

SEARCH GAMES ON A BROKEN WHEEL WITH TRAVELING AND SEARCH COSTS

Vic Baston
University of Southampton

Kensaku Kikuta
University of Hyogo

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Abstract The authors analyse a search game involving an immobile hider and a searcher in which play takes place in discrete time on a network comprising a cycle together with a node 0 adjacent to a specified set of nodes in the cycle. The hider chooses a node of the cycle. Unaware of the hider's choice, the searcher starts at the node 0 and, at each subsequent time instant, moves from the node he occupies to an adjacent node and decides whether to search it. Play terminates when the searcher is at the node chosen by the hider and searches there. The searcher incurs a cost of one in moving from a node to an adjacent one and a search cost depending on whether or not the node is in the specified set. The searcher wants to minimize the costs of finding the hider and the hider to maximize them.

We obtain upper bounds for the value of this game for the cases when the specified set has no adjacent nodes and when it is an interval. We show that these upper bounds are the value of the game in a number of cases.

Keywords: Game theory, two-person, cyclic network, ordering of search of nodes, search and traveling costs, optimal strategy

1. Introduction

The problem of finding someone who does not want to be found is an age-old one. Nowadays variations of it have come into prominence and include, amongst others, searching for arms and explosives hidden by terrorists, searching for drugs trafficked by cartels and, in the biological realm, behaviour of animals hiding food for future consumption. Although mathematical search games may not be able to give ready-made solutions to these problems, they can be a useful tool when addressing them. A particularly interesting example of this is detailed in Fokkink and Lindelauf [10] where the capture of Saddam Hussein is discussed. They illustrate how combining information about various social networks can be used to build a picture of likely hiding places so that the problem then becomes a search game.

The mathematical theory of search games is a rapidly expanding field with recent applications to security issues (see [5, 6, 16]) and biology (see [3, 17]). However there is still no comprehensive theory for the more traditional problems and many interesting and challenging ones remain unsolved, even for uncomplicated networks such as the cycle network. The earlier studies concentrated on games in which the hider can hide at any point of the network and the searcher starts from a designated node with the searcher incurring only travelling (time) costs. Details of this work can be found in the classic book of Alpern and Gal [4] published in 2003; for a more recent update on search games in various spaces including networks see Gal [11]. Even more recently Hohzaki [12] has produced a very comprehensive survey of search games. When the hider is constrained to hide at a node it is usual to include an inspection or search cost so that the searcher has both travelling and

search costs; problems of this type in which the searcher starts from a designated node are analysed in Kikuta and Ruckle [15], Kikuta [13] and Kikuta [14]. More recently there have been papers in which the searcher can choose the starting node (see Alpern, Baston and Gal [1] and [2], Dagan and Gal [9], Baston and Kikuta [7] and Baston and Kikuta [8]); in the first three of these papers the hider can hide at any point of the network whereas, in the last two, the hider must hide at a node.

In this paper we investigate search games on a cyclic graph in which the hider must hide at a node and the searcher incurs both travelling and search costs. In previous papers, the searcher has either complete freedom to choose the node at which to start the search or has to start at a designated node. However these two scenarios can be thought of as extreme cases of the situation in which the searcher has a limited number of options concerning the nodes at which a search can start and this is the scenario we investigate here. In general the case when there are arbitrary search costs at the nodes is an extremely difficult one, even when the distance between adjacent nodes is one. We therefore consider the situation in which the travelling cost between adjacent nodes is one and the nodes at which the searcher can start have one cost and the other nodes a separate cost. We show that, even for this case, the analysis is complicated when the graph is a cycle.

We now describe our results for the game on a cycle graph with n nodes when the nodes at which a searcher can start are denoted by S ; we call a game of this type a broken wheel search game. Let $|S|$ be the number of nodes in S .

- An upper bound for the value of the game when S has no adjacent nodes is obtained (Theorem 3.1); furthermore this upper bound is the value of the game for the cases when $|S| = 1$ and when $n = 2m$ and $|S| = m$ (Theorem 3.2);
- A complete solution of the game is found for the case when S is an interval and searching a node of S is more costly than searching a node not in S (Theorem 4.1);
- When S is an interval and the cost of searching a node not in S is more costly than the cost of searching a node of S , an upper bound for the value of the game is given and, when $|S| = 2$, this upper bound is the value of the game (Theorem 4.2).

The formulae for the upper bounds in the above cases differ markedly in form. Furthermore our optimal searcher strategies for games when S has no adjacent nodes and our optimal searcher strategies for games when S is an interval and the cost of searching a node in S is more costly than searching a node not in S have different supports. This suggests that it will be difficult to develop a comprehensive theory for broken wheel search games.

