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A STRUCTURAL GEOMETRICAL ANALYSIS OF WEAKLY INFEASIBLE SDPS

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Abstract In this article, we develop a detailed analysis of semidefinite feasibility problems (SDFPs) to understand how weak infeasibility arises in semidefinite programming. This is done by decomposing a SDFP into smaller problems, in a way that preserves most feasibility properties of the original problem. The decomposition utilizes a set of vectors (computed in the primal space) which arises when we apply the facial reduction algorithm to the dual feasible region. In particular, we show that for a weakly infeasible problem over $n \times n$ matrices, at most $n-1$ directions are required to approach the positive semidefinite cone. We also present a discussion on feasibility certificates for SDFPs and related complexity results.

Keywords: Optimization, weak infeasibility, semidefinite programming

1. Introduction

In this paper, we deal with the following semidefinite feasibility problem

$$\max\ 0 \ \text{ s.t. } \ x \in (L+c) \cap K_n, \tag{1}$$

where $L \subseteq \mathbb{S}_n$ is a vector subspace and $c \in \mathbb{S}_n$. By $\mathbb{S}_n$ we denote the linear space of $n \times n$ real symmetric matrices and $K_n \subseteq \mathbb{S}_n$ denotes the cone of $n \times n$ positive semidefinite matrices. We denote the problem (1) by (K_n, L, c).

We will focus on the case where (K_n, L, c) is weakly infeasible. It is known that every instance of a semidefinite program falls into one of the following four statuses:

- *strongly feasible*: $(L+c) \cap \operatorname{int}(K_n) \neq \emptyset$, where $\operatorname{int}(K_n)$ denotes the interior of K_n;
- *weakly feasible*: $(L+c) \cap \operatorname{int}(K_n) = \emptyset$, but $(L+c) \cap K_n \neq \emptyset$;
- *weakly infeasible*: $(L+c) \cap K_n = \emptyset$ and $\operatorname{dist}(K, L+c) = 0$;
- *strongly infeasible*: $(L+c) \cap K_n = \emptyset$ and $\operatorname{dist}(K, L+c) > 0$.

Among the four feasibility statuses, all but weak infeasibility afford simple finite certificates: an interior-feasible solution, a pair consisting of a feasible solution and a vector which is normal to a separating hyperplane, and a dual improving direction for strong feasibility, weak feasibility and strong infeasibility respectively. The last one is sometimes called a Farkas-type certificate, and plays an important role in optimization theory. However, it is not evident whether weak infeasibility affords such a finite certificate. A certificate of weak infeasibility reflecting its nature is the existence of a weakly infeasible sequence, i.e., a sequence $\{u^{(k)}\}$ in $L+c$ but not in K_n such that $\operatorname{dist}(u^{(k)}, K_n) \to 0$ as $k \to \infty$. Together with a certificate of infeasibility, the weakly infeasible sequence would constitute a certificate of weak infeasibility. But this is not a finite certificate.

In this paper, we study weak infeasibility in semidefinite programming from a constructive point of view. Specifically we demonstrate that it is possible to construct a "generator" of weakly infeasible sequences consisting of at most $n-1$ directions and a fixed point $x \in L + c$, i.e., there exists a set of at most $n-1$ directions such that a weakly infeasible sequence can be easily obtained by starting at x and adding a positive linear combination of those directions. Such directions would be used to replace "infinite sequence" in the above-mentioned certificate of weak infeasibility to obtain a finite certificate of weak infeasibility.

Regarding finite certificates, it is worthwhile mentioning that there is a way to obtain a finite and polynomially bounded certificate of weak infeasibility by using Ramana's extended Lagrangian dual [21] or the facial reduction algorithm [3, 4, 15, 26]. This result is based on the fact that

(K_n, L, c) *is weakly infeasible if and only if it is* infeasible *and* not strongly infeasible.

Ramana developed a generalized Farkas' Lemma for SDP which holds without any assumption. By using this result, the infeasibility of (K_n, L, c) is equivalent to the feasibility of another SDP. On the other hand, it is also known that strong infeasibility is equivalent to the feasibility of some other (polynomially bounded) SDP. By using Ramana's result again, it follows that there exists some SDP such that its feasibility is equivalent to (K_n, L, c) not being strongly infeasible. Thus, both infeasibility and *not* strong infeasibility admits finite certificates. Therefore, combining them we obtain a finite certificate of weak infeasibility. An analogous result is obtainable through facial reduction. Since we could not find any literature describing this result explicitly, we will explain it in more detail in Section 2. A drawback of this approach is that it tells little about the structure of weakly infeasible problems since this certificate has an existential flavor and is not related to weakly infeasible sequences in an obvious way.

Here we develop a constructive and direct approach for analyzing weakly infeasible problems in order to answer the following basic question:

Given a weakly infeasible SDFP, how can we generate a weakly infeasible sequence $\{u^{(i)} |\ u^{(i)} \in L + c, i = 1, \dots \infty\}$ *such that* $\lim_{i\to\infty} \mathrm{dist}(u^{(i)}, K_n) = 0$*?*

Due to the fact that the distance between K_n and $L + c$ is zero, we readily see that there exists a nonzero element a in $K_n \cap L$. However, it is not clear how a is related to the weak infeasibility of (K_n, L, c). Since the problem is infeasible, $\mathrm{dist}(ta + b, K_n) > 0$ for any $t > 0$ and $b \in L + c$. It would be natural to ask what happens as t goes to infinity. Can $\lim_{t\to\infty} \mathrm{dist}(ta + b, K_n) = \infty$ or a finite nonzero value, or zero? If we cannot find any $b \in L+c$ such that $\lim_{t\to\infty} \mathrm{dist}(ta+b, K_n) = 0$ holds, how can a be used to construct points close to the cone?

We will show in a constructive way that a alone is not enough to generate such a sequence, but $(n-1)$ directions including a are sufficient (with an appropriate choice of b), whenever the problem is weakly infeasible. In other words, if (K_n, L, c) is weakly infeasible then there exists a $(n-1)$ dimensional affine subspace $\mathcal{F} \subseteq L+c$ such that $\mathcal{F} \cap K_n = \emptyset$ but $\mathrm{dist}(\mathcal{F}, K_n) = 0$. This result is a bit surprising to us, because, in general, if K is a closed convex cone and (K, L, c) is weakly infeasible, then the number of directions necessary to approach the cone could be as large as the dimension of L, which could be up to $(\frac{n(n+1)}{2} - 1)$ in our context.

The proof is done by making use of a set of directions $d_1, d_2, \dots, d_\ell$, in L which is generated when applying the facial reduction algorithm to the dual system

$$\min\ \langle c, s\rangle \quad \text{s.t.} \quad s \in L^\perp \cap K_n. \tag{2}$$

The directions $d_1, d_2, \ldots, d_\ell$ are obtained recursively starting from a nonzero element in $K_n \cap L$. An important feature of this set is that, even though each direction is not necessarily positive (semi)definite, we can always find a positive linear combination which is almost positive semidefinite (the minimum eigenvalue can be made to be arbitrarily close to zero).

While facial reduction uses $d_1, d_2, \ldots, d_\ell$ to compute hyperplanes which contains the dual feasible region, we use the same vectors to analyze weak infeasibility. This is an important novel point of this paper.

