\top F(x, y) \geq 0, \quad \forall z \in Y(x). \tag{2.16}$$

Generally, the EPEC can be looked on as a generalization of the NEP or GNEP, where each player solves his/her own MPEC simultaneously, and equilibrium constraints consisting of a VI or CP parameterized by the decision variable x may be different from those of the other players. In particular, we may consider an EPEC with shared identical equilibrium constraints. More precisely, in such an EPEC, each leader solves the following optimization problem:

$$\begin{array}{ll} \underset{x^\nu, y}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} & (x^\nu, y) \in Z^\nu, \\ & y \text{ solves } \mathrm{VI}(Y(x), F(x, \cdot)), \end{array} \tag{2.17}$$

where $y \in \mathbb{R}^m$ is the shared response variable. For each leader $\nu = 1, \dots, N$, $\theta_\nu : \mathbb{R}^{n+m} \to \mathbb{R}$ and $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are given functions, $Z^\nu \subseteq \mathbb{R}^{n_\nu + m}$, and $Y : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ is a set-valued mapping from $\mathbb{R}^n$ to the set of non-empty closed convex subsets of $\mathbb{R}^m$.

2.4. Nash equilibrium problem and multi-leader-follower game with uncertainty

In this subsection, we describe the NEP and the multi-leader-follower game with uncertainty under the incomplete information assumption.

In the above two problems, Nash equilibrium, generalized Nash equilibrium, or L/F Nash equilibrium is well-defined when all players seek their own optimal strategies simultaneously by observing and estimating the opponents' strategies, as well as the values of their own objective functions, exactly. However, in many real-world models, such information may contain some uncertain parameters, because of observation errors and/or estimation errors.

To deal with some uncertainty in the NEP, Nishimura, Hayashi and Fukushima [55] considered a robust Nash equilibrium problem and defined the corresponding equilibrium called robust Nash equilibrium. Here we briefly explain it under the following assumption:

> A parameter $u^\nu \in \mathbb{R}^{l_\nu}$ is involved in player ν's objective function, which is now expressed as $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^{l_\nu} \to \mathbb{R}$. Although player ν does not know the exact value of parameter u^ν, yet he/she can confirm that it must belong to a given nonempty set $U^\nu \subseteq \mathbb{R}^{l_\nu}$.

Then, player ν solves the following optimization problem with parameter u^ν for his/her own variable x^ν:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} && \theta_\nu(x^\nu, x^{-\nu}, u^\nu) \\ &\text{subject to} && x^\nu \in X^\nu, \end{aligned} \tag{2.18}$$

where $u^\nu \in U^\nu$. According to the robust optimization paradigm, we assume that each player ν tries to minimize the worst value of his/her objective function. Under this assumption, each player ν considers the worst cost function $\tilde{\theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to (-\infty, +\infty]$ defined by

$$\tilde{\theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\theta_\nu(x^\nu, x^{-\nu}, u^\nu) \mid u^\nu \in U^\nu\}$$

and solves the following optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} && \tilde{\theta}_\nu(x^\nu, x^{-\nu}) \\ &\text{subject to} && x^\nu \in X^\nu. \end{aligned} \tag{2.19}$$

Since this is viewed as a NEP with complete information, we can define the equilibrium of the NEP with uncertain parameters as follows.

Definition 2.7. *A strategy tuple $x = (x^\nu)_{\nu=1}^N \in X$ is called a* robust Nash equilibrium *of the non-cooperative game comprised of problems* (2.18)*, if x is a Nash equilibrium of the NEP comprised of problems* (2.19)*. The problem of finding a robust Nash equilibrium is called the robust Nash equilibrium problem.*

Next, we further consider a multi-leader-follower game with N leaders and M followers under incomplete information, where leader $\nu = 1, \ldots, N$ tries to solve the following uncertain optimization problem for his/her own variable x^ν:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} && \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) \\ &\text{subject to} && x^\nu \in X^\nu. \end{aligned} \tag{2.20}$$

Here an uncertain parameter $u^\nu \in \mathbb{R}^{l_\nu}$ appears in the objective function $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^{m_\omega} \times \mathbb{R}^{l_\nu} \to \mathbb{R}$. We assume that although leader ν does not know the exact value of parameter u^ν, yet he/she can confirm that it must belong to a given nonempty set $U^\nu \subseteq \mathbb{R}^{l_\nu}$.