2. The Model and Some Notation

In this section we describe our model and create a formal mathematical model for it. In addition we introduce some notation and prove two elementary propositions which are used in the next section.

We first describe our model in general terms. A searcher wants to find a hider who is able to hide at one of a known number of locations. The searcher has a base from which he can move directly to some of the hider locations and the other locations have to be accessed via them. Travel between locations incurs a cost. To discover the hider, the searcher needs to be at the location where the hider is hiding and carry out a search there; this search involves a cost which can depend on the particular location. The searcher wants to minimize the expected cost of finding the hider and the hider to maximize it.

The particular model we consider here is one where the routes between the hider locations can be represented by a cyclic graph and the travelling cost between adjacent nodes is one.

Together with the routes from the searcher’s base, the network can therefore be thought of as a wheel with some of its spokes missing. Figure 1 represents the situation in which there are seven hider locations and the searcher can move from his base B to all nodes of S , 1 and 5 directly. In future we will use 0 to represent the searcher’s base.

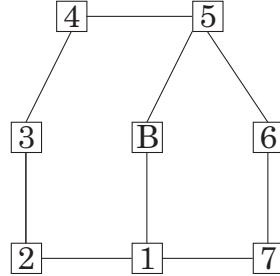


Figure 1: Network with $S = \{1, 5\}$

We now present our model formally.

Let $N = \{0, 1, 2, \dots, n\}$, $S \subseteq N \setminus \{0\}$ with $|S| \geq 1$ and $\mathcal{G}(N, S)$ denote the network with node set N and edge set E given by

$$E = \{(0, i) : i \in S\} \cup \{(i, i + 1) : 1 \leq i \leq n - 1\} \cup \{(1, n)\}.$$

The traveling cost between adjacent nodes is taken to be one and the shortest traveling cost between two nodes x and y is denoted by $d(x, y)$.

Put $T = N \setminus (S \cup \{0\})$. A search cost c is defined on $N \setminus \{0\}$ by

$$c_i = \begin{cases} c_S & \text{if } i \in S \\ c_T & \text{if } i \in T \end{cases}.$$

In relation to search and traveling costs we use the following quantities frequently

$$W = n + C, \text{ and } \epsilon = \frac{c_S - c_T}{2W}, \text{ where } C = \sum_{i=1}^n c_i.$$

We define a zero-sum game $\Gamma(N, S)$ on $\mathcal{G}(N, S)$ with two players called searcher and hider. A (mixed) hider strategy is a probability distribution p on $N \setminus \{0\}$ and a (pure) searcher strategy a permutation $\xi = (\xi(1), \xi(2), \dots, \xi(n))$ on $N \setminus \{0\}$ representing the order in which the nodes are to be searched.

Let Ξ denote the set of all permutations on $N \setminus \{0\}$. For $\xi \in \Xi$, let

$$d_i(\xi) = d(\xi(i - 1), \xi(i)) \text{ where } d_1(\xi) = d(0, \xi(1)), \tag{2.1}$$

$$D_j(\xi) = \sum_{i=1}^j d_i(\xi) \text{ and } C_j(\xi) = \sum_{i=1}^j c_{\xi(i)}. \tag{2.2}$$

The payoff function $f(p, \xi)$ to the hider is defined as

$$f(p, \xi) = \sum_{i=1}^n p_{\xi(i)} (D_i(\xi) + C_i(\xi)). \tag{2.3}$$

When the hider uses a pure strategy $j \in N \setminus \{0\}$ we write the payoff to the hider as $f(j, \xi)$, that is, $f(j, \xi) = D_{\xi^{-1}(j)}(\xi) + C_{\xi^{-1}(j)}(\xi)$.

We now introduce notation which will be used extensively in future sections. Pure searcher strategies which search a node and then traverse the cycle searching each node as they reach it play an important part in the analysis. For each $i \in S$, we define four searcher strategies \vec{i} , \overleftarrow{i} , i^\rightarrow and ${}^\leftarrow{i}$ by

$$\vec{i} = i, i + 1, \dots, n, 1, \dots, i - 1 \quad \overleftarrow{i} = i, i - 1, \dots, 1, n, \dots, i + 1 \quad (2.4)$$

$$i^\rightarrow = i + 1, i + 2, \dots, n, 1, \dots, i \quad {}^\leftarrow{i} = i - 1, i - 2, \dots, 1, n, \dots, i.$$

Comments relating to these strategies will be given later (See the beginning of Section 3.2.1 and the beginning of Section 3). The last two strategies look unnatural, but they must be understood in relation to search and traveling costs and strategies of the hider. Furthermore, each of them is not used as a pure strategy, but they are in mixed strategies. The searcher strategy which employs each of \vec{i} and \overleftarrow{i} with probability 1/2 is denoted by \overleftrightarrow{i} and the searcher strategy which employs each of i^\rightarrow and ${}^\leftarrow{i}$ with probability 1/2 is denoted by ${}^{\leftarrow i^\rightarrow}$. The expected costs for the nodes when the searcher uses these strategies are given in the following propositions which will be needed in the next section.