One possible application of our results is as follows. Consider the following SDP

$$\max \langle b, x\rangle \ \text{ s.t. } x \in (L + c) \cap K_n, \tag{P}$$

and suppose that the optimal value b^* is finite but not attained. The set $\{x \in L+c \mid \langle b, x\rangle = b^*\}$ is non-empty and is also an affine space. Denoting by $\widetilde{L}$ the underlying subspace and letting $\widetilde{c}$ be any point which belongs to the affine space, we have that $(K_n, \widetilde{L}, \widetilde{c})$ is weakly infeasible. Indeed such problems arise in many applications in semidefinite programming including control theory and polynomial optimization [25]. In this case, it is important to obtain feasible solutions whose values are arbitrarily close to optimality. The technique developed in this paper can be used to compute such an approximate optimal solution without solving SDPs repeatedly.

The main tool we use is Theorem 5, which implies that certain semidefinite feasibility problems (SDFPs) can be decomposed into smaller subproblems in a way that the feasibility properties are mostly preserved. This decomposition can be thought as a relaxation of the positive semidefinite constraint to another one in smaller dimension, with $L + c$ unchanged. This is then repeated until the relaxed system admits no recession direction. We show that the relaxed system is strongly feasible (strongly infeasible) iff the original system is strongly feasible (strongly infeasible, respectively). Hence, the relaxed system is in weak status iff the original system is, where "weak status" means either weakly infeasible or weakly feasible. Thus, the relaxation preserves (i) strong feasibility, (ii) strong infeasibility and (iii) weak status. In comparison, when applying the facial reduction algorithm (FRA) to the feasibility problem (K, L, c), we have that feasibility/infeasibility is preserved at each step, but weak status can be transformed to one of the strong statuses.

We also show that when no further relaxation is possible, the relaxed system can either be strongly feasible, strongly infeasible or weakly feasible, i.e., it can never be weakly infeasible. This is good because these are the states that afford easy certificates, see Proposition 1. It follows that we can also classify the original system as strongly feasible, strongly infeasible or in weak status. In contrast, FRA is able to tell whether the system is feasible or infeasible, but distinguishing between strong and weak infeasibility is not immediate.

Though it was not apparent initially, it turns out that the relaxation process can be thought as applying the conic expansion algorithm (CEA) to the dual system (2). Each step of CEA can be seen as an attempt to relax the original feasibility problem (K, L, c), and our analysis shows the possible changes of feasibility status at each step. This connection is explained in more detail in Section 3.3.

We now review related previous works. The existence of weak infeasibility/feasibility and nonzero duality gap is one of the main difficulties in semidefinite programming. These situations may occur in the absence of interior-feasible solutions to the primal and/or dual problems. Two possible techniques to recover interior-feasibility by reducing the feasible region of the problem or by expanding the feasible region of its dual counter-part are the facial reduction algorithm (FRA) and the conic expansion approach (CEA), respectively.

FRA was developed by Borwein and Wolkowicz [3] for problems more general than conic programming, whereas CEA was developed by Luo, Sturm and Zhang [19] for conic programming.

In the earlier stages of research of semidefinite programming, Ramana [21] developed an extended Lagrange-Slater dual (ELSD) that has no duality gap. ELSD has the remarkable feature that the size of the extended problem is bounded by a polynomial in terms of the size of the original problem. In [22], Ramana, Tunçel and Wolkowicz clarified the connection between ELSD and facial reduction, see also [15]. In [17], Pólik and Terlaky provided strong duals for conic programming over symmetric cones. Recently, Klep and Schweighofer developed another dual based on real algebraic geometry where the strong duality holds without any constraint qualification [6]. Like ELSD, their dual is just represented in terms of the data of original problem and the size of the dual is bounded by a polynomial in terms of the size of the original problem. Complexity of SDFP is yet a subtle issue. This topic was studied extensively by Porkolab and Khachiyan [18].

Waki and Muramatsu [26] considered a FRA for conic programming and showed that FRA can be regarded as a dual version of CEA. See the excellent article by Pataki [15] for an elementary exposition of FRA, where he points out the relation between facial reduction and extended duals. Pataki also found that all "badly behaved" semidefinite programs can be reduced to a common 2×2 semidefinite program [16]. Finally, we mention that Waki showed that weakly infeasible instances can be obtained from the semidefinite relaxation of polynomial optimization problems [25].

The problem of weak infeasibility is closely related to closedness of the image of K_n by a certain linear map. A comprehensive treatment of the subject was given by Pataki [14]. We will discuss the connection between Pataki's results and weak infeasibility in Section 2.

This paper is organized as follows. In Section 2, we discuss certificates for the different feasibility statuses and point the connections to previous works. In Section 3 we present Theorem 5 and discuss how certain SDFPs can be broken in smaller problems. We also prove the bound $n-1$ for the number of the directions needed to approach K_n. Finally, in Section 4 we discuss a few recent results and future research.

2. Characterization of Different Feasibility Statuses

In this section, we review the characterization of different feasibility statuses of semidefinite programs with an emphasis on weak infeasibility. The model of computation we use is the Blum-Shub-Smale model (BSS model) [1] of real computation. For the purposes of this paper, we only need to recall that operations on real numbers can be done in unit time and real numbers can be stored in unit space. Throughout this paper, we assume that L is represented as the set of solutions to a system of linear equations, where the coefficients and left hand side is explicitly given. We also note that checking positive semidefiniteness of a symmetric matrix can be done in polynomial time by using a variant of the LDL^T decomposition.

The main decision problem we are interested is: *given (K_n, L, c), what is its feasibility status?*[*] We start with the following proposition which characterizes strong feasibility, weak feasibility and strong infeasibility.

Proposition 1. *Let L be a subspace of $\mathbb{S}_n$ and $c \in \mathbb{S}_n$ then (K_n, L, c) is:*

[*]Strictly speaking, a decision problem should have "yes" or "no" as answers, but in our case the possible answers are strong/weakly feasible, strong/weakly infeasible. We could have broken down the decision problem in 4 different decision problems having "yes" or "no" as answers.

1. Strongly feasible*, if and only if there is $x \in L + c$ such that x is positive definite.*

2. Weakly feasible *if and only if there is*

i. $x \in L + c$ such that x is positive semidefinite

ii. $y \in L^\perp \cap K_n$ such that $y \neq 0$, $\langle y, c\rangle = 0$.

3. Strongly infeasible *if and only if $c \neq 0$ and $(K_n, L_{\mathrm{SI}}, c_{\mathrm{SI}})$ is feasible, where $L_{\mathrm{SI}} = L^\perp \cap c^\perp$, and $c_{\mathrm{SI}} = -\frac{c}{\|c\|^2}$.*

Proof. Item 1. is immediate. Items 2. and 3. follow easily from Theorem 11.3 and 11.4 of [23]. Item 2. correspond to the situation where K_n and $L + c$ can be properly separated but still a feasible point exists and item 3. to the case where they can be strongly separated. Also, for a proof of 3. see, for instance, Lemma 5 of [20]. □

Proposition 1 already implies that deciding strong feasibility, weak feasibility and strong infeasibility are in NP, in the BSS model. This means that each of these three states admits a finite certificate which can be checked in polynomial time. Note that (K_n, L, c) is not strongly feasible if and only if either item 2.*ii* (but not necessarily 2.*i*) or item 3. holds. This means that deciding strong feasibility lies in coNP as well†.

In Proposition 1, weak infeasibility is absent. When proving weak infeasibility, it is necessary to show that the distance between K_n and $L + c$ is 0. The obvious way is to produce a sequence $\{x_k\} \in L + c$ such that $\lim_{k\to+\infty} \mathrm{dist}(x_k, K_n) = 0$. In [19] it was shown that (K_n, L, c) is weakly infeasible if and only if there is no dual improving direction *and* there is a dual improving sequence (see Lemma 6 and Table 1 in [19]). But this is not a finite certificate of weak infeasibility.