On the other hand, given the leaders' strategies $x = (x^\nu)_{\nu=1}^N$, follower $\omega = 1, \ldots, M$ solves the following optimization problem for his/her own variable y^ω:

$$\begin{array}{ll} \underset{y^\omega}{\text{minimize}} & \gamma_\omega(x, y^\omega, y^{-\omega}) \\ \text{subject to} & y^\omega \in Y^\omega(x). \end{array}$$

Here we assume that, although all followers respond to the leaders' strategies with his/her optimal strategy, each leader cannot anticipate the response of the followers exactly because of some observation errors and/or estimation errors. Consequently, each leader ν estimates that follower ω solves the following uncertain optimization problem for variable $y^{\nu,\omega}$:

$$\begin{array}{ll} \underset{y^{\nu,\omega}}{\text{minimize}} & \gamma_{\nu,\omega}(x, y^{\nu,\omega}, y^{\nu,-\omega}, v^\nu) \\ \text{subject to} & y^{\nu,\omega} \in Y^\omega(x), \end{array} \tag{2.21}$$

where an uncertain parameter $v^\nu \in \mathbb{R}^{k_\nu}$ appears in the objective function $\gamma_{\nu,\omega} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{k_\nu} \to \mathbb{R}$ which is conceived by leader ν, and $y^{\nu,\omega} \in \mathbb{R}^{m_\omega}$ means the follower ω' strategy anticipated by leader ν. We assume that although leader ν cannot know the exact value of v^ν, yet he/she can estimate that it belongs to a given nonempty set $V^\nu \subseteq \mathbb{R}^{k_\nu}$. It should be emphasized that the uncertain parameter v^ν is associated with leader ν, which means the leaders may estimate the follower's problem differently. Hence, the followers' responses anticipated by a leader may be different from the one anticipated by another leader.

In the follower's problem (2.21) anticipated by leader ν, we assume that for any fixed $x \in X$ and $v^\nu \in V^\nu$, $\gamma_{\nu,\omega}(x, \cdot, v^\nu)$ is a strictly convex function, and $Y^\omega(x)$ is a nonempty, closed, convex set. That is, problem (2.21) is a strictly convex optimization problem parameterized by x and v^ν. We denote a Nash equilibrium in the lower-level game comprised of problems (2.21) by $y^\nu(x, v^\nu)$, which we assume to exist uniquely.

Consequently, the above multi-leader-follower game with uncertainty can be reformulated as a robust Nash equilibrium problem, where each player ν solves the following uncertain optimization problem for his/her own variable x^ν:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \\ \text{subject to} & x^\nu \in X^\nu, \end{array} \tag{2.22}$$

with uncertain parameters $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$.

By means of the robust optimization paradigm, we define the worst cost function $\tilde{\Theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to (-\infty, +\infty]$ for each player ν as follows:

$$\tilde{\Theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \,|\, u^\nu \in U^\nu, v^\nu \in V^\nu\}.$$

Thus, we obtain a NEP with complete information, where each player ν solves the following optimization problem:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \tilde{\Theta}_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & x^\nu \in X^\nu. \end{array} \tag{2.23}$$

Moreover, we can define an equilibrium for the multi-leader-follower game with uncertainty comprised of problems (2.20) and (2.21) as follows.

Definition 2.8. *A strategy tuple $x = (x^\nu)_{\nu=1}^N \in X$ is called a* robust L/F Nash equilibrium *of the multi-leader-follower game with uncertainty comprised of problems* (2.20) *and* (2.21), *if x is a robust Nash equilibrium of the NEP with uncertainty comprised of problems* (2.22), *i.e., a Nash equilibrium of the NEP comprised of problems* (2.23).

3. Applications of Multi-Leader-Follower Games

In this section, we introduce the applications of the multi-leader-follower game and the related EPEC by considering some examples from two specific aspects; electricity power markets and telecommunication industry.

3.1. Electricity power market

Privatization and restructuring of the deregulated electricity markets have become popular in many countries and areas. Several researchers have studied this kind of problems by means of the multi-leader-follower game and the equilibrium problem with equilibrium constraints; see [37, 42], Here, based on the model in [60], we introduce an approach to dealing with the electricity power market as the multi-leader-follower game and the equilibrium problem with equilibrium constraints, which is further extended in [39].