Proposition 1. *Let $i \in S$, then*

$$f(j, \overleftrightarrow{i}) = \begin{cases} 1 + c_i & \text{if } j = i \\ (W + 2 + c_i + c_j)/2 & \text{if } j \neq i \end{cases} \quad (2.5)$$

Proof. Let $i \in S$, then the searcher can move from 0 to i at a cost of one. If the hider is hiding at node i , each of \vec{i} and \overleftarrow{i} has an expected travel cost of one and an expected search cost of c_i . Thus $f(i, \overleftrightarrow{i}) = 1 + c_i$.

If the hider is hiding at node $j \neq i$, both node i and node j are searched by both \vec{i} and \overleftarrow{i} whereas every other node is searched by precisely one of them. Thus the expected search cost is $(C + c_i + c_j)/2$. If $j > i$, the travel cost for \vec{i} is $j + 1 - i$ and the travel cost for \overleftarrow{i} is $i + n + 1 - j$; if $j < i$, the travel cost for \vec{i} is $n + 1 + j - i$ and the travel cost for \overleftarrow{i} is $i - j + 1$. Thus, in either case, the expected travel cost is $(n + 2)/2$ and $f(j, \overleftrightarrow{i}) = (W + 2 + c_i + c_j)/2$. \square

Proposition 2. *Let $i \in S$, then*

$$f(j, {}^{\leftarrow i^\rightarrow}) = \begin{cases} W + 1 & \text{if } j = i \\ (W + 2 - c_i + c_j)/2 & \text{if } j \neq i \end{cases} \quad (2.6)$$

Proof. If the hider is hiding at node i , each of i^\rightarrow and ${}^\leftarrow{i}$ has an expected travel cost of $n + 1$ and an expected search cost of C so $f(i, {}^{\leftarrow i^\rightarrow}) = W + 1$.

If the hider is hiding at node $j \neq i$, node i is searched by neither i^\rightarrow nor ${}^\leftarrow{i}$, node j is searched by both i^\rightarrow and ${}^\leftarrow{i}$ while every other node is searched by precisely one of them. Thus the expected search cost is $(C - c_i + c_j)/2$. The travel costs for i^\rightarrow and ${}^\leftarrow{i}$ are the same as those for \overleftarrow{i} and \vec{i} in Proposition 1 so $f(j, {}^{\leftarrow i^\rightarrow}) = (C - c_i + c_j + n + 2)/2 = (W + 2 - c_i + c_j)/2$. \square

3. The Case when No Two Nodes of S Are Adjacent

It has been shown in Theorem 4 of [7] that, for the cycle graph when the searcher can choose at which node to start, an optimal searcher strategy is to choose the node i ($i = 1, \dots, n$) with an appropriate probability and then use \overleftrightarrow{i} ; this is so even for general search costs at the node. The situation is not quite so straightforward when the searcher must start at a restricted number of nodes in the cycle graph. However the idea that, once a node has been searched, the searcher should subsequently search every node at the first visit seems to be a sound one. In the subsection 3.1 we obtain a theorem (Theorem 3.1) which gives an upper bound for the value of the game $\Gamma(N, S)$ when no two nodes of S are adjacent. In Sections 3.2.1 and 3.2.2 we show that this upper bound is the value of the game when S is a singleton and when $n = 2m$ and $|S| = m$.

3.1. Upper bound for the value

Here we obtain a theorem (Theorem 3.1) which gives an upper bound for the value of the game $\Gamma(N, S)$ when no two nodes of S are adjacent.

Theorem 3.1. *Suppose no two vertices of S are adjacent in $\mathcal{G}(N, S)$. By choosing a node i of S at random and then using \overleftrightarrow{i} with probability $1/2 + \epsilon|S|$ and \overleftarrow{i} with probability $1/2 - \epsilon|S|$, the searcher can ensure that the expected cost of finding the hider does not exceed*

$$V = \frac{W + 2 + c_T}{2} + |S|c_S\epsilon. \tag{3.1}$$

Proof. Put $\gamma = 1/2 + \epsilon|S|$ and $\delta = 1/2 - \epsilon|S|$, then $\gamma + \delta = 1$.