A finite certificate for weak infeasibility can be obtained by using Ramana's results on an extended Lagrangian dual for semidefinite programming [21]. Ramana's dual has a number of key properties: it is written explicitly in terms of problem data, it has no duality gap and the optimal value is always attained when finite. With his dual, it was possible to develop an exact Farkas-type lemma for semidefinite programming. In Theorem 19 of [21], he constructed another SDFP $\mathcal{RD}(K_n, L, c)$ for which the following holds without any regularity conditions:

$$(K_n, L, c) \text{ is feasible if and only if } \mathcal{RD}(K_n, L, c) \text{ is infeasible.}$$

Furthermore, the size of $\mathcal{RD}(K_n, L, c)$ is bounded by a polynomial that depends only on the size of the system (K_n, L, c). Based on this strong result, we obtain a finite certificate of weak infeasibility as in the following proposition:

Proposition 2. *We have the following:*

i. (K_n, L, c) is weakly infeasible $\Leftrightarrow$ $c \neq 0$, $\mathcal{RD}(K_n, L, c)$ and $\mathcal{RD}(K_n, L_{\mathrm{SI}}, c_{\mathrm{SI}})$ are feasible.

ii. The problem of deciding whether a given (K_n, L, c) is weakly infeasible is in NP ∩ coNP *in the BSS model.*

Proof. A feasible solution to $\mathcal{RD}(K_n, L, c)$ attests the infeasibility of (K_n, L, c). As $\mathcal{RD}(K_n, L, c)$ has polynomial size, it is possible to check that a point is indeed a solution to it in polynomial time.

Note that a problem is weakly infeasible if and only if it is infeasible and is not strongly infeasible. Due to Proposition 1, we have that (K_n, L, c) is not strongly infeasible if and only if $c = 0$ or $c \neq 0$ and $\mathcal{RD}(K_n, L_{\mathrm{SI}}, c_{\mathrm{SI}})$ is feasible. Hence, feasible solutions to $\mathcal{RD}(K_n, L, c)$

†In this case, the decision problem has either "yes" or "no" as possible answers, so it makes sense to talk about coNP.

and $\mathcal{RD}(K_n, L_{\mathrm{SI}}, c_{\mathrm{SI}})$ can be used together as a certificate for weak infeasibility. Such a certificate can be checked in polynomial time, hence the problem is in NP.

Now, $c = 0$, a solution to $(K_n, L_{\mathrm{SI}}, c_{\mathrm{SI}})$ *or* to (K_n, L, c) can be used to certify that a system is not weakly infeasible. This shows that deciding weak infeasibility is indeed in coNP. □

The important point in the argument above is having both a certificate of infeasibility for the original system and a certificate of infeasibility for the system $(K_n, L_{\mathrm{SI}}, c_{\mathrm{SI}})$. Any method of obtaining finite certificates of infeasibility can be used in place of $\mathcal{RD}$, as long as it takes polynomial time to verify them. See the comments after Theorem 3.5 in Sturm [24] and also Theorem 7.5.1 of [12] for another certificate of infeasibility. Klep and Schweighofer [6] also developed certificates for infeasibility and a hierarchy of infeasibility in which 0-infeasibility corresponds to strong infeasibility and k-infeasibility to weak infeasibility, when $k > 0$. Liu and Pataki [7] also introduced an infeasibility certificate for semidefinite programming. They defined what is a reformulation of a feasibility system and showed that (K_n, L, c) is infeasible if and only if it admits a reformulation that converts the systems to a special format, see Theorem 1 therein.

We mention a few more related works on weak infeasibility. The feasibility problem (K_n, L, c) is weakly infeasible if and only if $c \in \mathrm{cl}\,(K_n + L) \setminus (K_n + L)$, where cl denotes the closure operator. Hence, a *necessary* condition for weak infeasibility is that $K_n + L$ fails to be closed. This problem is closely related to closedness of the image of K_n by a linear map which is the problem analyzed in detail by Pataki [14]. Theorem 1.1 in [14] provides a necessary and sufficient condition for the failure of closedness of $K_n + L$. Pataki's result implies that there is some $c \in \mathbb{S}_n$ such that (K_n, L, c) is weakly infeasible if and only if $L^\perp \cap (\mathrm{cl}\,\mathrm{dir}\,(x, K_n) \setminus \mathrm{dir}\,(x, K_n)) \neq \emptyset$, where x belongs to the relative interior of $L \cap K_n$ and $\mathrm{dir}\,(x, K_n)$ is the cone of feasible directions at x. This tells us whether K_n and L can accommodate a weakly infeasible problem. If it is indeed possible, Corollary 3.1 of [14] shows how to find an appropriate c. Bonnans and Shapiro [2] also discussed generation of weakly infeasible semidefinite programming problems. As a by-product of the proof of Proposition 2.193 therein, it is shown how to construct weakly infeasible problems.

In [16], Pataki introduced the notion of *well-behaved* system. (K_n, L, c) is said to be well-behaved if for all $b \in \mathbb{S}_n$, the optimal value of (P) and of its dual are the same and the dual is attained whenever it is finite. A SDP which is not well-behaved is said to be *badly-behaved*. Pataki showed that badly-behaved SDPs can be put into a special shape, see Theorem 6 in [16]. Then, a necessary condition for weak infeasibility is that the homogenized system $(K_n, \widetilde{L}, 0)$ be badly-behaved, where $\widetilde{L}$ is spanned by L and c. See the comments before Section 4 in [16].

3. A Decomposition Result

In this section, we develop a key decomposition result. Given an SDFP, we show how to construct a smaller dimensional SDFP which preserves most of the feasibility properties.

3.1. Preliminaries

First we introduce the notation. If $\mathcal{C}, \mathcal{D}$ are subsets of some real space, we write $\mathrm{dist}(\mathcal{C}, \mathcal{D}) = \inf\{\|x - y\| \mid x \in \mathcal{C}, y \in \mathcal{D}\}$, where $\|\cdot\|$ is the Euclidean norm or the Frobenius norm, in the case of subsets of $\mathbb{S}_n$. By $\mathrm{int}\,(\mathcal{C})$ and $\mathrm{ri}\,(\mathcal{C})$ we denote the interior and the relative interior of $\mathcal{C}$, respectively. We use I_n to denote the $n \times n$ identity matrix. Given (K_n, L, c) and a matrix $A \in K_n \cap L$ with rank k, we will call A a *recession direction* of rank k (a slightly abuse of the normal definition). We remark that when (K_n, L, c) is feasible, A is also a recession direction of the feasible region in the usual sense.

Let x be a $n \times n$ matrix, and $0 \leq k \leq n$. We denote by $\pi_k(x)$. the upper left $k \times k$ principal submatrix of x. For instance, if

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix},$$

then,

$$\pi_2(x) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

We define the subproblem $\pi_k(K_n, L, c)$ of (K_n, L, c) to be

$$\text{find } u \in \pi_k(L + c), \quad u \succeq 0.$$

In other words, it is the feasibility problem $(\pi_k(K_n), \pi_k(L), \pi_k(c))$. We denote by $\overline{\pi}_k(x)$, the lower right $(n-k) \times (n-k)$ principal submatrix. In the example above, we have $\overline{\pi}_2(x) = 6$. In a similar manner, we write $\overline{\pi}_k(K_n, L, c)$ for the feasibility problem $(\overline{\pi}_k(K_n), \overline{\pi}_k(L), \overline{\pi}_k(c))$. We remark that $\pi_n(x) = \overline{\pi}_0(x) = x$ and we define $\pi_0(x) = \overline{\pi}_n(x) = 0$.

The proposition belows summarizes the properties of the Schur Complement. For proofs, see Theorem 7.7.6 of [5].