In this model, there are several firms and one market maker, called the independent system operator (ISO for short), who employs a market clearing mechanism to collect the electricity from firms by paying the bid costs, determines the price of electricity, and sells it to consumers. For simplicity, we omit the problem of consumers, which means any quantity of electricity power can be consumed. The structure of the model can be described as follows. Again, for simplicity, we assume there are only two firms I and II. The two firms are competing for market power in an electricity network with M nodes. The vector $q = (q^{\mathrm{I}}, q^{\mathrm{II}}) \in \mathcal{Q}$ with $q^\nu = (q_1^\nu, \dots, q_M^\nu)^\top$, where each firm $\nu = \mathrm{I}, \mathrm{II}$ supplies electricity quantity q_i^ν to each node $i = 1, \dots, M$ and $\mathcal{Q}$ is the set of feasible supplies from the firms. Let $\rho^\nu = (\rho_1^\nu, \dots, \rho_M^\nu)^\top \in \Omega^\nu$ denote firm ν's bid parameter vector, where the component ρ_i^ν is the bid parameter from player ν to node $i = 1, \dots, M$ and Ω^ν is the admissible set of ρ^ν. Each firm will submit a bid function $b_\nu(q, \rho^\nu)$ to the ISO, which represents how much revenue firm ν will receive. At the same time, we assume that the transaction cost for player ν is $\omega_\nu(\rho^\nu)$. Then each firm $\nu = \mathrm{I}, \mathrm{II}$ tries to minimize the difference between its transaction cost and revenue by determining its bid parameter vector ρ^ν, and solves the following optimization problem:

$$\begin{aligned} \underset{\rho^\nu}{\text{minimize}} \quad & \omega_\nu(\rho^\nu) - b_\nu(q, \rho^\nu) \\ \text{subject to} \quad & \rho^\nu \in \Omega^\nu. \end{aligned} \tag{3.1}$$

We further assume that, at each node, the affine demand curves determine the prices p_i as the following function of the total quantity of electricity from firms I and II:

$$p_i(q_i^{\mathrm{I}}, q_i^{\mathrm{II}}) := \alpha_i - \beta_i(q_i^{\mathrm{I}} + q_i^{\mathrm{II}}), \; i = 1, \dots, M,$$

where α_i and β_i are given positive constants. Then the ISO tries to minimize its negative profit by solving the following optimization problem:

$$\begin{aligned} \underset{q=(q^{\mathrm{I}}, q^{\mathrm{II}})}{\text{minimize}} \quad & \sum_{i=1}^{M} \left[\frac{\beta_i}{2}(q_i^{\mathrm{I}} + q_i^{\mathrm{II}})^2 - \alpha_i(q_i^{\mathrm{I}} + q_i^{\mathrm{II}}) \right] + b_{\mathrm{I}}(q, \rho^{\mathrm{I}}) + b_{\mathrm{II}}(q, \rho^{\mathrm{II}}) \\ \text{subject to} \quad & q \in \mathcal{Q}. \end{aligned} \tag{3.2}$$

Altogether, (3.1) and (3.2) represent a multi-leader-follower game with two firms as leaders and the ISO as a single follower.

3.2. Telecommunication market

The second example comes from the telecommunication market, introduced by DeMiguel and Xu [16], where they considered a stochastic multi-leader-follower game. In this model, there are two types of telecommunications companies. Some well established companies (leaders) with sound assets run at full network capacity, without any spare capacity for some new service technology. Therefore, when a new technology, such as bandwidth, enters the telecommunication market, the leaders have to make a decision immediately as to whether they offer this new service to the customers by investing in expanding their current network or install a new network. Since the capacity expansion process takes up too much time, the leaders also have to make a decision on the quantity that they will supply to consumers in advance. It further induces that the leaders can only know their demand function with its probability distribution. On the other hand, the other newer and more fragile companies (followers) have sufficient network capacity for the new service and what they only need to do is to decide their capacity transformed from the existing services to the new one. Under this situation, the followers have a room to make this decision after observing the supplying quantities and the realized demand functions of leaders.

