Journal of the Operations Research Society of Japan
Vol. 59, No. 4, October 2016, pp. 291–311

DECOMPOSITION OF A MULTI-STATE SYSTEM BY SERIES SYSTEMS

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(Received August 19, 2015; Revised July 26, 2016)

Abstract In the theory and engineering of reliability, it is one of the important issues for reliability researchers to develop effective evaluation methods of reliability performance of systems. For the case of a binary state system, using the minimal-path or minimal-cut sets of the system, an effective method is given by decomposing a structure function into series or parallel systems. For multi-state systems with partially ordered state spaces, however, sufficient examinations of the decomposition and related subjects have not been given. In this paper, following the definition of a series system of Ohi [28], we show a necessary and sufficient condition for a multi-state system to be a series system, which denotes that a system is series system if and only if the serialisation at system's and component's levels are equivalent with each other, and then presenting the series-decomposition, we show the relationship among the stochastic bounds which is given by the decomposition. Furthermore, some examinations about the pattern of maximal state vectors of a series system are given. In this paper, we omit the discussions about the parallel system, since it is ordered set theoretically dual of the series system.

Keywords: Reliability, multi-state system, partially ordered state space, series system, parallel system, decomposition by series systems, stochastic bound

1. Introduction

One of the most important issues in reliability theory is to explain ordered set theoretical and probabilistic relations between a system and components, which gives us reliability evaluation methods useful for solving practical reliability evaluation problems. For binary state systems having the common state space $\{0, 1\}$, many works have been performed so far, and the fruits of these works are applied to practical problems. For example see Mine [14], Birnbaum and Esary [4], Birnbaum, Esary and Sauder [3], Esary and Proschan [5], and these works are totally summarised in a great book by Barlow and Proshcan [1].

Systems and their components, however, could practically take many intermediate performance levels between perfectly functioning and complete failure states, and furthermore several states sometimes can not be compared with each other.

For example, suppose a situation that the state of a component is taken to be temperature and the optimal temperature is T_o. If there are different two states T_1 and T_2 satisfying $T_1 < T_o < T_2$ in a numerical order, we may not define order relation between these T_1 and T_2 from the point of degradation, especially when these two temperatures correspond to different kinds of defects of the system.

Hence multi-state reliability models with partially ordered state spaces are required to understand and solve practical reliability problems, and some evaluation methods have been proposed and applied to real problems.

Multi-state systems with totally ordered state spaces have been mathematically studied by many authors. See Barlow and Wu [2], Griffith [6], El-Neweihi, Proschan and Sethuraman [17], Natvig [15], Ohi and Nishida [18–20, 22]. Huang, Zuo and Fang [7] extended a binary state consecutive k-out-of-n system which is well observed in a practical situation to the multi-state case.

Levitin [8–10] have extensively applied the universal generating function (UGF) method to solve reliability problems of multi-state systems and showed its effectiveness. UGF method was first proposed by Ushakov [30, 31] as a stochastic evaluation method of multi-state systems, and is thought to be effective especially for a system hierarchically composed of physical modules like series-parallel or parallel-series systems, which is practically well observed for a system treating flow of oil, gas, job shop and so on.

Ohi [25] has generally given stochastic upper and lower bounds for system's reliability performances via modular decompositions, which are convenient for systems designer and analyst, since the bounds are intuitively understood.

Assuming the state spaces to be totally ordered sets, Natvig [16], Lisnianski and Levitin [13], Lisnianski, Frenkel and Ding [12] have summarised the work about multi-state systems performed so far, and we may find examples of practical applications.

Studies in the case of partially ordered state spaces have started in recent years. Levitin [11], from a practical point of view, has proposed a multi-state vector-k-out-of-n system of which state spaces are supposed to be a subset of $\boldsymbol{R}^n$, a special type of partially ordered set. Yu, Koren and Guo [32], emphasising the case that the states are not necessarily totally ordered, have proposed a model of multi-state system having partially ordered state spaces. Ohi [23, 24] have been trying to build up a general theoretical frame work of multi-state systems. The former paper has given an existence theory of series and parallel systems, series-parallel decomposition of multi-state systems, when the state spaces are lattice sets. The latter work have given a characterisation of a module by φ-equivalent relation under the lattice set assumption for the state spaces. Furthermore Ohi [26, 27], continuations of Ohi [25], shows upper and lower bounds for $\boldsymbol{P}\{\varphi \geqq s\}$, the probability that the system's state is greater than or equals to s, when the state spaces are partially ordered sets.

One of the typical evaluation methods of the reliability performance of a binary state system is to utilise the decomposition of the structure function by minimal path series or minimal cut parallel systems, which is well-known to be max-min or min-max formulae of the structure function, i.e., for every state vector $\boldsymbol{x}$,

$$\varphi(\boldsymbol{x}) = \max_{1\leqq i\leqq p} \min_{j\in P_i} x_j, \tag{1.1}$$

where $\min_{j\in P_i} x_j$ means the series system composed of components of a minimal path set P_i. Considering the minimal cut sets, we have a min-max formula of the structure function φ, which is dual to the max-min formula. These formulae mean that two simplest structures, series and parallel, form basis and every binary state structure function can be expressed by taking max or min of series or parallel systems, respectively. Various stochastic evaluation methods and ageing properties of binary state systems are derived from these formulae. See Barlow and Proschan [1].

For multi-state systems, such a decomposition formulae have been given, when the state spaces are lattice sets, see Ohi [24], but not when they are generally partially ordered sets.

When the state spaces are totally ordered sets and the cardinal numbers of the state spaces are the same, a structure function φ of a multi-state series system is usually defined to be $\varphi(\boldsymbol{x}) = \min_{1\leqq i\leqq n} x_i$, where x_i means the state of the component i. But this definition has some problems. One of the most serious problems is that this formula supposes the possibility of comparing the states of different components. Usually a system is composed of many qualitatively different components. Even if, for example, the state spaces of components i is assumed to be $\{0, 1, \cdots, M_i\}$, the state 1 of the component 1 and the state 1 of the component 2 generally mean different physical situations, and may not be compared with each other. Ohi [18, 22, 24] have defined series and parallel systems for totally ordered and lattice state spaces, not using the above formula, and showed a necessary and sufficient condition for a system to be a series (parallel) system. Ohi [18, 22], assuming relevant property different from that shown in this paper, have proved that a structure function of a series system can be conventionally expressed as $\varphi(\boldsymbol{x}) = \min_{1 \leqq i \leqq n} x_i$, when the state spaces are totally ordered sets and have the same cardinal number. Ohi [23], for lattice state spaces, has examined a decomposition by series systems and stochastic bounds for the reliability of a multi-state system.

Definitions of series and parallel systems in a partially ordered case are given by Ohi [28]. In this paper, a necessary and sufficient condition for a system to be series is proved, which is a generalisation of the well known condition in the binary state case and means that the system is series if and only if the serialisation at the component and the system levels are equivalent with each other. The present paper lists important concepts as generalised infimum needed for examinations of the series systems. Some theorems proved in Ohi [28] are also listed in this paper in a shortened form because of its importance.

Using the definition of Ohi [28], Ohi [29] has shown a series decomposition and stochastic upper and lower bounds based on the decomposition. These decompositions are shown to relate to the variety of deterioration processes of the system and components. We give stochastic upper and lower bounds by these decompositions and relations of these bounds to those of Ohi [26], and show that lower bounds are coincident with each other but not are the upper bounds.

These considerations are easily transformed into the parallel case by the duality, and so we focus on series systems.

2. Notations

In this paper we use the following notations. Finite sets $C = \{1, 2, \cdots, n\}$, Ω_i $(i \in C)$ and S are respectively the set of the components, the state space of the i-th component and the state space of the system. φ is a mapping from the product ordered set $\Omega_C = \prod_{i \in C} \Omega_i$ to S, so-called a structure function of the system. The precise definition of a multi-state system is presented in Definition 4.1.

1. The symbol $\backslash$ denotes the difference between two sets A and B as

$$A \backslash B \overset{def}{=} \{ \ x \mid x \in A \ , \ x \notin B \ \}.$$

2. Ω_C is the product ordered set of Ω_i $(i \in C)$. A symbol $\times$ is also used to denote the product set as $\Omega_1 \times \Omega_2$.
3. An element $\boldsymbol{x} \in \Omega_C$ is precisely written as $\boldsymbol{x} = (x_1, \cdots, x_n),\ x_i \in \Omega_i\ (i = 1, \cdots n)$.
   For $\boldsymbol{x} \in \Omega_C$, $(u_i, \boldsymbol{x})$ $(i \in C)$ is used to emphasize that the i-th element of $\boldsymbol{x}$ is u. $(\cdot_i, \boldsymbol{x})$ is used to denote the element of $\Omega_{C \backslash \{i\}}$ given by eliminating x_i from $\boldsymbol{x}$, or sometimes denotes an combination of the states of components of $C \backslash \{i\}$, i.e., for $\boldsymbol{x} \in \Omega_{C \backslash \{i\}}$, the $\boldsymbol{x}$ is also written as $(\cdot_i, \boldsymbol{x})$.
4. For $A \subseteqq \Omega_i$ and $(\cdot_i, \boldsymbol{x}) \in \Omega_{C \backslash \{i\}}$, $A \otimes (\cdot_i, \boldsymbol{x})$ denotes

$$A \otimes (\cdot_i, \boldsymbol{x}) \overset{def}{=} \{(a_i, \boldsymbol{x}) \mid a \in A \ \}.$$

5. In this paper orders are commonly written as $\leqq$ except $\overset{\circ}{\leqq}$ defined in Section 3. For elements x, y of an ordered set W, $x < y$ means $x \leqq y$ and $x \neq y$.