Proposition 3 (Schur Complement). *Suppose $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is a symmetric matrix divided in blocks in a way that A is positive definite, then:*

- *M is positive definite if and only if $C - B^T A^{-1} B$ is.*
- *M is positive semidefinite if and only if $C - B^T A^{-1} B$ is.*

The properties of a semidefinite program are not changed when a congruence transformation is applied, i.e., for any nonsingular matrix P, we have that (K_n, L, c) and (K_n, PLP^T, PcP^T) have the same feasibility properties, where $PLP^T = \{PlP^T \mid l \in L\}$.

3.2. The main result

It will be convenient for now to collapse weak feasibility and weak infeasibility into a single status. We say that (K_n, L, c) is in *weak status* if it is either weakly feasible or weakly infeasible. We start with the following basic observation.

Proposition 4. *If (K_n, L, c) is weakly infeasible, there exists a nonzero vector in $K_n \cap L$.*

Proof. Due to weak infeasibility, there exists a sequence $\{l^k\} \subseteq L$ such that $\lim_{k \to +\infty} \operatorname{dist}(l^k + c, K_n) = 0$. Because there are no feasible solutions, the sequence $\{l^k + c\}$ can have no convergent subsequences, from which we conclude that $\lim_{k \to +\infty} \|l^k\| = +\infty$. Removing, if necessary, the l^k that are zero, we can consider the bounded sequence $\left\{\frac{l^k + c}{\|l^k\|}\right\}$. Passing to a subsequence if necessary, we may assume that it converges to some point z^*. The fact that K_n is a cone implies that $\lim_{k \to +\infty} \operatorname{dist}(\frac{l^k + c}{\|l^k\|}, K_n) = 0$, so we conclude that $z^* \in K_n$.

Hence, $z^* = \lim_{k \to +\infty} \frac{l^k + c}{\|l^k\|} = \frac{l^k}{\|l^k\|}$ and $z^* \in L$ too. □

Now we present a key result in our paper. The following theorem says that if (K_n, L, c) has a recession direction, then, we can construct another SDFP of smaller size whose feasibility status is *almost* identical to the original problem.

Theorem 5. *Let (K_n, L, c) be a SDFP, and consider a subproblem $\pi_k(K_n, L, c)$ for some $k > 0$. If the subproblem $\pi_k(K_n, L, c)$ admits an interior recession direction (i.e, $\operatorname{int} \pi_k(K_n) \cap \pi_k(L) \neq \emptyset$) then:*

1. (K_n, L, c) *is strongly feasible if and only if* $\overline{\pi}_k(K_n, L, c)$ *is.*
2. (K_n, L, c) *is strongly infeasible if and only if* $\overline{\pi}_k(K_n, L, c)$ *is.*
3. (K_n, L, c) *is in weak status if and only if* $\overline{\pi}_k(K_n, L, c)$ *is.*

Proof. Due to the assumption, there exists a $n \times n$ matrix

$$x = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where A is a $k \times k$ positive definite matrix.

We now prove items 1 and 2. Item 3. will follow by elimination.

(1) $\Rightarrow$) If $y \in L + c$ is positive definite, all its principal submatrices are also positive definite. Therefore, $\overline{\pi}_k(y)$ is positive definite.

(1) $\Leftarrow$) Suppose that $y \in L + c$ is such that $\overline{\pi}_k(y) \in \operatorname{int} K_{n-k}$. Then, we may write $y = \left(\begin{smallmatrix} F & E \\ E^T & G \end{smallmatrix}\right)$, where G is $(n-k) \times (n-k)$ and positive definite. For large and positive α, $F + \alpha A$ is positive definite and the Schur complement of $y + x\alpha$ is $G - E^T(F + \alpha A)^{-1}E$. Since G is positive definite, it is clear that, increasing α if necessary, the Schur complement is also positive definite. For such an α, $y + x\alpha \in (L + c) \cap \operatorname{int} K_n$.

(2) $\Rightarrow$). Suppose (K_n, L, c) strongly infeasible. Then there exists $s \in K_n$ such that $s \in L^\perp$ and $\langle s, c\rangle = -1$. As $x \in L$, we have $s \in K_n \cap \{x\}^\perp$. This means that s can be written as $\left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right)$, where D belongs to K_{n-k}. It follows that $\overline{\pi}_k(s) \in \overline{\pi}_k(L)^\perp$ and $\langle \overline{\pi}_k(s), \overline{\pi}_k(c)\rangle = -1$. By item *iii.* of Proposition 1, $\overline{\pi}_k(K_n, L, c)$ is strongly infeasible.

(2) $\Leftarrow$). Now, suppose $\overline{\pi}_k(K_n, L, c)$ is strongly infeasible. Note that $\overline{\pi}_k$ is a non-expansive map, i.e., $\|\overline{\pi}_k(y) - \overline{\pi}_k(z)\| \leq \|y - z\|$ holds. In particular, if $\inf_{y \in L+c, z \in K_n} \|\overline{\pi}_k(y) - \overline{\pi}_k(z)\| > 0$, then the same is true for $\inf_{y \in L+c, z \in K_n} \|y - z\|$. □

3.3. Forward procedure

Assume that (K_n, L, c) admits a recession direction $\widetilde{A}_1$ of rank k_1. Theorem 5 might not be directly applicable but after an appropriate congruence transformation by a nonsingular matrix P_1, we have that $(K_n, P_1^T L P_1, P_1^T c P_1)$ admits a recession direction of the form

$$A_1 = \begin{pmatrix} \widehat{A}_1 & 0 \\ 0 & 0 \end{pmatrix} = P_1^T \widetilde{A}_1 P_1,$$

where $\widehat{A}_1$ is a $k_1 \times k_1$ positive definite matrix. The feasibility status of (K_n, L, c) and

$$(K_{n-k_1}, \overline{\pi}_{k_1}(P_1^T L P_1), \overline{\pi}_{k_1}(P_1^T c P_1))$$

are mostly the same since Theorem 5 holds.

Now, suppose that $(K_{n-k_1}, \overline{\pi}_{k_1}(P_1^T L P_1), \overline{\pi}_{k_1}(P_1^T c P_1))$ admits a recession direction $\widetilde{A}_2$ of rank k_2. Then, after appropriate congruence transformation by $\widetilde{P}_2$, we obtain that

$$(K_{n-k_1}, \widetilde{P}_2^T \overline{\pi}_{k_1}(P_1^T L P_1)\widetilde{P}_2, \widetilde{P}_2^T \overline{\pi}_{k_1}(P_1^T c P_1)\widetilde{P}_2)$$

admits a recession direction of the form

$$\begin{pmatrix} \widehat{A}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\widehat{A}_2$ is $k_2 \times k_2$ positive definite matrix.