In this telecommunication market, there are N leaders and M followers. The cost functions of each leader $\nu = 1, \ldots, N$ and each follower $\omega = 1, \ldots, M$ are represented by $C_\nu(x^\nu)$ and $c_\omega(y^\omega)$, where $x^\nu \in \mathbb{R}^{n_\nu}$ and $y^\omega \in \mathbb{R}^{m_\omega}$ denote the variables of leader ν and follower ω, respectively. Let $\bar{x} := \sum_{\nu=1}^N x^\nu$ and $\bar{x}^{-\nu} := \sum_{i=1, i\neq\nu}^N x^i$ denote the aggregate supplies of all leaders and those excluding leader ν, respectively. Also let $\bar{y}$ and $\bar{y}^{-\omega}$ denote the corresponding quantities for the followers. Since the leaders have no capacity for the new service and they have to make decisions for the quantities they will supply in advance. We assume that the market price is denoted by $p(q, \xi(u))$, where q is the total quantities supplied by all leaders and followers, and $\xi : \Omega \to \mathbb{R}$ is a continuous stochastic variable, where Ω is a sample space. Then leader ν considers the following optimization problem:

$$\begin{array}{ll} \underset{x^\nu}{\text{maximize}} & \phi_\nu(x^\nu, \bar{x}^{-\nu}) := \mathbb{E}[x^\nu p(x^\nu + \bar{x}^{-\nu} + \bar{y}(x^\nu + \bar{x}^{-\nu}, \xi(u)), \xi(u))] - C_\nu(x^\nu) \\ \text{subject to} & x^\nu \geq 0, \end{array} \tag{3.3}$$

where $\mathbb{E}$ denotes the expectation with respect to the random variable ξ, and $\bar{y}(\bar{x}, \xi(u))$ is the aggregate quantities supplied by all followers for the given aggregate quantities of leaders $\bar{x}$ and a realization of the random variable $\xi(u)$.

Since the followers have enough capacity for the new service, they can choose their supply quantities after observing the aggregate quantities supplied by all leaders and the realized market price. Then follower ω considers the following optimization problem:

$$\begin{array}{ll} \underset{y^\omega}{\text{maximize}} & \psi_\omega(y^\omega, \bar{y}^{-\omega}, \xi(u)) := y^\omega p(\bar{x} + y^\omega + \bar{y}^{-\omega}, \xi(u)) - c_\omega(y^\omega) \\ \text{subject to} & y^\omega \geq 0. \end{array} \tag{3.4}$$

Altogether, (3.3) and (3.4) represent a multi-leader-following game with N leaders and M followers.

4. Methods for Multi-Leader-Follower Games

In this section, we introduce some recently proposed methods to solve the multi-leader-follower games.

4.1. Variational inequality formulation for multi-leader-follower games

In this subsection, we introduce a variational inequality approach to solve a class of multi-leader-follower games [39]. This game has N leaders (for simplicity of presentation, we set $N = 2$ below) and a single follower, who solve the following optimization problems with their own variables $x^{\mathrm{I}} \in \mathbb{R}^{n_{\mathrm{I}}}$, $x^{\mathrm{II}} \in \mathbb{R}^{n_{\mathrm{II}}}$ and $y \in \mathbb{R}^m$, respectively:

Leader I's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{I}}}{\text{minimize}} & f_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^\top D_{\mathrm{I}} y \\ \text{subject to} & g^{\mathrm{I}}(x^{\mathrm{I}}) \leq 0,\ h^{\mathrm{I}}(x^{\mathrm{I}}) = 0. \end{array}$$

Leader II's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{II}}}{\text{minimize}} & f_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ \text{subject to} & g^{\mathrm{II}}(x^{\mathrm{II}}) \leq 0,\ h^{\mathrm{II}}(x^{\mathrm{II}}) = 0. \end{array}$$

Follower's Problem.

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \frac{1}{2} y^\top B y + c^\top y - (x^{\mathrm{I}})^\top D_{\mathrm{I}} y - (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ \text{subject to} & Ay + a = 0. \end{array}$$

Here, for each leader $\nu = \mathrm{I}, \mathrm{II}$, $f_\nu : \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \mathbb{R}$ is twice continuously differentiable and convex with respect to the variable x^ν, $g^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}^{s_\nu}$ is convex, and $h^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}^{t_\nu}$ is affine. Matrix $B \in \mathbb{R}^{m \times m}$ is assumed to be symmetric and positive definite. $D_\nu \in \mathbb{R}^{n_\nu \times m}$, $c \in \mathbb{R}^m$, $a \in \mathbb{R}^p$, and matrix $A \in \mathbb{R}^{p \times m}$ has full row rank.