6. When each state space Ω_i $(i \in C)$ is endowed with an order $\leqq$, the product set Ω_C is assumed to be the product ordered set. Then, for state vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω_C, $\boldsymbol{x} \leqq \boldsymbol{y}$ means $x_i \leqq y_i$ for all $i \in C$, and $\boldsymbol{x} < \boldsymbol{y}$ means $\boldsymbol{x} \leqq \boldsymbol{y}$ and $x_i < y_i$ for some $i \in C$.
7. When an order $\leqq$ is assumed on S, for any state $s \in S$,

$$\begin{aligned} \varphi^{-1}[s, \to) &\stackrel{def}{=} \{ \boldsymbol{x} \in \Omega_C \mid s \leqq \varphi(\boldsymbol{x}) \}, \\ \varphi^{-1}(\leftarrow, s] &\stackrel{def}{=} \{ \boldsymbol{x} \in \Omega_C \mid \varphi(\boldsymbol{x}) \leqq s \}, \\ \varphi^{-1}(s) &\stackrel{def}{=} \{ \boldsymbol{x} \in \Omega_C \mid \varphi(\boldsymbol{x}) = s \}. \end{aligned}$$

8. $MI(W)$ and $MA(W)$ generally denotes the set of all the minimal and maximal elements of a finite ordered set W, respectively. In this paper a partially ordered set is called simply an ordered set. An element x of W is called a minimal (maximal) element when there exists no element y of W such that $y < x$ $(x < y)$. For example, $MI\,(\varphi^{-1}[s, \to))$ is the set of all the minimal elements of $\varphi^{-1}[s, \to)$, when orders are assumed on S and Ω_i $(i \in C)$. The order on the product set Ω_C is defined to be the product order assumed above on Ω_i $(i \in C)$.
9. When a set W is a lattice set, for $x, y \in W$, $x \wedge y$ and $x \vee y$ respectively denote

$$x \wedge y = \inf\{x, y\}, \quad x \vee y = \sup\{x, y\},$$

i.e., which are the infimum and the supremum of x and y, respectively. Generally, the infimum of $A \subseteqq W$ is denoted by $\inf A$, when the infimum exists.

3. Ordered Set Theoretical Preliminaries

In this section, assuming W to be an ordered set, we prepare some ordered set theoretical notions needed for our examination.

For $a, b \in W$ such that $a < b$, a is called a predecessor of b or b is called a successor of a, when there is no element $c \in W$ such that $a < c < b$. In this case we write $b = a + 1$ or $a = b - 1$. A predecessor and a successor are not generally unique.

$(s_0, \cdots, s_p)$ is called a path of length p from s_0 to s_p, when the following condition is satisfied.

$$s_0 < s_1 < \cdots < s_p, \quad s_{i+1} = s_i + 1, \quad i = 0, \cdots, p-1.$$

A path is called to be maximal, when the length of the path can not be extended by adding other elements to the path.

Since a finite ordered set has necessarily maximal and minimal elements, a maximal path on W is a path between a minimal and a maximal elements, and there is no other type of maximal path.

Example 3.1. In an ordered set which is defined by Hasse diagram of Figure 1, (a_2, b_4, b_2) is a maximal path, (b_4, b_2) is a path but not maximal and (a_1, b_2) is not a path. All the maximal paths are (a_1, b_4, b_1), (a_1, b_4, b_2), (a_2, b_4, b_1), (a_2, b_4, b_2), (a_1, b_3), (a_2, b_3), (a_3, b_3). Notice that (a_1, b_1) is not maximal, since there exists b_4 between them.

For an element $w \in W$ and a subsets $A, B \subseteq W$, we define the following relations.

$$\begin{aligned} w \leqq B &\stackrel{def}{\Longleftrightarrow} \forall b \in B,\ w \leqq b, \\ A \leqq B &\stackrel{def}{\Longleftrightarrow} \forall a \in A,\ a \leqq B \Longleftrightarrow \forall a \in A,\ \forall b \in B,\ a \leqq b, \\ A \stackrel{\circ}{\leqq} B &\stackrel{def}{\Longleftrightarrow} \forall a \in A,\ \exists b \in B,\ a \leqq b, \\ A \stackrel{\circ}{=} B &\stackrel{def}{\Longleftrightarrow} A \stackrel{\circ}{\leqq} B \text{ and } B \stackrel{\circ}{\leqq} A. \end{aligned}$$

$A \subseteqq B$ clearly implies $A \overset{\circ}{\leqq} B$, but we notice that $A = B$ is not generally coincident with $A \overset{\circ}{=} B$.

Example 3.2. Figures 1 and 2 give us examples of ordered sets for the above definitions by Hasse diagrams.

In Figure 1, for $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$, $A \leqq B$ holds. When $C = \{b_1, b_2, b_3, a_1\}$ and $D = \{b_1, b_2, b_3, b_4, a_3\}$, $C \overset{\circ}{=} D$ is easily verified to hold, even though $C = D$ does not hold.

In Figure 2, for $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$, $A \overset{\circ}{\leqq} B$ holds, but $A \leqq B$ does not hold, since, for example, $a_1 \leqq b_4$ does not hold.

It is easily proved that the following equivalent relation holds.

$$A \overset{\circ}{=} B \iff MA(A) = MA(B), \tag{3.1}$$

which means that when $A \overset{\circ}{=} B$, maximal elements of A and B are equivalent.

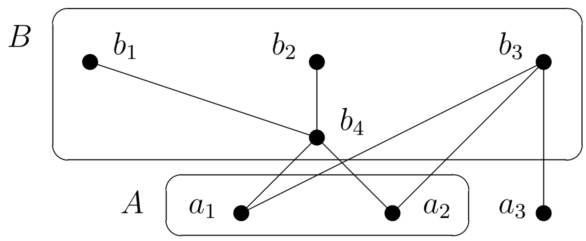

Figure 1: An ordered set having the $\leqq$ related subsets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$, and the $\overset{\circ}{=}$ related subsets $C = \{b_1, b_2, b_3, a_1\}$ and $D = \{b_1, b_2, b_3, b_4, a_4\}$

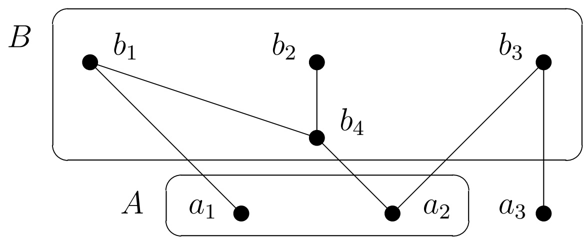

Figure 2: An ordered set having subsets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$ between which the relation $A \overset{\circ}{\leqq} B$ holds

Definition 3.1. LMA (lower maximal) of A, denoted by $LMA(A)$, is defined as

$$LMA(A) \overset{def}{=} MA\{\ z \mid z \leqq A\ \},$$

which is the set of all the maximal elements of $\{\ z \mid z \leqq A\ \}$.

Remark 3.1. (i) When A has the minimum element denoted by $\min A$, then $LMA(A) = \{\min A\}$. Hence, when W is a finite totally ordered set, every subset A of W has the minimum element and so $LMA(A) = \{\min A\}$.

(ii) When W is a finite lattice, every subset A of W has the infimum and then $LMA(A) = \{\inf A\}$.

By these remarks, we know that LMA is a natural extension of the concept of infimum from the lattice case to the general partially ordered case. See also Example 3.3 (iii).

Theorem 3.1. If

$$\exists a \in LMA(A),\ a \in A, \tag{3.2}$$

holds, then A has the minimum element and $LMA(A) = \{\min A\}$. By Remark 3.1 (i), the relation (3.2) is equivalent to that A has the minimum element.

The proof of Theorem 3.1 is easy and omitted here.

Example 3.3. We suppose that a finite ordered set W is given by the Hasse diagram of Figure 3.

(i) For a subset A of W given by $A = \{a, u, v, w, x, y, z\}$, a is the minimum element of A and then clearly $LMA(A) = \{a\}$, where $\{\ p \mid p \leqq A\ \} = \{a, b, c, d\}$ of which maximum element is also a.

(ii) For a subset $B = \{u, v\}$ of W, $LMA(B) = \{x, y\}$.

(iii) For a subset $C = \{v, w\}$, $LMA(C) = \{z\} = \{\inf C\}$.