If $c_S \geq c_T$, then $0 \leq |S|(c_S - c_T) < C < W$ so $\epsilon|S| < 1/2$ and γ satisfies $1/2 \leq \gamma < 1$.

If $c_T > c_S$, then, because no two nodes of S are adjacent, $|S| \leq |T|$ and $0 < |S|(c_T - c_S) < C < W$ so $0 > \epsilon|S| > -1/2$ and δ satisfies $1/2 < \delta < 1$.

Thus γ and δ are admissible as probabilities and we can take q_i to be the searcher strategy which uses \overleftrightarrow{i} with probability γ and \overleftarrow{i} with probability δ .

By Propositions 1 and 2, the expected cost $f(j, q_i)$ when the hider hides at j is given by

$$f(j, q_i) = \begin{cases} \gamma(1 + c_i) + \delta(W + 1) & \text{if } j = i \\ (W + 2 + c_j)/2 + (\gamma - \delta)c_i/2 & \text{if } j \neq i \end{cases}. \tag{3.2}$$

Now let q be the searcher strategy that chooses a member of $\{q_i : i \in S\}$ at random.

Suppose $j \in T$, then $c_j = c_T$, and $c_i = c_S$ for all $i \in S$. From (3.2), it follows that, for all $i \in S$,

$$f(j, q_i) = \frac{W + 2 + c_T}{2} + |S|c_S\epsilon = V.$$

Thus $f(j, q) = V$.

Suppose $j \in S$, then, when $i = j$, the first line of (3.2) gives

$$\begin{aligned} f(j, q_i) &= \gamma(1 + c_S) + \delta(W + 1) = \frac{W + 2 + c_S}{2} - |S|\epsilon(W - c_S) \\ &= \frac{W + 2 + c_S}{2} + \frac{|S|(c_T - c_S)}{2} + \epsilon c_S |S| \\ &= \frac{W + 2 + c_S}{2} + \frac{|S|(c_T - c_S)}{2} + (\gamma - \delta)c_S/2. \end{aligned}$$

So, from (3.2), the expectation of a node in S when the searcher uses strategy q is

$$\begin{aligned} \frac{1}{|S|} \sum_{i \in S} f(j, q_i) &= \frac{W + 2 + c_S}{2} + \frac{1}{|S|} \left(\frac{|S|(c_T - c_S)}{2} + \frac{(\gamma - \delta)c_S}{2} \right) + \frac{|S| - 1}{|S|} \left(\frac{(\gamma - \delta)c_S}{2} \right) \\ &= \frac{W + 2 + c_S}{2} + \frac{c_T - c_S}{2} + \epsilon|S|c_S = V. \end{aligned}$$

Thus $f(i, q) = V$ for all $i \in N \setminus \{0\}$ and the theorem is established. □

3.2. Value of $\Gamma(N, S)$ for special cases

In this subsection we show that this upper bound given in Theorem 3.1 is the value of the game when S is a singleton and when $n = 2m$ and $|S| = m$.

Theorem 3.2. *Suppose either $|S| = 1$ or $n = 2m$ and $|S| = m$. Then the value of the game $\Gamma(N, S)$ is V at (3.1).*

An optimal strategy for the hider is to hide at a node of S with probability c_S/W and at a node of T with probability $(n/|T| + c_T)/W$.

An optimal strategy for the searcher is to choose a node j of S at random and then use \overleftrightarrow{j} with probability $1/2 + |S|\epsilon$ and \overleftarrow{j} with probability $1/2 - |S|\epsilon$.

3.2.1. Proof of the theorem when $|S| = 1$

When $|S| = 1$, say $S = \{1\}$, a naive intuition might lead one to suspect that it would be advantageous for the hider to hide at nodes far away from 1 with greater probability than to hide at nodes near 1. However nodes near 1 can involve large costs if the searcher elects to traverse the graph in the “wrong” direction. Theorem 3.2 shows that it is optimal for the hider to hide with the same probability at each node of T . In order to do this, we introduce a new function F defined for a hider strategy p and a searcher strategy ξ by

$$F(p, \xi) = \sum_{i=1}^n p_{\xi(i)} (\Delta_i(\xi) + C_i(\xi)) \quad \text{where} \quad \Delta_i(\xi) = \begin{cases} i & \text{if } \xi(1) \in S \\ i + 1 & \text{if } \xi(1) \in T \end{cases}. \quad (3.3)$$

Because F has a similar form to that of our payoff function f , F can be thought of as the payoff function of a corresponding search game with search costs in which the travelling cost between two nodes is one except for travel from the starting node 0 to a node in T when the cost is two. It is easy to see that $\Delta_i(\xi) \leq D_i(\xi)$ so that

$$F(p, \xi) \leq f(p, \xi) \quad \text{for all } p \text{ and } \xi. \quad (3.4)$$

We will use the function F in the proofs of Theorems 4.1 and 4.2 as well as in the proof of Theorem 3.2.