Now, the feasibility status of $(K_{n-k_1}, \overline{\pi}_{k_1}(P_1^T L P_1), \overline{\pi}_{k_1}(P_1^T c P_1))$ and

$$(K_{n-k_1-k_2}, \overline{\pi}_{k_2}(\widetilde{P}_2^T \overline{\pi}_{k_1}(P_1^T L P_1)\widetilde{P}_2), \overline{\pi}_{k_2}(\widetilde{P}_2^T \overline{\pi}_{k_1}(P_1^T L P_1)\widetilde{P}_2))$$

are mostly the same. Note that instead of applying a congruence transformation by $\widetilde{P}_2$ to $(K_{n-k_1}, \overline{\pi}_{k_1}(P_1^T L P_1), \overline{\pi}_{k_1}(P_1^T c P_1))$, we can apply a congruence transformation by

$$P_2 = \begin{pmatrix} I_{k_1} & 0 \\ 0 & \widetilde{P}_2 \end{pmatrix}$$

to the original problem $(K_n, P_1^T L P_1, P_1^T c P_1)$, i.e., we consider

$$\left(K_n, P_2^T P_1^T L P_1 P_2, P_2^T P_1^T c P_1 P_2\right)$$

Then the subproblem defined by the $(n-k_1)\times(n-k_1)$ lower right block matrix is precisely

$$(K_{n-k_1}, \widetilde{P}_2^T \overline{\pi}_{k_1}(P_1^T L P_1)\widetilde{P}_2, \widetilde{P}_2^T \overline{\pi}_{k_1}(P_1^T c P_1)\widetilde{P}_2),$$

and we may pick $A_2 \in P_2^T P_1^T L P_1 P_2$ such that

$$\overline{\pi}_{k_1+k_2}(A_2) = \begin{pmatrix} \widehat{A}_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that A_2 has the following shape

$$A_2 = \begin{pmatrix} * & * & * \\ * & \widehat{A}_2 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Generalizing the process outlined above, we obtain the following procedure, which we call "forward procedure". Note that the congruence matrix at each step can be taken to be orthogonal. The set of matrices $\{A_1, \ldots, A_m\}$ obtained in this way will be called a *set of reducing directions*. We note that $\{A_1, \ldots, A_m\}$ is exactly the same as the set of matrices obtained when we apply FRA (or, equivalently, CEA), to the dual problem (2). The only caveat is that we "rotate" the problem to put it into a convenient shape before finding the next direction.

After each application of Theorem 5, the size of the matrices is reduced by at least one. This means that after at most n iterations, a subproblem with no nonzero reducing directions is found. At this point, no further directions can be added and we will say that the set is *maximal*.

We note that the problem of checking whether a SDFP $(K_{\tilde{n}}, \widetilde{L}, \tilde{c})$ has a nonzero reducing direction lies in $\mathsf{NP} \cap \mathsf{coNP}$, in the real computation model. In fact, by Gordan's Theorem, $(K_{\tilde{n}}, \widetilde{L}, \tilde{c})$ does not have a nonzero reducing direction if and only if $(K_n, \widetilde{L}^\perp, 0)$ is strongly feasible.

[Procedure FP]

Input: (K_n, L, c)

Output: an orthogonal P, a sequence $k_1, \ldots, k_m$ and a maximal set of reducing directions $\{A_1, \ldots, A_m\}$ contained in $P^T L P$. The A_i are such that

$$A_1 = \begin{pmatrix} \widehat{A}_1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} * & * & * \\ * & \widehat{A}_2 & 0 \\ * & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \widehat{A}_3 & 0 \\ * & * & 0 & 0 \end{pmatrix}$$

and so forth, where $\widehat{A}_i$ is positive definite and lies in K_{k_i}, for every i.

1. Set $i := 1$, $\widetilde{L} := L$, $\widetilde{c} := c$ $K := K_n$, $P := I_n$.
2. Find (i) $\widetilde{A}_i \in \widetilde{L} \cap K$, $\mathrm{tr}(\widetilde{A}_i) = 1$ or (ii) $\widetilde{B} \in \widetilde{L}^\perp \cap \mathrm{int}K$, $\mathrm{tr}(\widetilde{B}) = 1$. (Exactly one of (i) and (ii) is solvable.) If (ii) is solvable, then stop. (No nonzero reducing direction exists.)
3. Compute an orthogonal $\widetilde{P}$ such that,

$$\widetilde{P}^T \widetilde{A}_i \widetilde{P} = \begin{pmatrix} \widehat{A}_i & 0 \\ 0 & 0 \end{pmatrix}$$

   where $\hat{A}_i$ is a positive definite matrix. Let $k_i := \mathrm{rank}(\widetilde{A}_i)$.
4. Compute $M = \begin{pmatrix} I_{k_1+\ldots+k_{i-1}} & 0 \\ 0 & \widetilde{P} \end{pmatrix}$ and set $P^T := M^T P^T$. (If $i = 1$, take $M = \widetilde{P}$)
5. Let A_i be any matrix in $P^T L P$ such that $\overline{\pi}_{k_1+\ldots+k_{i-1}}(A_i) = \widetilde{P}^T \widetilde{A}_i \widetilde{P}$. For each $1 \leq j < i$ exchange A_j for $M^T A_j M$.
6. Set $\widetilde{L} := \overline{\pi}_{k_i}(\widetilde{P}^T \widetilde{L} \widetilde{P}), \widetilde{c} := \overline{\pi}_{k_i}(\widetilde{P}^T \widetilde{c} \widetilde{P})$, $K := \overline{\pi}_{k_i}(K_n)$, $i := i + 1$ and return to Step 2. (This step is just to pick the lower-right block after the congruence transformation.)

We now discuss how **FP** affects the feasibility problem (2).

Proposition 6. *Suppose that (K_n, L, c) is such that there is a nonzero element in $K_n \cap L$. Applying **FP** to (K_n, L, c) we have that:*

1. (K_n, L, c) is strongly feasible if and only if $\overline{\pi}_{k_1+\ldots+k_m}(K_n, P^T LP, P^T cP)$ is.

2. (K_n, L, c) is strongly infeasible if and only if $\overline{\pi}_{k_1+\ldots+k_m}(K_n, P^T LP, P^T cP)$ is.

3. (K_n, L, c) is in weak status if and only if $\overline{\pi}_{k_1+\ldots+k_m}(K_n, P^T LP, P^T cP)$ is weakly feasible.

Proof. If $m = 0$, then the proposition follows because $\overline{\pi}_0$ is equal to the identity map. In the case $m = 1$, the result follows from Theorem 5.

Note that at the i-th iteration, if a direction A_i is found then, after applying the congruence transformation $\widetilde{P}$, $\overline{\pi}_{k_i}(K, \widetilde{P}^T \widetilde{L} \widetilde{P}, \widetilde{P}^T \widetilde{c} \widetilde{P})$ preserves feasibility properties in the sense of Theorem 1. Note that it is a SDFP over $\mathbb{S}_{n-k_1-\ldots-k_i}$. Also, due to the way M is selected, we have that equation $\overline{\pi}_{k_i}(K, \widetilde{P}^T \widetilde{L} \widetilde{P}, \widetilde{P}^T \widetilde{c} \widetilde{P}) = \overline{\pi}_{k_1+\ldots+k_i}(K_n, P^T LP, P^T cP)$ holds after Line 4 and before $\widetilde{L}$ and K are updated. This justifies items 1. and 2..

Consider the case where (K_n, L, c) is in weak status. When $(K, \widetilde{L}, \widetilde{c})$ is weakly infeasible we can always find a new direction A_i and the size of problem decreases by a positive amount, so that $(K, \widetilde{L}, \widetilde{c})$ cannot be weakly infeasible for all iterations. The only other possibility is weak feasibility, which justifies item 3. □

The matrices $A_1, \ldots, A_m$ obtained through **FP** have the shape

$$\begin{pmatrix} \widehat{A}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & \widehat{A}_2 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \widehat{A}_3 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \ldots$$

where $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \ldots$ are positive definite. The matrix A_i are referred to as *reducing directions*, since the $\widehat{A}_i$ are reducing directions. The problem $\overline{\pi}_{k_1+\ldots+k_m}(K_n, P^T LP, P^T cP)$ will be referred to as the *last subproblem* of (K_n, L, c).

We obtain the following alternative characterization of weak infeasibility based on **FP**.