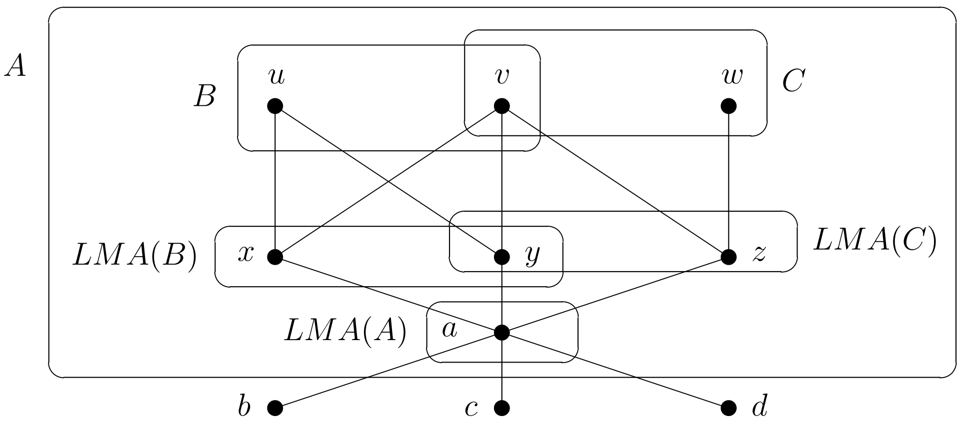

Figure 3: An ordered set of Example 3.3

4. Multi-state Systems

Definition 4.1. A multi-state system composed of n components is defined to be a triplet $(\prod_{i \in C} \Omega_i, S, \varphi)$ satisfying the following conditions.

(i) $C = \{1, \cdots, n\}$ is a set of components.

(ii) $\Omega_i (i \in C)$ and S are finite ordered sets, denoting the state spaces of the i-th component and the system, respectively. Each state space has the minimum and the maximum elements, denoted by m_i and M_i for Ω_i, and m and M for S. The minimum and maximum elements mean the perfect failure and the perfect functioning states, respectively.

(iii) $\prod_{i \in C} \Omega_i$ is the product ordered set of Ω_i $(i \in C)$, which is simply written as Ω_C. An element $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega_C$ is called a state vector.

(iv) φ is a surjection from Ω_C to S, which is called a structure function and reflects an inner structure of the system.

A multi-state system (Ω_C, S, φ) is simply called a system φ, when there is no confusion.

Definition 4.2. A system φ is called increasing when

$$\forall \boldsymbol{x}, \forall \boldsymbol{y} \in \Omega_C, \quad \boldsymbol{x} \leqq \boldsymbol{y} \Longrightarrow \varphi(\boldsymbol{x}) \leqq \varphi(\boldsymbol{y})$$

The increasing property means that the state of the system does not degrade when the states of the components are improved.

Definition 4.3. A system φ is called relevant when the following condition is satisfied.

$$\forall i \in C,\ \forall k, \forall l \in \Omega_i \text{ such that } k \neq l,\ \exists(\cdot_i, \boldsymbol{x}),\ \varphi(k_i, \boldsymbol{x}) \neq \varphi(l_i, \boldsymbol{x}).$$

This relevant property is not a strict condition, since any non-relevant system is easily converted to a relevant one. If a system φ is not relevant, we have

$$\exists i \in C,\ \exists k, \exists l \in \Omega_i \text{ such that } k \neq l,\ \forall(\cdot_i, \boldsymbol{x}),\ \varphi(k_i, \boldsymbol{x}) = \varphi(l_i, \boldsymbol{x}),$$

which means that these two sates k and l contribute to the system's performance in the same way and may be merged into one state. Hence we have a relevant system equivalent to the original non-relevant system.

For an increasing system, the relevant property of the component i means that for states k and l of Ω_i such that $k < l$, we have

$$\exists(\cdot_i, \boldsymbol{x}),\ \varphi(k_i, \boldsymbol{x}) < \varphi(l_i, \boldsymbol{x}),$$

denoting that the improvement of the component's state implies strict improvement of the system's state in some circumstance $(\cdot, \boldsymbol{x})$. When $k < l$ and $l < k$ do not hold, we may only say that these two states differently contribute to the system's performance at a circumstance, but not excluding the improvement of the system's state.

Definition 4.4. Let (Ω_C, S, φ) be an increasing system.

(i) The system is called minimally normal, when the following condition is satisfied.

$$MI\left(\varphi^{-1}[s, \rightarrow)\right) = MI\left(\varphi^{-1}(s)\right).$$

(ii) The system is called maximally normal, when the following condition is satisfied.

$$MA\left(\varphi^{-1}(\leftarrow, s]\right) = MA\left(\varphi^{-1}(s)\right).$$

(iii) The system is called simply normal, when it is minimally and maximally normal.

The normal property is introduced by Ohi [25, 26, 29], when stochastic bounds via a modular decomposition is given for the stochastic performance of multi-state systems. When the state spaces are totally ordered sets, it is proved in Ohi [25] that the system's state does not drastically change by the change of the components' state.

Theorem 4.1. Suppose that all the state spaces of components of an increasing and relevant system φ are totally ordered sets. We have the following relation.

$$\forall i \in C,\ \forall k \in \Omega_i,\ \exists s \in S,\ \exists \boldsymbol{x} \in MI(\varphi^{-1}(s)),\ x_i = k,$$

which means that every state of every component is an element of some minimal state vector.

Proof. First, we notice that Ω_i may be assumed to be $\{0, 1, \cdots, N_i\}$ without loss of generality. If a state $k \in \Omega_i$ of a component i does not satisfy the condition, we have

$$\forall(k_i, \boldsymbol{x}) \in \Omega_C, \forall \boldsymbol{y} \in MI\left(\varphi^{-1}(\varphi(k_i, \boldsymbol{x}))\right) \text{ such that } \boldsymbol{y} \leqq (k_i, \boldsymbol{x}),\ y_i \leqq k - 1 < k,$$

and so for every $(\cdot_i, \boldsymbol{x})$, $\varphi(k_i, \boldsymbol{x}) = \varphi((k-1)_i, \boldsymbol{x})$ holds and the system is not relevant. □

Our Theorem 4.1 weakened the assumption of Theorem 3.1 of Ohi [28] that all the state spaces are totally ordered sets to that all the state spaces of the components are totally ordered sets.

In the sequel, a system is assumed to be increasing, and is simply called a system.

Since $LMA(A) \leqq A$, $\varphi(LMA(A)) \leqq \varphi(A)$ holds by the increasing property of φ. Then we have easily the next Theorem.

Theorem 4.2. For a system φ, we have the following relation.

$$\forall A \subseteqq \Omega_C,\ \varphi(LMA(A)) \overset{\circ}{\leqq} LMA(\varphi(A))$$

When $A = \{\boldsymbol{x}, \boldsymbol{y}\}$ for $\boldsymbol{x}, \boldsymbol{y} \in \Omega_C$, Theorem 4.2 means $\varphi(LMA\{\boldsymbol{x}, \boldsymbol{y}\}) \overset{\circ}{\leqq} LMA\{\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y})\}$, and when W is a lattice, this relation is the well-known inequality $\varphi(\boldsymbol{x} \wedge \boldsymbol{y}) \leqq \varphi(\boldsymbol{x}) \wedge \varphi(\boldsymbol{y})$, which tells us that the serialisation at component level is generally worse than that at system level.

The next theorem, which is proved in Ohi [28], plays an important role for our examination, we again present here it and a shortened proof is shown in Appendices for convenience of readers.

Theorem 4.3. Let φ be a system. For a subset $A \subseteqq \Omega_C$, we have the following equivalent relations.

$$\begin{aligned} LMA(\varphi(A)) \overset{\circ}{=} \varphi(LMA(A)) &\iff LMA(\varphi(A)) = MA(\varphi(LMA(A))) \\ &\iff LMA(\varphi(A)) \subseteq \varphi(LMA(A)) \end{aligned}$$

Since the next theorem is also important, we show the proof of Ohi [28] in Appendices.

Theorem 4.4. For a system φ,

$$\forall A \subseteqq \Omega_C,\ \varphi(LMA(A)) \overset{\circ}{=} LMA(\varphi(A)) \tag{4.1}$$

hods if and only if

$$\forall s \in S,\ \varphi^{-1}[s, \rightarrow) \text{ has the minimum element.} \tag{4.2}$$

When all the state spaces are lattice sets, the necessary condition (4.1) of Theorem 4.4 is reduced to

$$\varphi(\boldsymbol{x} \wedge \boldsymbol{y}) = \varphi(\boldsymbol{x}) \wedge \varphi(\boldsymbol{y}), \tag{4.3}$$

which means that the serialisations at component and system levels are equivalent with each other. The equality (4.3) has been proved to be a necessary and sufficient condition for a multi-state system to be a series system, when the state spaces are lattices. See Ohi [24]. In the next section, adopting the notions of series and parallel systems given by Ohi [28, 29], we show a necessary and sufficient condition for a system to be a series system when the state spaces are partially ordered sets.

5. Series and Parallel Systems

Definition 5.1. (i) A system φ is called a series system, when for every $s \in S$, $\varphi^{-1}[s, \rightarrow)$ has the minimum element.

(ii) A system φ is called a parallel system, when for every $s \in S$, $\varphi^{-1}(\leftarrow, s]$ has the maximum element.

Since a structure function is a surjection, a series system is necessarily minimally normal. And then, a parallel system is maximally normal, because of duality. The next example shows us that a series system is not necessarily maximally normal.