For a given vector $x = (x^{\mathrm{I}}, x^{\mathrm{II}}) \in \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}}$, by substituting the unique optimal response $y(x^{\mathrm{I}}, x^{\mathrm{II}})$ for y in the leaders' problems, the above multi-leader-follower game can be reformulated as the following NEP denoted by $\mathrm{NEP}(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$, where the strategy sets X^ν are defined by $X^\nu = \{x^\nu : g^\nu(x^\nu) \leq 0, h^\nu(x^\nu) = 0\}, \nu = \mathrm{I}, \mathrm{II}$.

Leader I's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{I}}}{\text{minimize}} & \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} & g^{\mathrm{I}}(x^{\mathrm{I}}) \leq 0,\ h^{\mathrm{I}}(x^{\mathrm{I}}) = 0. \end{array}$$

Leader II's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{II}}}{\text{minimize}} & \Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} & g^{\mathrm{II}}(x^{\mathrm{II}}) \leq 0,\ h^{\mathrm{II}}(x^{\mathrm{II}}) = 0. \end{array}$$

Here, the leaders' objective functions are expressed as follows:

$$\begin{aligned} \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &:= f_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^\top D_{\mathrm{I}} r + (x^{\mathrm{I}})^\top D_{\mathrm{I}} G x^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} H x^{\mathrm{II}}, \\ \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &:= f_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^\top D_{\mathrm{II}} r + (x^{\mathrm{II}})^\top D_{\mathrm{II}} G x^{\mathrm{I}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} H x^{\mathrm{II}}, \end{aligned}$$

where $G \in \mathbb{R}^{m \times n_{\mathrm{I}}}$, $H \in \mathbb{R}^{m \times n_{\mathrm{II}}}$, and $r \in \mathbb{R}^m$ are given by

$$\begin{aligned} G &= B^{-1}(D_{\mathrm{I}})^T - B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{I}})^T, \\ H &= B^{-1}(D_{\mathrm{II}})^T - B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{II}})^T, \\ r &= -B^{-1}c - B^{-1}A^T(AB^{-1}A^T)^{-1}(a - AB^{-1}c). \end{aligned}$$

By Proposition 2.4, the above NEP can be further reformulated as the following VI denoted by VI$(X, \hat{F})$: Find a vector $x^* \in X := X^{\rm I} \times X^{\rm II}$ such that

$$\hat{F}(x^*)^\top (x - x^*) \geq 0 \quad \forall\, x \in X,$$

where function $\hat{F} : \mathbb{R}^{n_{\rm I}+n_{\rm II}} \to \mathbb{R}^{n_{\rm I}+n_{\rm II}}$ is defined by

$$\hat{F}(x) := \begin{pmatrix} \nabla_{x^{\rm I}} \Theta_{\rm I}(x^{\rm I}, x^{\rm II}) \\ \nabla_{x^{\rm II}} \Theta_{\rm II}(x^{\rm I}, x^{\rm II}) \end{pmatrix} = \begin{pmatrix} \nabla_{x^{\rm I}} f_{\rm I}(x^{\rm I}, x^{\rm II}) + D_{\rm I} r + 2D_{\rm I} G x^{\rm I} + D_{\rm I} H x^{\rm II} \\ \nabla_{x^{\rm II}} f_{\rm II}(x^{\rm I}, x^{\rm II}) + D_{\rm II} r + D_{\rm II} G x^{\rm I} + 2D_{\rm II} H x^{\rm II} \end{pmatrix}.$$

Then one can establish some existence and uniqueness results on the Nash equilibrium, as well as the L/F Nash equilibrium for the multi-leader-follower game, by Propositions 2.1 and 2.2.

Theorem 4.1. *If function $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$F_0(x) = F_0(x^{\rm I}, x^{\rm II}) := \begin{pmatrix} \nabla_{x^{\rm I}} f_{\rm I}(x^{\rm I}, x^{\rm II}) \\ \nabla_{x^{\rm II}} f_{\rm II}(x^{\rm I}, x^{\rm II}) \end{pmatrix} \tag{4.1}$$

is strictly monotone, and NEP$(\Theta_\nu, X^\nu)_{\nu={\rm I}}^{\rm II}$ has at least one Nash equilibrium, then the Nash equilibrium is unique.