Proposition 7. *(K, L, c) is weakly infeasible if and only if it is in weak status and is infeasible. Therefore, weak infeasibility is detected by executing **FP** for checking weak status and FRA for checking infeasibility.*

Example 8. *Let*

$$L+c=\left\{\begin{pmatrix} t & v & 1 & u \\ v & z+2 & v+1 & z+1 \\ 1 & v+1 & u-1 & s \\ u & z+1 & s & 0 \end{pmatrix} \mid t,u,v,s,z\in\mathbb{R}\right\}. \tag{3}$$

and let us apply ***FP*** *to* (K_4, L, c). *The first direction can be, for instance,*

$$A_1=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $k_1 = 1$ *and* $\widetilde{P}$ *is the identity, at this step. At the next iteration, we have* $K = K_3$ *and*

$$\widetilde{L}=\left\{\begin{pmatrix} z & v & z \\ v & u & s \\ z & s & 0 \end{pmatrix} \mid u,s,v,z\in\mathbb{R}\right\}.$$

Then, $\widetilde{A}_2$ *can be taken as*

$$\widetilde{A}_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and k_2 *is* 1. *A possible choice of* $\widetilde{P}$ *and* P *is*

$$\widetilde{P}=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can then take

$$A_2=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\tilde{L}$ *is then updated and it becomes* $\{\left(\begin{smallmatrix} z & z \\ z & 0 \end{smallmatrix}\right) \mid z\in\mathbb{R}\}$. *The procedure stops here, because* 0 *is the only positive semidefinite matrix in* $\tilde{L}$.

Now,

$$\overline{\pi}_2(P^T(L+c)P)=\left\{\begin{pmatrix} z+2 & z+1 \\ z+1 & 0 \end{pmatrix} \mid z\in\mathbb{R}\right\},$$

so $\overline{\pi}_2(K_4, P^TLP, P^TcP)$ *is a weakly feasible system. Therefore, by Proposition 6,* (K_4, L, c) *has weak status and is either weakly infeasible or weakly feasible. The* 0 *in the lower right corner of* (3) *forces* $u = 0$, $z = -1$ *and* $s = 0$, *but this assignment produces a negative element in the diagonal. This tells us that* (K_4, L, c) *is infeasible so it must be weakly infeasible.*

Corollary 9. *The matrices* $A_1, \ldots, A_m, P, \widetilde{B}$ *as described in* ***FP*** *together with a finite weak feasibility certificate for* $\overline{\pi}_{k_1+\ldots k_m}(K_n, P^TLP, P^TcP)$ *form a finite certificate that* (K_n, L, c) *is in weak status. If no such a certificate exists, then either item* $1.$ *or item* $3.$ *of Proposition 1 holds. This shows that deciding whether a SDFP is in weak status is in* $\mathsf{NP}\cap\mathsf{coNP}$.

Proof. Follows directly from Proposition 6. □

Now, we present a procedure to distinguish the four statuses of (K_n, L, c).

1. Apply **FP** to check whether (K_n, L, c) is strongly feasible, strongly infeasible or in weak status.
2. If (K_n, L, c) is in weak status, then apply the facial reduction algorithm to check the feasibility of (K_n, L, c).
   (a) If (K_n, L, c) is infeasible, then, the problem is weakly infeasible. Otherwise, it is weakly feasible.

We now take a closer look at the connection between FRA/CEA and **FP**. More details on FRA can be found in [15] and [26]. In [26], it is also explained the connection between FRA and CEA, see section 4 therein. Roughly speaking, FRA and CEA can be seen as dual approaches to one another. We now give a very brief explanation about FRA. Let K be an arbitrary closed convex cone. Whenever (K, L, c) is not strongly feasible, it can be proven that there exists $A \in K^* \cap L^\perp$ such that either: (i) $\langle c, A\rangle = 0$ and $A \notin K^\perp$ or (ii) $\langle c, A\rangle < 0$. If (ii) holds, (K, L, c) must be strongly infeasible, by item 3. Proposition 1. If (i) holds, we substitute K by the face $K \cap \{A\}^\perp$, which is proper since $A \notin K^\perp$ and also contains $K \cap (L + c)$. Then, we consider the problem $(K \cap \{A\}^\perp, L, c)$ and repeat. We can continue for as long as $(K \cap \{A\}^\perp, L, c)$ is not strongly feasible. Ultimately, FRA will either detect that the problem is infeasible or find the minimal face of K which contains $K \cap (L + c)$, thus restoring strong feasibility. Note that the feasible region is unchanged throughout the process.

Suppose that we apply a single facial reduction step to the problem $(K_n, L^\perp, 0)$, which is a dual of sorts for (1). If $(K_n, L^\perp, 0)$ is not strongly feasible, then we will obtain a nonzero matrix $A \in K_n \cap (L^{\perp\perp}) = K_n \cap L$. Note that $K_n \cap L$ is precisely the set where we look for the first direction in **FP**. The meaning of A for the problem $(K_n, L^\perp, 0)$ is clear: it defines a proper face of K_n which contains $K_n \cap L^\perp$. However, our analysis also shows that A has meaning for (K, L, c) as well. Namely, after an appropriate transformation, Theorem 5 implies that A can be used to define a smaller problem that shares most of the feasibility properties. In the next section, we will also see that the set of reducing directions can be used to approach K_n. We will now explain the connection between the smaller problem in Theorem 5 and $(K_n \cap \{A\}^\perp, L, c)$, where the latter arises from a single facial reduction step.

Suppose that A has rank k and has the following shape.

$$A = \begin{pmatrix} \widehat{A} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

where $\widehat{A} \in K_k$ and is positive definite. Per the FRA recipe, the next step would be to replace $(K_n, L^\perp, 0)$ by $(K_n \cap A^\perp, L^\perp, 0)$. We have

$$K_n \cap A^\perp = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \in \mathbb{S}_n \;\middle|\; C \in K_{n-k} \right\}$$
$$(K_n \cap A^\perp)^* = \left\{ \begin{pmatrix} E & F \\ F^T & C \end{pmatrix} \in \mathbb{S}_n \;\middle|\; C \in K_{n-k} \right\}.$$

It follows that $((K_n \cap A^\perp)^*, L, c)$ and $\overline{\pi}_k(K_n, L, c)$ are equivalent problems. Namely, if $\overline{\pi}_k(x)$ is feasible for $\overline{\pi}_k(K_n, L, c)$, then x must be feasible for $((K_n \cap A^\perp)^*, L, c)$ and vice-versa. By using the projection $\overline{\pi}_k$ we can focus on the essential part, which is the semidefinite constraint that still remains at the lower $(n-k) \times (n-k)$ part. As $(K_n \cap A^{\perp})^*$ properly contains K_n, we do not expect that (K_n, L, c) and $((K_n \cap A^{\perp})^*, L, c)$ have exactly the same feasibility properties. Nevertheless, Theorem 5 shows that they are *almost* the same, which is not something apparent from the classical FRA analysis. Finally, we mention that the connection to CEA is due to the fact we are taking the dual of the cone $K_n \cap A^{\perp}$.

3.4. Maximum number of directions required to approach the positive semidefinite cone

According to Proposition 4, there is always a nonzero element in $K_n \cap L$ when (K_n, L, c) is weakly infeasible. Therefore, a natural question is, given a weakly infeasible (K_n, L, c), whether it is always possible to select a point in $x \in L + c$ and then a nonzero direction $d \in K_n \cap L$ such that $\lim_{t\to+\infty} \mathrm{dist}(x+td, K_n) = 0$ or not. We call weakly infeasible problems having this property *directionally weakly infeasible* (DWI). The simplest instance of DWI problem is

$$\max\ 0 \text{ s.t. } \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \in K_2,\ t \in \mathbb{R}.$$

Unfortunately, not all weakly infeasible problems are DWI, as shown in the following instance.