Example 5.1. We consider a two-unit system having the state spaces as $\Omega_1 = \Omega_2 = \{0, 1, 2\}$ and $S = \{m, A, B, M\}$, where $0 < 1 < 2$, $m < A < M$ and $m < B < M$. There is no order relation between A and B. Ω_1 and Ω_2 are totally ordered state spaces and S is a partially ordered state space. A structure function $\varphi : \Omega_1 \times \Omega_2 \to S$ is defined as

$$\begin{aligned}
\varphi^{-1}(M) &= \{(2,2)\}, \\
MI\left(\varphi^{-1}(M)\right) &= \{(2,2)\},\ MA\left(\varphi^{-1}(M)\right) = \{(2,2)\}, \\
\varphi^{-1}(A) &= \{(2,1),(1,1)\}, \\
MI\left(\varphi^{-1}(A)\right) &= \{(1,1)\},\ MA\left(\varphi^{-1}(A)\right) = \{(2,1)\}, \\
\varphi^{-1}(B) &= \{(1,2),(0,2)\}, \\
MI\left(\varphi^{-1}(B)\right) &= \{(0,2)\},\ MA\left(\varphi^{-1}(B)\right) = \{(1,2)\}, \\
\varphi^{-1}(m) &= \{(2,0),(1,0),(0,1),(0,0)\}, \\
MI\left(\varphi^{-1}(m)\right) &= \{(0,0)\},\ MA\left(\varphi^{-1}(m)\right) = \{(2,0),(0,1)\}.
\end{aligned}$$

See Figure 4, where $\Omega_1 \times \Omega_2$ and S are given by Hasse diagrams. This system is a series system and not maximally normal, since

$$MA\left(\varphi^{-1}(\leftarrow, B]\right) = \{(1,2),(2,0)\},\ MA\left(\varphi^{-1}(B)\right) = \{(1,2)\}.$$

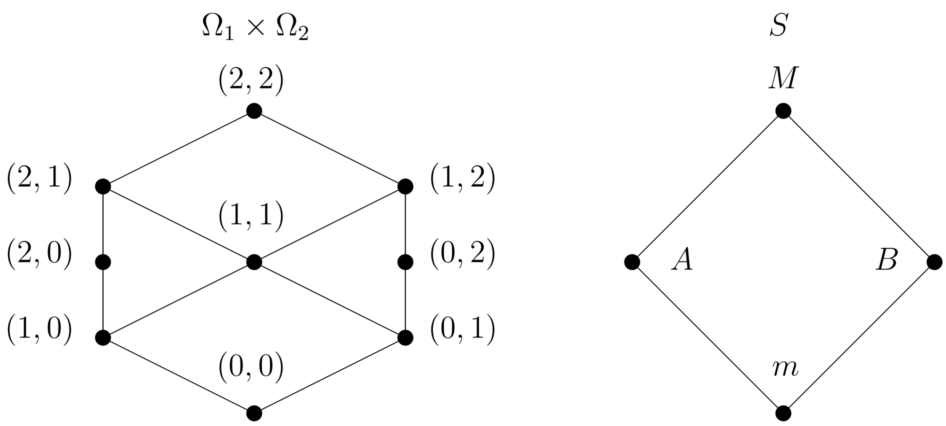

Figure 4: The Hasse diagrams of the state spaces $\Omega_1 \times \Omega_2$ and S of Example 5.1

Theorem 5.1. A system φ is a series system if and only if

$$\forall \boldsymbol{x}, \forall \boldsymbol{y} \in \Omega_C,\ \varphi(LMA\{\boldsymbol{x}, \boldsymbol{y}\}) \overset{\circ}{=} LMA\{\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y})\}. \tag{5.1}$$

Proof. If the system φ is series, then the equality (5.1) clearly holds from Theorem 4.4 by setting $A = \{\boldsymbol{x}, \boldsymbol{y}\}$. We prove that (5.1) implies that the system φ is a series system, that is to say, $\varphi^{-1}[s, \to)$ has the minimum element for every $s \in S$. We divide the proof into some steps.

(i) Since the structure function φ is a surjection, we have $\varphi(\boldsymbol{x}) = s$, for some $\boldsymbol{x} \in MI\left(\varphi^{-1}[s, \to)\right)$. So writing $MI\left(\varphi^{-1}[s, \to)\right) \equiv \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_k\}$, we may assume $\varphi(\boldsymbol{x}_1) = s$ without loss of generality.

(ii) First we consider $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$. Since $\varphi(\boldsymbol{x}_2) \geqq s$, from (7), we have

$$\varphi\left(LMA\{\boldsymbol{x}_1, \boldsymbol{x}_2\}\right) \overset{\circ}{=} LMA\{\varphi(\boldsymbol{x}_1), \varphi(\boldsymbol{x}_2)\} = \{\varphi(\boldsymbol{x}_1)\} = \{s\}.$$

(iii) Since $\{s\} \overset{\circ}{\leqq} \varphi\left(LMA\{\boldsymbol{x}_1, \boldsymbol{x}_2\}\right)$, from the definition of $\overset{\circ}{\leqq}$, we have $\varphi(\boldsymbol{y}) \geqq s$ for some $\boldsymbol{y} \in LMA\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$. On the other hand, from the definition of LMA, $\boldsymbol{y} \leqq \boldsymbol{x}_1$ and $\boldsymbol{y} \leqq \boldsymbol{x}_2$ hold. Then, $\varphi(\boldsymbol{y}) = s$ and so $\boldsymbol{y} \in \varphi^{-1}[s, \rightarrow)$ follows. Hence $\boldsymbol{y} = \boldsymbol{x}_1$ or $\boldsymbol{x}_2$ hold because of that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are minimal elements of $\varphi^{-1}[s, \rightarrow)$. As a consequence,

$$\boldsymbol{y} \in LMA\{\boldsymbol{x}_1, \boldsymbol{x}_2\},\ \boldsymbol{y} \in \{\boldsymbol{x}_1, \boldsymbol{x}_2\},$$

which means that $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ has the minimum element from Theorem 3.1 and thus $\boldsymbol{x}_1 = \boldsymbol{x}_2$ holds.

(iv) Applying the same argument to $\{\boldsymbol{x}_1, \boldsymbol{x}_3\}$, we have $\boldsymbol{x}_1 = \boldsymbol{x}_3$. Hence the inductive argument gives us $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_k$, which means that $\varphi^{-1}[s, \rightarrow)$ has the minimum element. □

For a parallel system, we may easily prove a necessary and sufficient condition in a dual way to the series case.

6. The Case of Totally Ordered State Spaces

6.1. The pattern of maximal state vectors of series systems

Next Theorem 6.1 shows us the pattern of maximal elements of $MI\left(\varphi^{-1}[s, \rightarrow)\right), s \in S$ of a series system, when the state space of the system is a totally ordered set.

Lemma 6.1. Let (Ω_C, S, φ) be a multi-state series system with a totally ordered state space S. Assuming $s \in S, s \neq M, \boldsymbol{b} \in MI\left(\varphi^{-1}(s+1)\right), b_1 \neq m_1$, for $\boldsymbol{x} = (x_1, M_2, \cdots, M_n)$, $\varphi(\boldsymbol{x}) \leqq s$ holds if and only if $x_1 \ngeqq b_1$.

Proof. If $x_1 \geqq b_1$, then $\boldsymbol{b} \leqq \boldsymbol{x}$ and so $\varphi(\boldsymbol{x}) \geqq \varphi(\boldsymbol{b}) = s + 1$. Thus

$$\varphi(\boldsymbol{x}) \leqq s \implies x_1 \ngeqq b_1.$$

If $\varphi(\boldsymbol{x}) \nleqq s$, in other words, $\varphi(\boldsymbol{x}) \geqq s + 1$, since the state space S is a totally ordered set, then $\boldsymbol{x} \geqq \boldsymbol{b}$ because of that the system is series. Thus $x_1 \geqq b_1$ and so

$$x_1 \ngeqq b_1 \implies \varphi(\boldsymbol{x}) \leqq s.$$

Hence the lemma is proved. □

From Lemma 6.1, we have easily the next Theorem 6.1 which gives us the pattern of the maximal state vectors of a series system.

Theorem 6.1. Under the same assumptions of Lemma 6.1, we have the following equality about the pattern of the maximal state vectors of a series system. For $s \in S$,

$$MA\left(\varphi^{-1}(\leftarrow, s]\right) = \bigcup_{i=1}^{n} MA\{\ (M_1, \cdots, M_{i-1}, x_i, M_{i+1}, \cdots, M_n) \mid b_i \ngeqq x_i\ \},$$

$$MA\{\ (M_1, \cdots, M_{i-1}, x_i, M_{i+1}, \cdots, M_n) \mid b_i \ngeqq x_i\ \}$$
$$= MA\{\ x_i \mid b_i \ngeqq x_i\ \} \otimes (M_1, \cdots, M_{i-1}, \cdot, M_{i+1}, \cdots, M_n).$$

Theorem 6.1 tells us that the problem to find the maximal state vectors of $\varphi^{-1}(\leftarrow, s]$ is the one to find the maximal elements of $\{\ x_i \mid b_i \ngeqq x_i\ \}$.

6.2. A necessary and sufficient condition for the existence of a series system

For a series system φ,

$$\forall s \in S,\ \min \varphi^{-1}[s, \rightarrow) = \min \varphi^{-1}(s),$$

since the structure function is generally a surjection.

Reminding that we have assumed φ to be relevant, we have for every $i \in C$

$$\forall k \in \Omega_i, \exists s \in S,\ \left(\min \varphi^{-1}(s)\right)_i = k \tag{6.1}$$

by Theorem 4.1, when the state spaces of the components are totally ordered sets. Then we have the first inequality of the next Theorem 6.2, clearly.