Theorem 4.2. *If function F_0 defined by* (4.1) *is strongly monotone, then NEP$(\Theta_\nu, X^\nu)_{\nu={\rm I}}^{\rm II}$ has a unique Nash equilibrium.*

4.2. Generalized variational inequality formulation for multi-leader-follower games with uncertainty

In this subsection, the results in the previous subsection are generalized to a multi-leader-follower game with incomplete information. For more details, the reader may refer to [41].

For simplicity, we concentrate on a multi-leader-follower game with uncertainty comprised of two leaders and one follower. It can be extended to the case of more than two leaders and multiple followers in a straightforward manner. In this multi-leader-follower game, each leader $\nu = {\rm I}, {\rm II}$ solves the following optimization problem for his/her own variable x^ν:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) \\ &\text{subject to} \quad x^\nu \in X^\nu. \end{aligned} \tag{4.2}$$

Here, uncertainty parameter $u^\nu \in U^\nu \subseteq \mathbb{R}^{l_\nu}$ appears in leader ν's objective function due to lack of complete information. For a given strategy vector $x = (x^{\rm I}, x^{\rm II})$ of the leaders, the follower chooses his/her strategy by solving the following optimization problem for variable y:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \gamma(x, y) \\ &\text{subject to} \quad y \in K(x). \end{aligned}$$

However, due to lack of information again, each leader ν can only estimate that the follower solves the following optimization problem for variable y:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \gamma^\nu(x, y, v^\nu) \\ &\text{subject to} \quad y \in K(x). \end{aligned} \tag{4.3}$$

Here, uncertainty parameter $v^\nu \in V^\nu \subseteq \mathbb{R}^{k_\nu}$ appears in the follower's objective function.

In the follower's problem anticipated by leader ν, it is further assumed that for any fixed strategy vector $x \in X$ and uncertainty parameter $v^\nu \in V^\nu$, the anticipated follower's objective function $\gamma^\nu(x, \cdot, v^\nu)$ is a strictly convex function, and $K(x)$ is a nonempty, closed, convex set. That is, problem (4.3) is a strictly convex optimization problem parameterized by x and v^ν. Its unique optimal solution is denoted by $y^\nu(x, v^\nu)$, which is assumed to exist.

Therefore, the above multi-leader-follower game with uncertain data can be reformulated as a Nash equilibrium problem where each player ν solves the following uncertain optimization problem for his/her own variable x^ν:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned}$$

with uncertain parameters $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$.

By means of the robust optimization paradigm, the worst cost function $\tilde{\Theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to (-\infty, +\infty]$ for each player ν is defined as follows:

$$\tilde{\Theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \,|\, u^\nu \in U^\nu, v^\nu \in V^\nu\}.$$

This yields a NEP with complete information, where each player $\nu = \mathrm{I}, \mathrm{II}$ solves the following optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \tilde{\Theta}_\nu(x^\nu, x^{-\nu}) \\ &\text{subject to} \quad x^\nu \in X^\nu. \end{aligned}$$

In order to discuss the existence of robust L/F Nash equilibrium for the above multi-leader-follower game, the following assumption is made:

Assumption 4.1. *For each leader ν, the following conditions hold.*
(a) *The functions $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^m \times \mathbb{R}^{l_\nu} \to \mathbb{R}$ and $y^\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^{k_\nu} \to \mathbb{R}^m$ are both continuous.*
(b) *The uncertainty sets $U^\nu \subseteq \mathbb{R}^{l_\nu}$ and $V^\nu \subseteq \mathbb{R}^{k_\nu}$ are both nonempty and compact.*
(c) *The strategy set X^ν is nonempty, compact and convex.*
(d) *The function $\Theta_\nu(\cdot, x^{-\nu}, v^\nu, u^\nu) : \mathbb{R}^{n_\nu} \to \mathbb{R}$ is convex for any fixed $x^{-\nu}$, v^ν, and u^ν.*

Then the existence of a robust L/F Nash equilibrium is established as follows.

Theorem 4.3. *If Assumption 4.1 holds, then the robust multi-leader-follower game comprised of problems* (4.2) *and* (4.3) *has at least one robust L/F Nash equilibrium.*

In the remainder of this subsection, the uniqueness of a robust L/F Nash equilibrium is discussed for the special class of multi-leader-follower games with uncertainty, where each leader $\nu = \mathrm{I}, \mathrm{II}$ is assumed to solve the following optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top E_\nu x^{-\nu} + (x^\nu)^\top R_\nu u^\nu + (x^\nu)^\top D_\nu y \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned} \tag{4.4}$$

where y is an optimal solution of the following follower's problem anticipated by leader ν:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \frac{1}{2}y^\top B y + (c + v^\nu)^\top y - (x^{\mathrm{I}})^\top D_{\mathrm{I}} y - (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ &\text{subject to} \quad Ay + a = 0, \end{aligned} \tag{4.5}$$

where $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$, $\nu = \mathrm{I}, \mathrm{II}$.