Example 10 (A weakly infeasible problem that is not directionally weakly infeasible)**.** *Let* (K_3, L, c) *be such that*

$$L + c = \left\{ \begin{pmatrix} t & 1 & s \\ 1 & s & 1 \\ s & 1 & 0 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}$$

and let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying Theorem 1 twice, we see that the problem is in weak status. Looking at its 2×2 *lower right block, we see this problem is infeasible and hence is weakly infeasible. But this problem is not DWI. If* (K_3, L, c) *were DWI, we would have* $\lim_{t\to+\infty} \mathrm{dist}(tA_1 + c', K_3) = 0$*, for some* $c' \in L + c$*. To show this does not hold, we fix* s*. Regardless of the value of* $t \geq 0$*, the minimum eigenvalue of the matrix is uniformly negative, since its* 2×2 *lower right block is strongly infeasible.*

Thus, a weakly infeasible problem is not DWI in general. If we let s sufficiently large in the example, then the minimum eigenvalue of the lower 2×2 matrix gets very close to zero. This will make the $(1, 3)$ and $(3, 1)$ elements large. But we can let t much larger than s. Then, the minimum eigenvalue of the submatrix $\left(\begin{smallmatrix} t & s \\ s & 0 \end{smallmatrix}\right)$ is close to zero. Intuitively, this neutralize the effect of big off-diagonal elements, and we obtain points in $L + c$ arbitrarily close to K_3, by taking s to be large and t to be much larger than s.

Generalizing this intuition, in the following, we show that $n-1$ directions are enough to approach the positive semidefinite cone. First we discuss how the set of reducing directions $\{A_1, \ldots, A_m\}$ of **FP** fits in the concept of tangent cone. We recall that for $x \in K_n$ the cone of feasible directions is the set $\mathrm{dir}\,(x, K_n) = \{d \in \mathbb{S}_n \mid \exists t > 0 \text{ s.t } x + td \in K_n\}$. Then the tangent cone at x is the closure of $\mathrm{dir}\,(x, K_n)$ and is denoted by $\mathrm{tanCone}\,(x, K_n)$. It can be shown that if $d \in \mathrm{tanCone}\,(x, K_n)$ then $\lim_{t\to+\infty} \mathrm{dist}(tx + d, K_n) = 0$.

We remark that if $x = \left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where D is positive definite $k \times k$ matrix, then $\mathrm{tanCone}\,(x, K_n)$ consists of all symmetric matrices $\left(\begin{smallmatrix} * & * \\ * & E \end{smallmatrix}\right)$, where $*$ denotes arbitrary entries and E is a positive semidefinite $(n-k) \times (n-k)$ matrix. See [13] for more details.

The output $\{A_1, \ldots, A_m\}$ of **FP** is such that $A_2 \in \text{tanCone}\,(A_1, K_n)$. This is clear from the shape of A_1 and A_2, and from a simple argument using the Schur Complement. Now, A_3 is such that $\overline{\pi}_{k_1+k_2}(A_3)$ is positive semidefinite. We have

$$A_2 = \begin{pmatrix} * & * & * & * \\ * & \widehat{A}_2 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \widehat{A}_3 & 0 \\ * & * & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} * & * & * \\ * & \widehat{A}_3 & 0 \\ * & 0 & 0 \end{pmatrix} \in \text{tanCone} \left(\begin{pmatrix} \widehat{A}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_{n-k_1} \right),$$

i.e., $\overline{\pi}_{k_1}(A_3) \in \text{tanCone}\,(\overline{\pi}_{k_1}(A_2), K_{n-k_1})$. Denote $k_1 + \ldots + k_i$ by N_i and set $N_0 = 0$. Then, for $i > 2$, we have:

$$\overline{\pi}_{N_{i-2}}(A_i) \in \text{tanCone}\,(\overline{\pi}_{N_{i-2}}(A_{i-1}), K_{n-N_{i-2}}).$$

Moreover, if the last subproblem $\overline{\pi}_{N_m}(K_n, P^T LP, P^T cP)$ has a feasible solution, we can pick some $P^T c' P \in P^T(L + c)P$ such that $\overline{\pi}_{N_m}(P^T c' P)$ is positive semidefinite. Then

$$\overline{\pi}_{N_{m-1}}(P^T c' P) \in \text{tanCone}\,(\overline{\pi}_{N_{m-1}}(A_m), K_{n-N_{m-1}}).$$

Given $\epsilon > 0$, by picking $\alpha_m > 0$ sufficiently large we have

$$\text{dist}(\overline{\pi}_{N_{m-1}}(P^T c' P + \alpha_m A_m), K_{n-N_{m-1}}) < \epsilon.$$

Now, $\overline{\pi}_{N_{m-2}}(P^T c' P + \alpha_m A_m)$ does not necessarily lie on the tangent cone of $\overline{\pi}_{N_{m-2}}(A_{m-1})$ at $K_{n-N_{m-2}}$, but still it is possible to pick $\alpha_{m-1} > 0$ such that

$$\text{dist}(\overline{\pi}_{N_{m-2}}(P^T c' P + \alpha_m A_m + \alpha_{m-1} A_{m-1}), K_{n-N_{m-2}}) < 2\epsilon.$$

In order to show this, let $h \in K_{n-N_{m-1}}$ be such that

$$\|\overline{\pi}_{N_{m-1}}(P^T c' P + \alpha_m A_m) - h\| = \text{dist}(\overline{\pi}_{N_{m-1}}(P^T c' P + \alpha_m A_m), K_{n-N_{m-1}}).$$

Now, define $\widetilde{h}$ to be the matrix $\overline{\pi}_{N_{m-2}}(P^T c' P + \alpha_m A_m)$, except that the lower right $(n - k_m) \times (n - k_m)$ block is replaced by h. It follows readily that $\widetilde{h}$ lies on the tangent cone of $\overline{\pi}_{N_{m-2}}(A_{m-1})$. Then, we may pick $\alpha_{m-1} > 0$ sufficiently large such that $\text{dist}(\overline{\pi}_{N_{m-2}}(\alpha_m A_m) + h, K_{n-N_{m-2}}) < \epsilon$. Let $y_1 = \overline{\pi}_{N_{m-2}}(P^T c' P + \alpha_m A_m)$, $y_2 = \overline{\pi}_{N_{m-2}}(\alpha_{m-1} A_{m-1})$. We then have the following implications:

$$\begin{aligned} \text{dist}(y_1 + y_2, K_{n-N_{m-2}}) &\le \text{dist}(y_1 - \widetilde{h}, K_{n-N_{m-2}}) + \text{dist}(y_2 + \widetilde{h}, K_{n-N_{m-2}}) \\ &\le \|\overline{\pi}_{N_{m-1}}(P^T c' P + \alpha_m A_m) - h\| + \epsilon \le 2\epsilon. \end{aligned}$$

If we continue in this way, it becomes clear that $\alpha_1, \ldots, \alpha_m$ can be selected such that $\text{dist}(P^T c' P + \alpha_m A_m + \alpha_{m-1} A_{m-1} + \ldots + \alpha_1 A_1, K_n) < m\epsilon$. This shows how the directions $\{A_1, \ldots, A_m\}$ can be used to construct points that are arbitrarily close to K_n, when the last subproblem is feasible. This leads to the next theorem.