Theorem 6.2. When the state spaces of a system and components are finite totally ordered sets, a series system $\varphi : \Omega_C \rightarrow S$ exists if and only if the following inequality holds.

$$\max_{i \in C} |\Omega_i| \leqq |S| \leqq \sum_{i=1}^{n} \{|\Omega_i| - 1\} + 1,$$

where $|\cdot|$ means the cardinal number of a set.

The second inequality is clear by considering the maximum number of the state vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m$ such that

$$\boldsymbol{x}_1 \leqq \cdots \leqq \boldsymbol{x}_m,\ \boldsymbol{x}_i \neq \boldsymbol{x}_{i+1},\ i = 1, \cdots, m-1.$$

When $|\Omega_i| = |S|$ $(i \in C)$, a relevant series system is uniquely determined as

$$\varphi(\boldsymbol{x}) = \min_{1 \leqq i \leqq n} x_i, \tag{6.2}$$

where we assume $\Omega_i = S = \{0, 1, \cdots, N\}$ without loss of generality. In this case, from the relevant property and the equality of the cardinal numbers of the state spaces, we have necessarily $MI\left(\varphi^{-1}[s, \rightarrow)\right) = (s, \cdots, s),\ s \in S$, and so the formula (6.2) holds. This formula is commonly used as a definition of series system, when the state spaces are the same. We may say that our definition of series system is reasonable, since it include the formula in the special case of the same totally ordered state spaces used for physical systems as pipeline systems. See Lisnianski and Levitin [13], Natvig [16].

7. Decomposition along with the Variety of System's Deterioration Patterns

Let (Ω_C, S, φ) be an increasing system and $\mathcal{P}$ be the set of all the paths from the minimum element m to the maximum element M of S, in other words, maximal paths. For a path $\boldsymbol{p} = (s_0, \cdots, s_p) \in \mathcal{P}$, we define $\alpha_{\boldsymbol{p}} : \Omega_C \rightarrow S_{\boldsymbol{p}} = \{s_0, \cdots, s_p\}$ as the following.

$$\boldsymbol{x} \in \Omega_C,\ \alpha_{\boldsymbol{p}}(\boldsymbol{x}) \stackrel{def}{=} \max\{\ t \in S_{\boldsymbol{p}} \mid t \leqq \varphi(\boldsymbol{x})\ \}, \tag{7.1}$$

where we notice that $s_0 = m$ and $s_p = M$, and the set in the right hand side of (7.1) is not empty since for every $\boldsymbol{x} \in \Omega_C$, $\varphi(\boldsymbol{x}) \geqq m$ holds. In the sequel, we let $\boldsymbol{p}$ denote also the set $S_{\boldsymbol{p}}$ for simplicity.

Clearly we may have

$$\boldsymbol{x} \in \Omega_C,\ \varphi(\boldsymbol{x}) = s \implies \alpha_{\boldsymbol{p}}(\boldsymbol{x}) \begin{cases} = s, & \text{if } s \in \boldsymbol{p}, \\ \leqq s, & \text{if } s \notin \boldsymbol{p}. \end{cases}$$

Since there exists a path containing s for every $s \in S$, then when $\varphi(\boldsymbol{x}) = s$,

$$\forall \boldsymbol{p} \in \mathcal{P},\ \alpha_{\boldsymbol{p}}(\boldsymbol{x}) \leqq s,$$
$$\exists \boldsymbol{p} \in \mathcal{P} \text{ such that } s \in \boldsymbol{p},\ \alpha_{\boldsymbol{p}}(\boldsymbol{x}) = s$$

holds and so $\{\alpha_{\boldsymbol{p}}(\boldsymbol{x})\}_{\boldsymbol{p}\in\mathcal{P}}$ has the maximal element which is s. Thus we have the following decomposition of the structure function.

$$\boldsymbol{x} \in \Omega_C, \quad \varphi(\boldsymbol{x}) = \max_{\boldsymbol{p}\in\mathcal{P}} \alpha_{\boldsymbol{p}}(\boldsymbol{x}). \tag{7.2}$$

When S is a totally ordered set, a maximal path on S is unique, and so $\alpha_{\boldsymbol{p}}$ is unique and is φ itself.

A maximal path on S is a deterioration path from the maximum state to the minimum state. The number of the maximal paths corresponds to the variety of deteriorating paths. Then the formula (7.2) shows us that the structure function may be decomposed along with a deterioration process of the system. In the next section, the structure function $\alpha_{\boldsymbol{p}}$ is decomposed into a family of series systems.

8. Decomposition of a Structure Function into a Family of Series Systems

For a maximal path $\boldsymbol{p} = (s_0, \cdots, s_p) \in \mathcal{P}$, $\mathcal{P}_{\boldsymbol{p}}$ denotes the set of all maximal paths on $\bigcup_{s\in\boldsymbol{p}} MI\left(\varphi^{-1}(s)\right)$, where we notice $s_0 = m$.

For every $\boldsymbol{k} = (\boldsymbol{u}_0, \cdots, \boldsymbol{u}_l) \in \mathcal{P}_{\boldsymbol{p}}$, we define $\alpha^{\boldsymbol{p}}_{\boldsymbol{k}} : \Omega_C \to \{s_0, \cdots, s_p\}$ as the following.

$$\boldsymbol{x} \in \Omega_C,\ \alpha^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}) \overset{def}{=} \varphi\left(\max\{\boldsymbol{u}_i \in \boldsymbol{k} | \boldsymbol{u}_i \leqq \boldsymbol{x}\}\right),$$

where $\boldsymbol{u}_0 = \boldsymbol{m} = (m_0, \cdots, m_n)$. Then for $\boldsymbol{x} \in \Omega_C$, we have the next equivalent relation

$$\alpha^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}) \geqq \varphi(\boldsymbol{u}_i) \iff \boldsymbol{x} \geqq \boldsymbol{u}_i,$$

which means that $\alpha^{\boldsymbol{p}}_{\boldsymbol{k}}$ is a series system.

We have the following Theorem 8.1 which gives us a decomposition of $\alpha_{\boldsymbol{p}}$ by $\alpha^{\boldsymbol{p}}_{\boldsymbol{k}}$ $\left(\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}\right)$.

Theorem 8.1. For a maximal path $\boldsymbol{p} \in \mathcal{P}$ on S, we have

$$\boldsymbol{x} \in \Omega_C,\ \alpha_{\boldsymbol{p}}(\boldsymbol{x}) = \max_{\boldsymbol{k}\in\mathcal{P}_{\boldsymbol{p}}} \alpha^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}).$$

Proof. Supposing $\alpha_{\boldsymbol{p}}(\boldsymbol{x}) = s_l \in \{s_0, \cdots, s_p\}(= S_{\boldsymbol{p}})$, we have from the definition (7.1) of $\alpha_{\boldsymbol{p}}$

$$s_l \leqq \varphi(\boldsymbol{x}),\ s_{l+1} \nleqq \varphi(\boldsymbol{x}), \cdots, s_p \nleqq \varphi(\boldsymbol{x}).$$

Then

$$\forall j\ (l+1 \leqq j \leqq p),\ \forall \boldsymbol{a} \in MI\left(\varphi^{-1}(s_j)\right),\ \boldsymbol{a} \nleqq \boldsymbol{x},$$

and furthermore

$$\exists \boldsymbol{b} \in MI\left(\varphi^{-1}(s_l)\right),\ \boldsymbol{b} \leqq \boldsymbol{x}$$

hold. Then for $\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}$,

$$\varphi(\max\{\ \boldsymbol{u} \in \boldsymbol{k} \mid \boldsymbol{u} \leqq \boldsymbol{x}\ \}) \leqq s_l.$$

Especially for $\boldsymbol{k}$ containing $\boldsymbol{b}$,

$$\varphi(\max\{\ \boldsymbol{u} \in \boldsymbol{k} \mid \boldsymbol{u} \leqq \boldsymbol{x}\ \}) = s_l$$

holds. Thus the Theorem 8.1 is proved. □

A structure function φ may be decomposed by these series systems as the next Corollary, which corresponds to (1.1) in the binary case. The min of (1.1) is $\alpha^{\boldsymbol{p}}_{\boldsymbol{k}}$, and the max of (1.1) becomes to be the duplicated max, since the system's state space is an ordered set. When the system's state space is totally ordered set, the duplicated max comes to be single.

Corollary 8.1.

$$\boldsymbol{x} \in \Omega_C,\ \varphi(\boldsymbol{x}) = \max_{\boldsymbol{p}\in\mathcal{P}} \max_{\boldsymbol{k}\in\mathcal{P}_{\boldsymbol{p}}} \alpha^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}). \tag{8.1}$$

Then we have apparently the following inequality.

$$\forall \boldsymbol{p} \in \mathcal{P},\ \forall \boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}},\ \forall \boldsymbol{x} \in \Omega_C,\ \varphi(\boldsymbol{x}) \geqq \alpha^{\boldsymbol{p}}_{\boldsymbol{k}}(\boldsymbol{x}). \tag{8.2}$$

When all the state spaces of a system are the binary state space as $\{0,1\}$, the above equality (8.1) is equivalent to (1.1), the decomposition by the usual minimal path sets. A min formula of a structure function by parallel system is given by dual examination of (8.1).

From the above examination, we know that a structure function may be decomposed hierarchically along with the variety of deterioration paths of the system and the components.

Example 8.1. $(\Omega_1 \times \Omega_2, S, \varphi)$ is a two-unit system, each state space of which is given by

$$\Omega_i = \{m_i, A_i, B_i, M_i\},\ S = \{m, A, B, M\},$$

where A_i and B_i are not ordered, and A and B are not also ordered. See Figure 5.