Proof of Theorem 3.2 for $|S| = 1$. Without loss of generality we may assume that $S = \{1\}$. Let α denote the hider strategy given by

$$\alpha_i = \begin{cases} \alpha_S = c_S/W & \text{if } i \in S \\ \alpha_T = (n/(n - 1) + c_T)/W & \text{if } i \in T \end{cases}. \quad (3.5)$$

We will show that $\min_{\xi \in \Xi} F(\alpha, \xi) = V$ where F and V are given by (3.3) and (3.1) respectively.

First suppose $\xi(1) \in S$, then $\alpha_{\xi(1)} = \alpha_S$, $\alpha_{\xi(i)} = \alpha_T$ if $i \neq 1$, $\Delta_i(\xi) = i$ and $C_i(\xi) = c_S + (i - 1)c_T$ so that

$$\begin{aligned} F(\alpha, \xi) &= 1 + c_S + \alpha_T(1 + c_T + \dots + (n - 1) + (n - 1)c_T) \\ &= 1 + c_S + \alpha_T(1 + c_T)n(n - 1)/2 = 1 + c_S + \frac{(n + (n - 1)c_T)n(1 + c_T)}{2W} \\ &= 1 + c_S + \frac{(W - c_S)(W - c_S + c_T)}{2W} = \frac{2 + W + c_T}{2} + c_S\epsilon = V. \end{aligned}$$

Thus

$$F(\alpha, \xi) = V \quad \text{if } \xi \in \Xi \text{ with } \xi(1) \in S. \tag{3.6}$$

Now suppose $\xi(j) \in S$ with $2 \leq j < n$ then $\alpha_{\xi(j)} = \alpha_S$ and $\alpha_{\xi(i)} = \alpha_T$ for $i \neq j$. Define $\theta \in \Xi$ by $\theta(j + 1) = \xi(j)$, $\theta(j) = \xi(j + 1)$ and $\theta(i) = \xi(i)$ otherwise, then

$$\begin{aligned} F(\alpha, \xi) - F(\alpha, \theta) &= \alpha_S(1 + j + C_j(\xi)) + \alpha_T(2 + j + C_{j+1}(\xi)) \\ &\quad - \alpha_T(1 + j + C_j(\theta)) - \alpha_S(2 + j + C_{j+1}(\theta)) \\ &= \alpha_T(1 + c_S) - \alpha_S(1 + c_T) = \frac{(n/(n - 1) + c_T)(1 + c_S) - c_S(1 + c_T)}{W} > 0. \end{aligned}$$

By repeating the argument an appropriate number of times, it follows that

$$\min\{F(\alpha, \xi) : \xi \in \Xi \text{ and } \xi(1) \in T\} = F(\alpha, \phi) \tag{3.7}$$

where $\phi(n) \in S$ and $\phi(i) \in T$ for $i < n$. We then have, using $(n - 1)\alpha_T = 1 - \alpha_S$,

$$\begin{aligned} F(\alpha, \phi) &= 1 + \alpha_T(1 + c_T)n(n - 1)/2 + \alpha_S(n + c_S + (n - 1)c_T) \\ &= 1 + (1 - \alpha_S)n(1 + c_T)/2 + \alpha_S W = 1 + (1 - \alpha_S)(W + c_T - c_S)/2 + \alpha_S W \\ &= \frac{(2 + W + c_T - c_S)}{2} + \frac{\alpha_S W}{2} + \frac{\alpha_S(c_S - c_T)}{2} = \frac{(2 + W + c_T)}{2} + c_S\epsilon = V. \end{aligned}$$

Thus, by (3.7),

$$F(\alpha, \xi) \geq V \quad \text{if } \xi \in \Xi \text{ with } \xi(1) \in T. \tag{3.8}$$

Hence, by (3.6) and (3.8), $F(\alpha, \xi) \geq V$ for all $\xi \in \Xi$.