Theorem 11. *If (K_n, L, c) is weakly infeasible then there exists an affine space of dimension at most $n-1$ such that $L' + c' \subseteq L + c$ and (K_n, L', c') is weakly infeasible.*

Proof. The construction above shows that if $P^T L' P \subseteq P^T LP$ is the space spanned by $\{A_1, \ldots, A_m\}$ and $P^T c' P$ is taken as above, then $(K_n, P^T L' P, P^T c' P)$ is weakly infeasible, which implies that (K_n, L', c') is weakly infeasible as well . As (K_n, L, c) is weakly infeasible, we have $m > 0$. We also have $k_1 + \ldots + k_m \leq n$, which implies $m \leq n$. Notice that $\overline{\pi}_n(K_n, P^T LP, P^T cP)$ is strongly feasible, because it is equal to the system $(\{0\}, \{0\}, 0)$. Therefore $k_1 + \ldots + k_m < n$, which forces $m < n$. □

In the presence of strong feasibility, the discussion above shows how to construct an interior feasible solution for (K_n, L, c) starting from an interior feasible solution to

$$\overline{\pi}_{k_1+\ldots+k_m}(K_n, P^T LP, P^T cP)$$

and the directions $\{A_1, \ldots, A_m\}$. This is summarized in the following proposition.

Proposition 12. *Suppose we apply* ***FP*** *to (K_n, L, c). Suppose also that there exists some $c' \in L + c$ such that $\overline{\pi}_{k_1+\ldots+k_m}(P^T c' P)$ is positive definite. Then, there exists a subspace L' of dimension m such that $L' + c' \subseteq L + c$ and (K_n, L', c') is strongly feasible. We also have the bound $m \leq n$.*

Proof. As in Theorem 11, we take L' to be the space spanned by $\{A_1, \ldots, A_m\}$. The difference, however, is that, this time, it is possible that $m = n$. As before, a constructive way of showing the existence of $\alpha_1, \ldots, \alpha_m$ such that $P^T c' P + \sum_{i=1}^m \alpha_i A_i$ is positive definite is by first selecting α_m such that $\overline{\pi}_{k_1+\ldots+k_{m-1}}(P^T c' P + \alpha_m A_m)$ is positive definite and then recursively working our way up to the top. To see that we can indeed take α_m such that $\overline{\pi}_{k_1+\ldots+k_{m-1}}(c + \alpha_m A_m)$ is positive definite, it suffices to do as in the proof of "1) $\Leftarrow$" of Theorem 5. □

4. Recent Developments and Conclusions

In this article we presented an analysis of weakly infeasible problems via the procedure **FP**, which outputs a finite set of directions and for weakly infeasible problem, they can be used to construct points in $L + c$ arbitrarily close to K_n. Extension of our analysis to blockwise SDPs and to other classes of conic linear programs is an interesting topic for future research.

Since our original technical report on weakly infeasible SDPs appeared [9], there have been a few developments. Recently, Liu and Pataki [8] extended Theorem 11 to general closed convex cones. They showed that if (K, L, c) is weakly infeasible, then there is always a sub affine space of dimension at most $\ell_{K^*} - 1$ such that the corresponding system is weakly infeasible, where ℓ_{K^*} is the longest chain of faces in the dual cone K^*. Moreover, they showed that the bound can be improved to $\ell_K - 2$ when $K = K_n = K_n^*$, which matches our bound since $\ell_{K_n} = n + 1$. We also developed an analogous bound for second order cone programs. If K is a product of m Lorentz cones, then we have the bound m for the dimension of the affine space [11]. We note that this is tighter than the bound predicted by [8].

In another work [10], we used the results described here to show how to completely solve an arbitrary SDP by a finite sequence of problems that have primal and dual interior feasible points. By "completely solve", we mean that the feasibility status is determined exactly together with the optimal value. We also are able to detect unboundedness and if the optimal value is finite, we discover whether its attained or not. If not, then we use the reducing directions to construct feasible points approaching optimality.

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References

[1] L. Blum, F. Cucker, M. Shub, and S. Smale: *Complexity and Real Computation* (Springer, New York, 1998).

[2] J.F. Bonnans and A. Shapiro: *Perturbation Analysis of Optimization Problems* (Springer Science & Business Media, 2000).

[3] J.M. Borwein and H. Wolkowicz: Facial reduction for a cone-convex programming problem. *Journal of the Australian Mathematical Society (Series A)*, **30-03** (1981), 369–380.

[4] J.M. Borwein and H. Wolkowicz: Regularizing the abstract convex program. *Journal of Mathematical Analysis and Applications*, **83-02** (1981), 495–530.

[5] R.A. Horn and C.R. Johnson: *Matrix Analysis* (Cambridge University Press, 1990).

[6] I. Klep and M. Schweighofer: An exact duality theory for semidefinite programming based on sums of squares. *Mathematics of Operations Research*, **38-3** (2013), 569–590.

[7] M. Liu and G. Pataki: Exact duality in semidefinite programming based on elementary reformulations. *SIAM Journal on Optimization*, **25-3** (2015), 1441–1454.

[8] M. Liu and G. Pataki: Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming. Optimization Online, July (2015).

[9] B.F. Lourenço, M. Muramatsu, and T. Tsuchiya: A structural geometrical analysis of weakly infeasible SDPs. Optimization Online, November (2013).

[10] B.F. Lourenço, M. Muramatsu, and T. Tsuchiya: Solving SDP completely with an interior point oracle. Optimization Online, July (2015).

[11] B.F. Lourenço, M. Muramatsu, and T. Tsuchiya: Weak infeasibility in second order cone programming. Optimization Online (to appear in Optimization Letters), September (2015).

[12] Z. Luo and J. Sturm: Error analysis. In H. Wolkowicz, R. Saigal, and L. Vandenberghe (eds.): *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications* (Kluwer Academic Publishers, 2000), 163–189.

[13] G. Pataki. The geometry of semidefinite programming: In H. Wolkowicz, R. Saigal, and L. Vandenberghe (eds.): *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications* (Kluwer Academic Publishers, 2000), 29–65.

[14] G. Pataki: On the closedness of the linear image of a closed convex cone. *Mathematics of Operations Research*, **32-2** (2007), 395–412.

[15] G. Pataki: Strong duality in conic linear programming: Facial reduction and extended duals. In *Computational and Analytical Mathematics* (Springer, New York, 2013), volume 50, 613–634.

[16] G. Pataki: Bad semidefinite programs: they all look the same. Technical report, Department of Statistics and Operations Research, University of North Carolina at Chapel Hill (2014).

[17] I. Pólik and T. Terlaky: Exact duality for optimization over symmetric cones. AdvOL Report 2007/10, McMaster University, Advanced Optimization Lab, Hamilton, Canada (2007).

[18] L. Porkolab and L. Khachiyan: On the complexity of semidefinite programs. *Journal of Global Optimization*, **10** (1997), 351–365.

[19] Z.Q. Luo, J.F. Sturm, and S. Zhang: Duality and self-duality for conic convex programming. Technical report, Econometric Institute, Erasmus University Rotterdam, The Netherlands (1996).

[20] Z.Q. Luo, J.F. Sturm, and S. Zhang: Duality results for conic convex programming. Technical report, Econometric Institute, Erasmus University Rotterdam, The Netherlands (1997).

[21] M.V. Ramana: An exact duality theory for semidefinite programming and its complexity implications. *Mathematical Programming* **77**, (1997), 129–162.

[22] M.V. Ramana, L. Tunçel, and H. Wolkowicz: Strong duality for semidefinite programming. *SIAM Journal on Optimization*, **7-3** (1997), 641–662.

[23] R.T. Rockafellar: *Convex Analysis* (Princeton University Press, 1997).

[24] J.F. Sturm: Error bounds for linear matrix inequalities. *SIAM Journal on Optimization*, **10-4** (2000), 1228–1248.

[25] H. Waki: How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization. *Optimization Letters*, **6-8** (2012), 1883–1896.

[26] H. Waki and M. Muramatsu: Facial reduction algorithms for conic optimization problems. *Journal of Optimization Theory and Applications*, **158-1** (2013), 188–215.

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