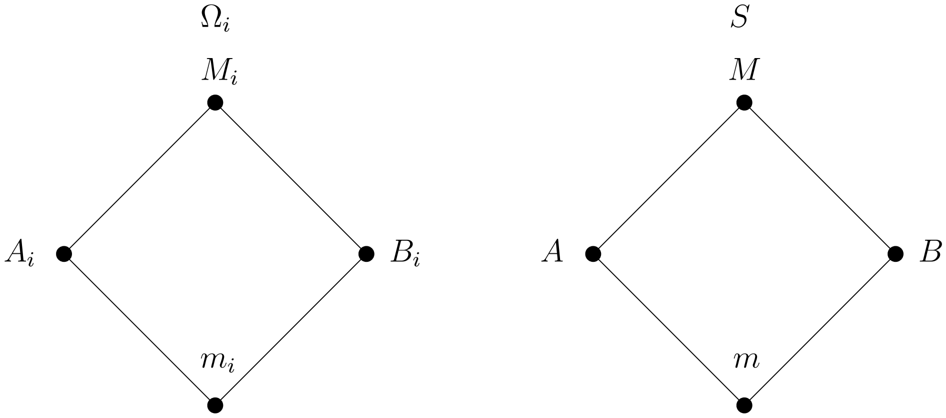

Figure 5: The Hasse diagrams of the state spaces $\Omega_i, i = 1,2$ and S of Example 8.1

The structure function φ is defined as

$$\begin{aligned}
\varphi^{-1}(M) &= \{(M_1,B_2),(B_1,M_2)\},\\
\varphi^{-1}(A) &= \{(A_1,A_2),(B_1,m_2),(B_1,A_2),(M_1,m_2),(M_1,A_2)\},\\
\varphi^{-1}(B) &= \{(m_1,M_2),(A_1,B_2),(A_1,M_2),(B_1,B_2)\},\\
\varphi^{-1}(m) &= \{(m_1,m_2),(m_1,A_2),(m_1,B_2),(A_1,m_2)\}.
\end{aligned}$$

The minimal state vectors are given by

$$\begin{aligned}
MI\left(\varphi^{-1}(M)\right) &= \{(M_1,B_2),(B_1,M_2)\},\\
MI\left(\varphi^{-1}(A)\right) &= \{(A_1,A_2),(B_1,m_2)\},\\
MI\left(\varphi^{-1}(B)\right) &= \{(m_1,M_2),(B_1,B_2),(A_1,B_2)\},\\
MI\left(\varphi^{-1}(m)\right) &= \{(m_1,m_2)\}.
\end{aligned}$$

On the state space S, there exist two maximal paths as

$$\boldsymbol{p}_1 = (m, A, M),\ \boldsymbol{p}_2 = (m, B, M).$$

For the maximal path $\boldsymbol{p}_1$, there are three maximal paths on $MI\left(\varphi^{-1}(m)\right) \cup MI\left(\varphi^{-1}(A)\right) \cup MI\left(\varphi^{-1}(M)\right)$,

$$\begin{aligned}
\boldsymbol{k}_{11} &= ((m_1, m_2), (A_1, A_2)),\\
\boldsymbol{k}_{12} &= ((m_1, m_2), (B_1, m_2), (M_1, B_2)),\\
\boldsymbol{k}_{13} &= ((m_1, m_2), (B_1, m_2), (B_1, M_2)).
\end{aligned}$$

For the maximal path $\boldsymbol{p}_2$, there are four maximal paths on $MI\left(\varphi^{-1}(m)\right) \cup MI\left(\varphi^{-1}(B)\right) \cup MI\left(\varphi^{-1}(M)\right)$,

$$\begin{aligned}
\boldsymbol{k}_{21} &= ((m_1, m_2), (m_1, M_2), (B_1, M_2)),\\
\boldsymbol{k}_{22} &= ((m_1, m_2), (B_1, B_2), (M_1, B_2)),\\
\boldsymbol{k}_{23} &= ((m_1, m_2), (B_1, B_2), (B_1, M_2)),\\
\boldsymbol{k}_{24} &= ((m_1, m_2), (A_1, B_2), (M_1, B_2)).
\end{aligned}$$

Then, for the path $\boldsymbol{p}_1$, we have the series systems as

$$\begin{aligned}
\alpha^{\boldsymbol{p}_1}_{\boldsymbol{k}_{11}}(x_1, x_2) &= \begin{cases} \varphi(m_1, m_2) = m, & (x_1, x_2) \geqq (m_1, m_2) \text{ and } (x_1, x_2) \ngeqq (A_1, A_2),\\ \varphi(A_1, A_2) = A, & (x_1, x_2) \geqq (A_1, A_2), \end{cases}\\
\alpha^{\boldsymbol{p}_1}_{\boldsymbol{k}_{12}}(x_1, x_2) &= \begin{cases} \varphi(m_1, m_2) = m, & (x_1, x_2) \geqq (m_1, m_2) \text{ and } (x_1, x_2) \ngeqq (B_1, m_2),\\ \varphi(B_1, m_2) = A, & (x_1, x_2) \geqq (B_1, m_2) \text{ and } (x_1, x_2) \ngeqq (M_1, B_2),\\ \varphi(M_1, B_2) = M, & (x_1, x_2) \geqq (M_1, B_2), \end{cases}\\
\alpha^{\boldsymbol{p}_1}_{\boldsymbol{k}_{13}}(x_1, x_2) &= \begin{cases} \varphi(m_1, m_2) = m, & (x_1, x_2) \geqq (m_1, m_2) \text{ and } (x_1, x_2) \ngeqq (B_1, m_2),\\ \varphi(B_1, m_2) = A, & (x_1, x_2) \geqq (B_1, m_2) \text{ and } (x_1, x_2) \ngeqq (B_1, M_2),\\ \varphi(B_1, M_2) = M, & (x_1, x_2) \geqq (B_1, M_2). \end{cases}
\end{aligned}$$

For the path $\boldsymbol{p}_2$, $\alpha^{\boldsymbol{p}_2}_{\boldsymbol{k}_{21}} \sim \alpha^{\boldsymbol{p}_2}_{\boldsymbol{k}_{24}}$ are similarly determined. $\alpha^{\boldsymbol{p}_2}_{\boldsymbol{k}_{21}}$ is, for example, given as the following.

$$\alpha^{\boldsymbol{p}_2}_{\boldsymbol{k}_{21}}(x_1, x_2) = \begin{cases} \varphi(m_1, m_2) = m, & (x_1, x_2) \geqq (m_1, m_2) \text{ and } (x_1, x_2) \ngeqq (m_1, M_2),\\ \varphi(m_1, M_2) = B, & (x_1, x_2) \geqq (m_1, M_2) \text{ and } (x_1, x_2) \ngeqq (B_1, M_2),\\ \varphi(B_1, M_2) = M, & (x_1, x_2) \geqq (B_1, M_2). \end{cases}$$

Here we notice how the decomposition (8.2) of a series system is. Supposing a system φ to be a series system, we denote the minimum element of $MI\left(\varphi^{-1}(s)\right)\ (s \in S)$ by $\boldsymbol{m}_s$. For a maximal path $\boldsymbol{p} = (s_0, \cdots, s_p) \in \mathcal{P}$, $\mathcal{P}_{\boldsymbol{p}}$ is composed of only one path $(\boldsymbol{m}_{s_0}, \cdots, \boldsymbol{m}_{s_p})$ by the normal property of the series system, where $\boldsymbol{m}_{s_0} = (m_1, \cdots, m_n)$.

We may have more strong assertion which is given by the following Theorem, of which proof is easy and omitted.

Theorem 8.2. For a system φ, a necessary and sufficient condition for the system φ to be a series system is that for each $\boldsymbol{p} \in \mathcal{P}$, $\mathcal{P}_{\boldsymbol{p}}$ is composed of only one path.

9. Stochastic Bounds

Let $\mathbf{P}$ be a probability on Ω_C. From (8.1) and (8.2), we have easily stochastic bounds for the stochastic performance of a system given by the following Theorem 9.1. See Ohi [29] and a simple proof is also shown in Appendices.

Theorem 9.1. (i) From (7.2), for every $s \in S$, we have

$$\mathbf{P}(\varphi \geqq s) \geqq \max_{\boldsymbol{p}\in\mathcal{P}} \max_{\boldsymbol{k}\in\mathcal{P}_{\boldsymbol{p}}} \mathbf{P}\left(\alpha_{\boldsymbol{k}}^{\boldsymbol{p}} \geqq s\right). \tag{9.1}$$

(ii) When $\mathbf{P}$ is associated, from (8.1), we have for every $s \in S$

$$\mathbf{P}(\varphi \geqq s) \leqq \coprod_{\boldsymbol{p}\in\mathcal{P},\boldsymbol{k}\in\mathcal{P}_{\boldsymbol{p}}} \mathbf{P}\left(\alpha_{\boldsymbol{k}}^{\boldsymbol{p}} \geqq s\right). \tag{9.2}$$

For an associated probability, refer Ohi [21].

(9.1) and (9.2) give us lower and upper bounds for $\mathbf{P}(\varphi \geqq s)$ $(s \in S)$, respectively. We may make the bounds (9.1) and (9.2) more precise by using the min formula of φ by parallel systems, but we omit them here.