By (3.4) we therefore have $f(\alpha, \xi) \geq V$ for all $\xi \in \Xi$. Thus the hider can ensure a cost of at least V by using the strategy α . However $\overleftarrow{1}$ Theorem 3.1 ensures that the searcher can restrict the cost to at most V by choosing $\overleftarrow{1}$ with probability $(W + c_S - c_T)/2W$ and $\overleftarrow{1}^{\rightarrow}$ with probability $(W + c_T - c_S)/2W$. Thus the value of the game is V and the optimal strategies for the players are as stated in the theorem. \square

3.2.2. Proof of the theorem when $n = 2m$ and $|S| = m$

The case analysed in the previous section is the one for which S contains the least possible number of nodes because $|S| \geq 1$. In this section a case when S contains the greatest number of nodes consistent with it having no two nodes adjacent is investigated. In Section 3.2.1 we introduced the function F which had a similar form to that of our payoff function f and this enabled us to prove that the hider strategy in Theorem 3.2 is optimal. However a more complicated function than F is needed to prove the optimality of the hider strategy in Theorem 3.2. We define a function H as follows:

Given $\xi \in \Xi$, let $v_1(\xi) = 1$ if $\xi(1) \in S$ and $v_1(\xi) = 2$ if $\xi(1) \in T$,

$$v_i(\xi) = \begin{cases} 2 & \text{if } \xi(i-1) \text{ and } \xi(i) \text{ are both in } S \text{ or both in } T \\ 1 & \text{otherwise} \end{cases} \quad \text{for } i > 1,$$

and $\Upsilon_j(\xi) = \sum_{i=1}^j v_i(\xi)$.

For a hider strategy p and a searcher strategy ξ define $H(p, \xi)$ by

$$H(p, \xi) = \sum_{i=1}^n p_{\xi(i)} (\Upsilon_i(\xi) + C_i(\xi)). \tag{3.9}$$

Proof of Theorem 3.2 for $n = 2m$ and $|S| = m$. Because $S = \{1, 3, 5, \dots, 2m - 1\}$, $D_i(\xi)$ given by (2.2) satisfies $\Upsilon_i(\xi) \leq D_i(\xi)$ for all $\xi \in \Xi$ and all i . Thus, for all hider strategies p and $\xi \in \Xi$,

$$H(p, \xi) \leq f(p, \xi). \tag{3.10}$$

Let α denote the hider strategy given by

$$\alpha_i = \begin{cases} \alpha_S = c_S/W & \text{if } i \in S \\ \alpha_T = (2 + c_T)/W & \text{if } i \in T \end{cases}. \tag{3.11}$$

We will show that $L = \min_{\xi \in \Xi} H(\alpha, \xi)$ equals V where H and V are given by (3.9) and (3.1) respectively. Let $\Pi = \{\xi \in \Xi : H(\alpha, \xi) = L\}$. We have the next lemma, the proof of which is given in Appendix (I).

Lemma 3.1. *Permutations arising from $\vec{1}$ and from 1^\rightarrow are in Π .*

Now

$$\begin{aligned} H(\alpha, \vec{1}) &= \alpha_S(1 + c_S + (3 + 2c_S + c_T) + \dots + 2m - 1 + mc_S + (m - 1)c_T) \\ &\quad + \alpha_T(2 + c_S + c_T + (4 + 2c_S + 2c_T) + \dots + 2m + mc_S + mc_T) \\ &= m\alpha_S(1 + c_S) + (\alpha_S + \alpha_T)(W/m)(1 + 2 + \dots + m - 1) + \alpha_T W \\ &= \frac{mc_S(1 + c_S)}{W} + \frac{W(m - 1)}{2m} + 2 + c_T = \frac{mc_S(1 + c_S)}{W} + \frac{W}{2} - \frac{2 + c_S + c_T}{2} + 2 + c_T \\ &= \frac{mc_S(1 + c_S)}{W} + \frac{W + 2 + c_T}{2} - \frac{c_S}{2} = \frac{W + 2 + c_T}{2} + mc_S\epsilon = V. \end{aligned}$$

Comparing the costs of nodes in $H(\alpha, 1^\rightarrow)$ with the costs of the corresponding nodes in $H(\alpha, \vec{1})$, each node other than node 1 is c_S less costly whereas node 1 is $W + 1 - 1 - c_S$ more costly. Thus

$$H(\alpha, 1^\rightarrow) - H(\alpha, \vec{1}) = \alpha_S(W - mc_S) - m\alpha_T c_S = \alpha_S(W - mc_S - m(2 + c_T)) = 0.$$

Thus $L = V$ and, by (3.10), $f(\alpha, \xi) \geq V$ for all $\xi \in \Xi$. Thus the hider can ensure a cost of at least V by using the strategy α . However Theorem 3.1 ensures that the searcher can restrict the cost to at most V by choosing $\vec{1}$ with probability $1/2 + m\epsilon$ and 1^\rightarrow with probability $1/2 - m\epsilon$. Thus the value of the game is V and the optimal strategies for the players are as stated in the theorem. \square

4. The Case when S Is an Interval

In the case when no two nodes in S are adjacent, we have seen that the searcher has a strategy which guarantees an expected cost of at most \bar{V} given by (3.1) and which is optimal in at least two cases. The support of this strategy is the same irrespective of whether $c_S > c_T$ or $c_S < c_T$. This does not appear to be the case when S does have adjacent nodes. In this section we obtain an expression (v_{int}) for the value of $\Gamma(N, S)$ when S is an interval and $c_S \geq c_T$. We also obtain an upper bound for the game when $c_S < c_T$ and show that this upper bound is the value of the game when S comprises two adjacent nodes.