The follower's problems estimated by two leaders are both strictly convex quadratic programming problems with equality constraints. The Karush-Kuhn-Tucker conditions of those problems are systems of linear equations, which can be solved uniquely for y^ν, yielding the unique optimal response $y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu)$ of the follower anticipated by each leader ν. Then, by substituting $y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu)$ for y in the respective leader's problem, the above multi-leader single-follower game with uncertainty can be formulated as a NEP with uncertainty, where, as the νth player, leader ν solves the following optimization problem:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) \\ \text{subject to} & x^\nu \in X^\nu, \end{array}$$

where $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$, $\nu = \mathrm{I}, \mathrm{II}$, and leader ν's objective function can be rewritten as

$$\begin{aligned} \Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) &:= \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \\ &= \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top D_\nu G_\nu x^\nu + (x^\nu)^\top R_\nu u^\nu + (x^\nu)^\top D_\nu r \\ &\quad + (x^\nu)^\top (D_\nu G_{-\nu} + E_\nu) x^{-\nu} - (x^\nu)^\top D_\nu B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^\nu. \end{aligned}$$

Here, $G_{\mathrm{I}} \in \mathbb{R}^{m \times n_{\mathrm{I}}}$, $G_{\mathrm{II}} \in \mathbb{R}^{m \times n_{\mathrm{II}}}$, and $r \in \mathbb{R}^m$ are given by

$$\begin{aligned} G_{\mathrm{I}} &= B^{-\frac{1}{2}} P B^{-\frac{1}{2}} (D_{\mathrm{I}})^\top, \\ G_{\mathrm{II}} &= B^{-\frac{1}{2}} P B^{-\frac{1}{2}} (D_{\mathrm{II}})^\top, \\ r &= -B^{-\frac{1}{2}} P B^{-\frac{1}{2}} c - B^{-1} A^\top (A B^{-1} A^\top)^{-1} a, \end{aligned}$$

and matrix P is defined as

$$P := I - B^{-\frac{1}{2}} A^\top (A B^{-1} A^\top)^{-1} A B^{-\frac{1}{2}}.$$

By means of the robust optimization technique, one can construct the robust counterpart of the above NEP with uncertainty, which is a NEP with complete information, where each leader ν solves the following optimization problem:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \tilde{\Theta}_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & x^\nu \in X^\nu. \end{array}$$

Here, functions $\tilde{\Theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ are defined by

$$\begin{aligned} \tilde{\Theta}_\nu(x^\nu, x^{-\nu}) &:= \sup\{\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) \mid u^\nu \in U^\nu, v^\nu \in V^\nu\} \\ &= \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top D_\nu G_\nu x^\nu + (x^\nu)^\top D_\nu r \\ &\quad + (x^\nu)^\top (D_\nu G_{-\nu} + E_\nu) x^{-\nu} + \phi_\nu(x^\nu), \end{aligned}$$

where $\phi_\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ are given by

$$\begin{aligned} \phi_\nu(x^\nu) &:= \sup\{(x^\nu)^\top R_\nu u^\nu \mid u^\nu \in U^\nu\} \\ &\quad + \sup\{-(x^\nu)^\top D_\nu B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^\nu \mid v^\nu \in V^\nu\}. \end{aligned}$$

The following theorem shows the existence of a robust L/F Nash equilibrium.