Example 9.1. Using Example 8.1, we demonstrate the calculation process of Theorem 9.1. Let $\mathbf{P}$ be an associated probability on $\Omega_1 \times \Omega_2$.

$$\begin{aligned}
\mathbf{P}\{\varphi \geqq A\} \;&\geqq\; \max\Big\{\, \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{11}}^{\boldsymbol{p}_1} \geqq A\right\}, \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{12}}^{\boldsymbol{p}_1} \geqq A\right\}, \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{13}}^{\boldsymbol{p}_1} \geqq A\right\}, \\
&\qquad \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{21}}^{\boldsymbol{p}_2} \geqq A\right\}, \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{22}}^{\boldsymbol{p}_2} \geqq A\right\}, \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{23}}^{\boldsymbol{p}_2} \geqq A\right\}, \mathbf{P}\left\{\alpha_{\boldsymbol{k}_{24}}^{\boldsymbol{p}_2} \geqq A\right\} \Big\} \\
&= \max\{\, \mathbf{P}\{\boldsymbol{x} \geqq (A_1, A_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (B_1, m_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (B_1, m_2)\}, \\
&\qquad \mathbf{P}\{\boldsymbol{x} \geqq (B_1, M_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (M_1, B_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (B_1, M_2)\}, \\
&\qquad \mathbf{P}\{\boldsymbol{x} \geqq (M_1, B_2)\} \,\} \\
&= \max\{\, \mathbf{P}\{\boldsymbol{x} \geqq (A_1, A_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (B_1, m_2)\} \,\}, \\
\mathbf{P}\{\varphi \geqq A\} \;&\leqq\; 1 - (1 - \mathbf{P}\{\boldsymbol{x} \geqq (A_1, A_2)\})(1 - \mathbf{P}\{\boldsymbol{x} \geqq (B_1, m_2)\})^2 \\
&\qquad (1 - \mathbf{P}\{\boldsymbol{x} \geqq (B_1, M_2)\})^2 (1 - \mathbf{P}\{\boldsymbol{x} \geqq (M_1, B_2)\})^2,
\end{aligned}$$

where, for example, $\mathbf{P}\{\boldsymbol{x} \geqq (A_1, A_2)\} = \mathbf{P}\{\boldsymbol{x} \in \Omega_1 \times \Omega_2 | \boldsymbol{x} \geqq (A_1, A_2)\}$. Similarly we have upper and lower bounds for $\mathbf{P}\{\varphi \geqq B\}$ as

$$\begin{aligned}
\mathbf{P}\{\varphi \geqq B\} \;&\geqq\; \max\{\, \mathbf{P}\{\boldsymbol{x} \geqq (m_1, M_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (B_1, B_2)\}, \mathbf{P}\{\boldsymbol{x} \geqq (A_1, B_2)\} \,\}, \\
\mathbf{P}\{\varphi \geqq B\} \;&\leqq\; 1 - (1 - \mathbf{P}\{\boldsymbol{x} \geqq (M_1, B_2)\})(1 - \mathbf{P}\{\boldsymbol{x} \geqq (B_1, M_2)\})(1 - \mathbf{P}\{\boldsymbol{x} \geqq (m_1, M_2)\}) \\
&\qquad (1 - \mathbf{P}\{\boldsymbol{x} \geqq (B_1, B_2)\})^2 (1 - \mathbf{P}\{\boldsymbol{x} \geqq (A_1, B_2)\}).
\end{aligned}$$

We know, from this example, that the upper bounds include some needless terms in comparison with the lower bounds. Such a situation will made clear by comparing these bounds with those given by Ohi [26, 29].

An increasing system is uniquely determined by $MI\left(\varphi^{-1}[s, \to)\right)$ $(s \in S)$ as

$$\varphi^{-1}[s, \to) = \bigcup_{\boldsymbol{s}\in MI(\varphi^{-1}[s,\to))} [\boldsymbol{s}, \to).$$

From which, we have the following stochastic inequalities (9.3) and (9.4) of Ohi [26, 29]. Noticing $\varphi^{-1}[s, \to) = (\varphi \geqq s)$, for $s \in S$,

$$\mathbf{P}(\varphi \geqq s) \geqq \max_{\boldsymbol{s}\in MI(\varphi^{-1}[s,\to))} \mathbf{P}[\boldsymbol{s}, \to). \tag{9.3}$$

When $\mathbf{P}$ is associated, for $s \in S$,

$$\mathbf{P}(\varphi \geqq s) \leqq \coprod_{\boldsymbol{s} \in MI(\varphi^{-1}[s,\rightarrow))} \mathbf{P}[\boldsymbol{s}, \rightarrow). \tag{9.4}$$

We have the next Theorem for relationships among (9.1), (9.2), (9.3) and (9.4).
Theorem 9.2. (i) For $s \in S$,

$$\max_{\boldsymbol{s} \in MI(\varphi^{-1}[s,\rightarrow))} \mathbf{P}[\boldsymbol{s}, \rightarrow) = \max_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}} \mathbf{P}\left(\alpha_{\boldsymbol{k}}^{\boldsymbol{p}} \geqq s\right). \tag{9.5}$$

(ii) For $s \in S$,

$$\coprod_{\boldsymbol{s} \in MI(\varphi^{-1}[s,\rightarrow))} \mathbf{P}[\boldsymbol{s}, \rightarrow) \leqq \coprod_{\boldsymbol{p} \in \mathcal{P}, \boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}} \mathbf{P}\left(\alpha_{\boldsymbol{k}}^{\boldsymbol{p}} \geqq s\right). \tag{9.6}$$

Proof. In the sequel, we show three propositions, from which (9.5) is easily verified. (9.6) is clear and the proof is omitted here.

(1) We first suppose $\boldsymbol{p} \in \mathcal{P}$, $\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}$, $\boldsymbol{s} \in MI\left(\varphi^{-1}[s,\rightarrow)\right)$ and $\boldsymbol{s} \in \boldsymbol{k}$. $\varphi(\boldsymbol{s}) \in \boldsymbol{p}$ clearly holds. For $\boldsymbol{x} \in \Omega_C$, noticing the values which $\alpha_{\boldsymbol{k}}^{\boldsymbol{p}}$ may take, we have the following equivalent relations.

$$\begin{aligned} \boldsymbol{x} \geqq \boldsymbol{s} &\iff \max\{\boldsymbol{u} \in \boldsymbol{k} | \boldsymbol{x} \geqq \boldsymbol{u}\} \geqq \boldsymbol{s} \\ &\iff \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) = \varphi(\max\{\boldsymbol{u} \in \boldsymbol{k} | \boldsymbol{x} \geqq \boldsymbol{u}\}) \geqq \varphi(\boldsymbol{s}) \geqq s. \end{aligned}$$

Then noticing $\boldsymbol{s} \in MI\left(\varphi^{-1}[s,\rightarrow)\right)$ and $\boldsymbol{s} \in \boldsymbol{k}$, in this case we have the following equality.

$$\{\ \boldsymbol{x} \mid \boldsymbol{x} \geqq \boldsymbol{s}\ \} = \left\{\ \boldsymbol{x}\ \middle|\ \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq s\ \right\}. \tag{9.7}$$

There necessarily exit $\boldsymbol{p} \in \mathcal{P}$ and $\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}$ which contains $\boldsymbol{s}$. Then, $\alpha_{\boldsymbol{k}}^{\boldsymbol{p}}$ which satisfies (9.7) exists, but not unique, since the existence of $\boldsymbol{k}$ containing $\boldsymbol{s}$ is not uniquely assured.

(2) For $\boldsymbol{u} \in \left(\bigcup_{t \geqq s} MI\left(\varphi^{-1}(t)\right)\right) \setminus MI\left(\varphi^{-1}[s,\rightarrow)\right)$,

$$\exists \boldsymbol{s} \in MI\left(\varphi^{-1}[s,\rightarrow)\right),\ \boldsymbol{s} < \boldsymbol{u}$$

holds. In this case,

$$\{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}'}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq\ s\} \subseteqq \{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq\ s\}$$

follows, where $\boldsymbol{u} \in \boldsymbol{k}'$, $\boldsymbol{s} \in \boldsymbol{k}$. Thus we have

$$\boldsymbol{P}\{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}'}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq\ s\} \leqq \boldsymbol{P}\{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq\ s\}.$$

(3) The next if and only if relation holds.

$$\forall \boldsymbol{u} \in \bigcup_{t \geqq s} MI\left(\varphi^{-1}(t)\right),\ \boldsymbol{u} \notin \boldsymbol{k} \iff \{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq s\ \} = \phi. \tag{9.8}$$

(4) From the above three propositions, we have

$$\begin{aligned} \max_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}} \boldsymbol{P}\{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq s\ \} &= \max_{\boldsymbol{s} \in MI(\varphi^{-1}[s,\rightarrow))} \max_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}, \boldsymbol{s} \in \boldsymbol{k}} \boldsymbol{P}\{\ \boldsymbol{x} \mid \alpha_{\boldsymbol{k}}^{\boldsymbol{p}}(\boldsymbol{x}) \geqq s\ \} \\ &= \max_{\boldsymbol{s} \in MI(\varphi^{-1}[s,\rightarrow))} \boldsymbol{P}\{\ \boldsymbol{x} \mid \boldsymbol{x} \geqq \boldsymbol{s}\ \}, \end{aligned}$$

and then (9.5) is proved. □

From this theorem, we know that the upper stochastic bounds by the series decomposition is worse than that given simply by the minimal state vectors in Ohi [26, 29], in other words, the structure function is too decomposed. And so we have a possibility to make the upper stochastic bounds to be the same by choosing appropriate series systems, which is remained to be a future work.