Theorem 4.1. *Suppose the nodes of S comprise an interval with $|S| \geq 2$ and $c_S \geq c_T$, then the value of $\Gamma(N, S)$ is*

$$v_{int} = \frac{W + 1 + c_T}{2} + |S|(1 + c_S)\epsilon. \tag{4.1}$$

Assuming the interval is $[1, k]$, an optimal strategy for the searcher is to use each of \overrightarrow{k} and $\overleftarrow{k-1}$ with probability $1/2 - \epsilon$ and each of $\overrightarrow{1}$ and \overleftarrow{k} with probability ϵ .

An optimal strategy for the hider is to hide at each node of S with probability $(1 + c_S)/W$ and at each node of T with probability $(1 + c_T)/W$.

Proof. We first show that the searcher can make the expected cost equal to v_{int} . Let G_1 denote the searcher strategy which uses each of \overrightarrow{k} and $\overleftarrow{k-1}$ with probability $1/2$.

If the hider hides at node i , the expected cost for G_1 is

$$\frac{n + 1 + C + c_i}{2} = \frac{W + 1 + c_i}{2}.$$

Let G_2 denote the searcher strategy which uses each of $\overrightarrow{1}$ and \overleftarrow{k} with probability $1/2$.

If the hider hides at a node of S , the expected cost for G_2 is

$$\frac{(k + 1)(1 + c_S)}{2}.$$

If the hider hides at a node of T , the expected cost for these nodes is

$$\frac{n + k + 1 + C + c_T + kc_S}{2} = \frac{W + k(1 + c_S) + 1 + c_T}{2}.$$

Let G^* denote the searcher strategy which employs G_1 with probability $1 - 2\epsilon$ and G_2 with probability 2ϵ .

If the hider hides at a node of S , the expected cost for G^* is

$$\frac{W + 1 + c_S}{2} - \epsilon[W - k(1 + c_S)] = \frac{W + 1 + c_T}{2} + k(1 + c_S)\epsilon = v_{int}.$$

If the hider hides at a node of T , the expected cost for G^* is

$$\frac{W + 1 + c_T}{2} + \epsilon[k(1 + c_S)] = v_{int}.$$

Thus G^* restricts the hider to (4.1).

We now show that the hider can ensure that the expected cost is at least v_{int} . Put

$$\alpha_i = \begin{cases} \alpha_S = (1 + c_S)/W & \text{if } i \in S \\ \alpha_T = (1 + c_T)/W & \text{if } i \in T \end{cases} \tag{4.2}$$

and let $L = \min_{\xi \in \Xi} F(\alpha, \xi)$ where F is given by (3.3).

Let $\Pi = \{\xi \in \Xi : F(\alpha, \xi) = L\}$. Suppose $\phi \in \Pi$ and there is a j with $1 < j < n$ satisfying $\phi(j) \in T$ and $\phi(j + 1) \in S$. Define ζ by $\zeta(j) = \phi(j + 1)$, $\zeta(j + 1) = \phi(j)$ and $\zeta(i) = \phi(i)$ otherwise. Because $\zeta(1) = \phi(1)$, the corresponding terms in the sums for $F(\alpha, \phi)$ and $F(\alpha, \zeta)$ are equal except possibly when $i = j$ or $j + 1$. Hence, putting $v = v_1(\phi) = v_1(\xi)$,

$$\begin{aligned} F(\alpha, \phi) - F(\alpha, \zeta) &= \alpha_T(j - 1 + v + C_j(\phi)) + \alpha_S(j + v + C_{j+1}(\phi)) \\ &\quad - \alpha_S(j - 1 + v + C_j(\zeta)) - \alpha_T(j + v + C_{j+1}(\zeta)) \\ &= \alpha_S(1 + c_T) - \alpha_T(1 + c_S) = 0. \end{aligned}$$

Because $F(\alpha, \phi) = F(\alpha, \zeta)$, given any $\xi \in \Xi$ and i with $1 < i < n$, $\xi(i) \in T, \xi(i + 1) \in S$ we can interchange the elements $\xi(i)$ and $\xi(i + 1)$ and leave the value of F unchanged. The analysis therefore divides into two cases depending on whether $\xi(1) \in S$ or $\xi(1) \in T$.