Theorem 4.4. *Suppose that for each $\nu = \mathrm{I}, \mathrm{II}$, the strategy set X^ν is nonempty, compact and convex, the matrix $H_\nu \in \mathbb{R}^{n_\nu \times n_\nu}$ is symmetric and positive semidefinite, and the uncertainty sets U^ν and V^ν are nonempty and compact. Then, the multi-leader single-follower game with uncertainty comprised of problems* (4.4) *and* (4.5) *has at least one robust L/F Nash equilibrium.*

By the convexity of objective function $\tilde{\Theta}_\nu$ of each leader ν with respect to x^ν, one can investigate the uniqueness of a robust L/F Nash equilibrium by considering the following GVI problem which is formulated by concatenating the first-order optimality conditions of all leaders' problems: Find a vector $x^* = (x^{*,\mathrm{I}}, x^{*,\mathrm{II}}) \in X := X^\mathrm{I} \times X^\mathrm{II}$ such that

$$\exists\, \xi \in \tilde{\mathcal{F}}(x^*), \quad \xi^\top (x - x^*) \geq 0, \quad \forall x \in X,$$

where $\xi = (\xi^\mathrm{I}, \xi^\mathrm{II}) \in \mathbb{R}^n$, $x = (x^\mathrm{I}, x^\mathrm{II}) \in \mathbb{R}^n$, and the set-valued mapping $\tilde{\mathcal{F}} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is defined by $\tilde{\mathcal{F}}(x) := \partial_{x^\mathrm{I}} \tilde{\Theta}_\mathrm{I}(x^\mathrm{I}, x^\mathrm{II}) \times \partial_{x^\mathrm{II}} \tilde{\Theta}_\mathrm{II}(x^\mathrm{I}, x^\mathrm{II})$.

If mapping $\tilde{\mathcal{F}}$ is strictly monotone, then Proposition 2.3 ensures the uniqueness of a robust L/F Nash equilibrium. Since the subdifferentials $\partial \phi_\mathrm{I}$ and $\partial \phi_\mathrm{II}$ are monotone, $\tilde{\mathcal{F}}$ is strictly monotone if the following mapping $T : \mathbb{R}^{n_\mathrm{I} + n_\mathrm{II}} \to \mathbb{R}^{n_\mathrm{I} + n_\mathrm{II}}$ is strictly monotone:

$$T(x) := \begin{pmatrix} T_\mathrm{I}(x^\mathrm{I}, x^\mathrm{II}) \\ T_\mathrm{II}(x^\mathrm{I}, x^\mathrm{II}) \end{pmatrix},$$

where the mappings $T_\mathrm{I} : \mathbb{R}^{n_\mathrm{I}} \times \mathbb{R}^{n_\mathrm{II}} \to \mathbb{R}^{n_\mathrm{I}}$ and $T_\mathrm{II} : \mathbb{R}^{n_\mathrm{I}} \times \mathbb{R}^{n_\mathrm{II}} \to \mathbb{R}^{n_\mathrm{II}}$ are expressed as

$$\begin{aligned} T_\mathrm{I}(x^\mathrm{I}, x^\mathrm{II}) &:= H_\mathrm{I} x^\mathrm{I} + D_\mathrm{I} r + 2 D_\mathrm{I} G_\mathrm{I} x^\mathrm{I} + (D_\mathrm{I} G_\mathrm{II} + E_\mathrm{I}) x^\mathrm{II}, \\ T_\mathrm{II}(x^\mathrm{I}, x^\mathrm{II}) &:= H_\mathrm{II} x^\mathrm{II} + D_\mathrm{II} r + (D_\mathrm{II} G_\mathrm{I} + E_\mathrm{II}) x^\mathrm{I} + 2 D_\mathrm{II} G_\mathrm{II} x^\mathrm{II}. \end{aligned}$$

In fact, the strict monotonicity of mapping T is ensured if the matrix

$$\mathcal{J} := \begin{pmatrix} H_\mathrm{I} & E_\mathrm{I} \\ E_\mathrm{II} & H_\mathrm{II} \end{pmatrix} \tag{4.6}$$

is positive definite.

Consequently, one can establish the uniqueness of a robust L/F Nash equilibrium.

Theorem 4.5. *Suppose that matrix $\mathcal{J}$ defined by* (4.6) *is positive definite, and the uncertainty sets U^ν and V^ν are nonempty and compact. Then the multi-leader single-follower game with uncertainty comprised of problems* (4.4) *and* (4.5) *has a unique robust L/F Nash equilibrium.*

5. Final Remarks

The multi-leader-follower game is a vigorous tool to model many real-world problems. However, the study of this field is still in its infancy, since the complex structure of the multi-leader-follower game makes it difficult to deal with general problems. We believe that to study those problems which have certain particular structures coming from some real-world applications will be a bright avenue. Finally, we admit that the materials of the paper are biased and some important results are omitted due to lack of space. Nevertheless, we hope that the readers catch something about this interesting and important problem from the paper.

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