10. On a Definition of a k-out-of-n System

Series and parallel systems are complete opposite with each other in the class of all the k-out-of-n systems. It is remained to be an open problem how to define a k-out-of-n system in the case of partially ordered state spaces. When the state spaces are totally ordered sets, a definition of a k-out-of-n system has been given by Ohi [18, 22], which is however seemed to be a working hypothesis.

k-out-of-n system implies that for each state vector $\boldsymbol{x}$, the state of the system is determined by $\boldsymbol{x}^A$ for some $A \subsetneqq C$ such that $|A| = k$ and is independent of $\boldsymbol{x}^{C\setminus A}$. Then, generalising Ohi [18, 22], when the state space of the system S is lattice, we may define a k-out-of-n:G system to satisfy

$$\forall A \subsetneqq C \text{ such that } |A| = k,\ \exists \text{a series system } (\Omega_A, S, \varphi_A),\ \varphi = \sup \varphi_A, \tag{10.1}$$

which includes the series system as a special case. Where $\boldsymbol{x}^A$ means a state vector composed of the states x_i $(i \in A)$ taken out from $\boldsymbol{x}$. For example, when $A = \{1, 2, 4\}$, $\boldsymbol{x}^A = (x_1, x_2, x_4)$. In the lattice case, we easily conjecture that for each $s \in S$, $\varphi^{-1}[s, \rightarrow) = \cup_{A \subsetneqq C,\ |A|=k} \varphi_A^{-1}[s, \rightarrow)$ holds. This equality, however, does not hold generally for the lattice case. Then, the examination about the minimal state vectors will be performed in a somewhat complicated situation, if we adopt (10.1) as the definition.

Following faithfully the above mentioned image of a k-out-of-n:G system, we may give another definition as a system φ is called a k-out-of-n system when the following (10.2) is satisfied.

$$\forall A \subsetneqq C \text{ such that } |A| = k,\ \exists \text{a series system } (\Omega_A, S, \varphi_A),$$
$$\varphi^{-1}(s) = \bigcup_{A \subsetneqq C,\ |A|=k} \varphi_A^{-1}(s) \times \Omega_{C\setminus A}, \tag{10.2}$$

which also includes the series system as a special case. In this case we do not need any calculation as taking supremum, and can be used for the general partially ordered case for the state space of the system. $MI\left(\varphi^{-1}(s)\right)$ may be determined by the minimum state vectors of $\varphi_A^{-1}(s)$ $(A \subsetneqq C, |A| = k)$. But, we have a question whether there is a consistency between (10.2) and Ohi [18, 22].

Corresponding to (10.1) or (10.2), A k-out-of-n:F system is defined in a dual manner to satisfy respectively

$$\forall A \subsetneqq C \text{ such that } |A| = k,\ \exists \text{a paralle system } \varphi_A : \Omega_A \rightarrow S,\ \varphi = \inf \varphi_A, \tag{10.3}$$

or

$$\forall A \subsetneqq C \text{ such that } |A| = k,\ \exists \text{a paralle system } (\Omega_A, S, \varphi_A),$$
$$\varphi^{-1}(s) = \bigcup_{A \subsetneqq C,\ |A|=k} \varphi_A^{-1}(s) \times \Omega_{C\setminus A}, \tag{10.4}$$

Detailed examinations of k-out-of-n:G and k-out-of-n:F systems are remained for future work.

A definition of a consecutive k-out-of-n:G system is easy after the completion of the definition of k-out-of-n:G system. For example, when the system's state space is a lattice set, we may define the notion of the system by restricting A in (10.1) to be $\{i, i+1, \cdots, i+k-1\}$, where $i = 1, \cdots, n-k+1$ or the periodic boundary condition is considered.

11. Conclusions

In this paper, following Ohi [28, 29], we have shown a definition of series and parallel multi-state systems for the case of partially ordered state spaces, and shown a necessary and sufficient condition given in Theorem 5.1 for a system to be a series system and that the definition is reasonable from the point of view of serialisation. Furthermore the structure function of a series system may be conveniently expressed as a min-formula. We also have given a decomposition of a multi-state system by series systems, by which stochastic bounds for the reliability of multi-state systems are given and compared with the previously given stochastic bounds.

Precise numerical examinations of these stochastic bounds are remained for future work.

For the parallel systems, we may easily present similar results, since parallel is a dual notion of series.

In the case of binary state spaces, a system is decomposed into series systems defined by minimal path sets and this decomposition is used to give stochastic evaluation of the binary state system. Our results in this paper are entire extension of the results in the binary case.

When all the state spaces of a system are given as totally ordered sets, in Ohi [19, 22] another definition of relevant property is given ; for every component $i \in C$,

$$\forall r, \forall s \in S \text{ such that } r \neq s,\ \exists k, \exists l \in \Omega_i,\ \exists (k_i, \boldsymbol{x}), \exists (l_i, \boldsymbol{x}) \in \Omega_C,$$
$$\varphi(k_i, \boldsymbol{x}) = r,\ \varphi(l_i, \boldsymbol{x}) = s.$$

This definition of relevancy is given from the system level's point of view, and Ohi [22] have proved the min formula for a series system, starting from this relevant property. On the other hand, Definition 4.3 of this paper is from the component level's point of view. It is remained to be a future work to explain what practical differences between these relevant properties are.

For a definition of a k-out-of-n system which includes a series and parallel systems as special cases, we have proposed tow kind of candidates and an assertion about minimal state vectors.

Acknowledgements

This work is supported by Grant #25350441, Grant-in-Aid for Scientific Research (c) from JSPS(2013 - 2015)

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Appndices

Proof of Theorem 4.3

proof of the first equivalent relation From (3.1), the next equivalent relation holds.

$$LMA(\varphi(A)) \overset{\circ}{=} \varphi(LMA(A)) \iff MA(LMA(\varphi(A))) = MA(\varphi(LMA(A))).$$

Noticing $MA(LMA(\varphi(A))) = LMA(\varphi(A))$, since every two different elements of $LMA(\varphi(A))$ have no order relation, the first equivalent relation is clear.

proof of the second equivalent relation From Theorem 4.2, $\varphi(LMA(A)) \overset{\circ}{\leqq} LMA(\varphi(A))$ holds automatically, then it is sufficient to prove the next equivalent relation.

$$LMA(\varphi(A)) \overset{\circ}{\leqq} \varphi(LMA(A)) \iff LMA(\varphi(A)) \subseteq \varphi(LMA(A)),$$

which is clear by the first equivalent relation.

Proof of Theorem 4.4

proof of (4.1) $\Longrightarrow$ (4.2) First we notice that

$$LMA(\varphi(\varphi^{-1}[s, \rightarrow))) = LMA([s, \rightarrow)) = \{s\}$$

holds. Then setting $A = \varphi^{-1}[s, \rightarrow)$ in Theorem 4.3, we have $MA(\varphi(LMA(A))) = \{s\}$ and

$$\exists \boldsymbol{a} \in LMA(\varphi^{-1}[s, \rightarrow)),\ \varphi(\boldsymbol{a}) = s.$$

So we have $\boldsymbol{a} \in \varphi^{-1}[s, \rightarrow)$, which means that $\boldsymbol{a}$ is the minimum element of $\varphi^{-1}[s, \rightarrow)$ by Theorem 3.1.

proof of (4.1) $\Longleftarrow$ (4.2) From Theorem 4.3, (4.1) is equivalent to

$$LMA(\varphi(A)) \subseteqq \varphi(LMA(A)), \tag{11.1}$$

and so we here prove the relation (4.2) $\Longrightarrow$ (11.1). $s \in LMA(\varphi(A))$ implies $\varphi(A) \subseteqq [s, \rightarrow)$ and so $A \subseteqq \varphi^{-1}[s, \rightarrow)$. Since $\varphi^{-1}[s, \rightarrow)$ has the minimum element, we have, from the definition of LMA,

$$\exists \boldsymbol{a} \in LMA(A), \ \min \varphi^{-1}[s, \rightarrow) \leqq \boldsymbol{a} \leqq A$$

and for this $\boldsymbol{a}$, using the increasing property of φ,

$$\varphi(\min \varphi^{-1}[s, \rightarrow)) = s \leqq \varphi(\boldsymbol{a}) \leqq \varphi(A).$$

Since $s \in LMA(\varphi(A))$, $s = \varphi(\boldsymbol{a})$ holds. Thus, reminding $\boldsymbol{a} \in LMA(A)$, $s \in \varphi(LMA(A))$ follows.

Proof of Theorem 9.1

(9.1) is clear by (8.2). (9.2) is also clear by noticing

$$\{\varphi \ngeqq s\} \;=\; \{\max_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}} \alpha_{\boldsymbol{k}}^{\boldsymbol{p}} \ngeqq s\} = \bigcap_{\boldsymbol{p} \in \mathcal{P}} \bigcap_{\boldsymbol{k} \in \mathcal{P}_{\boldsymbol{p}}} \{\alpha_{\boldsymbol{k}}^{\boldsymbol{p}} \ngeqq s\}$$

and associated property of $\mathbf{P}$.

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