If $\xi(1) \in S$, all the S members of ξ can be brought forward so that $F(\alpha, \xi) = F(\alpha, \theta)$ where $\theta(i) \in S$ for $i = 1, \dots, k$ and $\theta(i) \in T$ otherwise.

Now

$$\begin{aligned} W = n + C &= \frac{(n + C)^2}{n + C} = \frac{(k(1 + c_S) + (n - k)(1 + c_T))^2}{W} \\ &= \alpha_S k^2(1 + c_S) + 2k(1 + c_S)(n - k)\alpha_T + \alpha_T(n - k)^2(1 + c_T) \end{aligned}$$

and

$$F(\alpha, \theta) = \alpha_S \frac{k(k + 1)}{2}(1 + c_S) + \alpha_T k(1 + c_S)(n - k) + \alpha_T \frac{(n - k)(n - k + 1)}{2}(1 + c_T)$$

so

$$F(\alpha, \theta) = \frac{W}{2} + \frac{\alpha_S(k + kc_S)}{2} + \frac{\alpha_T(n + C - k - kc_S)}{2} = \frac{W + 1 + c_T}{2} + k(1 + c_S)\epsilon = v_{int}.$$

If $\xi(1) \in T$, all the T members of ξ can be brought forward so that $F(\alpha, \xi) = F(\alpha, \mu)$ where $\mu(i) \in T$ for $i = 1, \dots, n - k$ and $\mu(i) \in S$ otherwise. Now

$$\begin{aligned} F(\alpha, \mu) &= 1 + \alpha_T \frac{(n - k)(n - k + 1)}{2}(1 + c_T) + \alpha_S(n - k)(1 + c_T)k + \alpha_S \frac{k(k + 1)}{2}(1 + c_S) \\ &= 1 + F(\alpha, \theta). \end{aligned}$$

Hence, for $\zeta \in \Xi$, by (3.3)

$$f(\alpha, \zeta) \geq F(\alpha, \zeta) \geq \min\{F(\alpha, \theta), F(\alpha, \mu)\} = F(\alpha, \theta) = v_{int}$$

which is equivalent to (4.1). Thus the hider can ensure an expected cost of at least (4.1) by using the strategy α given by (4.2) and the proof is complete. □

Theorem 4.2. *Suppose the nodes of S comprise the interval $[1, k]$ where $k \geq 2$ and $c_T \geq c_S$. By using the strategy which chooses each of \overrightarrow{k} and $\overleftarrow{k - 1}$ with probability $1/2 - \epsilon$ and each of $\overleftarrow{1}$ and $k \rightarrow$ with probability ϵ , the searcher can make the expected cost in $\Gamma(N, S)$ equal to*

$$V_{INT} = \frac{W + 1 + c_T}{2} - [2 - k(1 + c_S)]\epsilon. \tag{4.3}$$

When $k = 2$, V_{INT} is the value of the game and an optimal hider strategy is α where $\alpha_i = c_S/W$ if $i \in S$ and $\alpha_i = (n/(n - 2) + c_T)/W$ if $i \in T$.

Proof. (Outline) We first show that the searcher can make the expected cost equal to (4.3). Let G_1 denote the searcher strategy which uses each of \overrightarrow{k} and $\overleftarrow{k-1}$ with probability $1/2$. Let G_2 denote the searcher strategy which uses each of $\overleftarrow{1}$ and \overrightarrow{k} with probability $1/2$. Let G^* denote the searcher strategy which employs G_1 with probability $1 + 2\epsilon$ and G_2 with probability -2ϵ . Then G^* restricts the hider to V_{INT} given by (4.3).

Next we prove that the value of the game is given by V_{INT} when $k = 2$. Let α be the strategy for the hider given in the statement of the theorem. We show that $L = \min_{\xi \in \Xi} F(\alpha, \xi)$ equals V_{INT} where F is given by (3.3). By a similar argument to that of Theorem 4.1 we arrive at the fact that $L = F(\alpha, \overrightarrow{2}) = F(\alpha, \overleftarrow{1}) = f(\alpha, \overrightarrow{2}) = f(\alpha, \overleftarrow{1})$. By symmetry $f(\alpha, \overleftarrow{1}) = f(\alpha, \overrightarrow{2})$ and $f(\alpha, \overrightarrow{2}) = f(\alpha, \overleftarrow{1})$. These imply $f(\alpha, G^*) = L$ when $k = 2$. From the first part of the proof, we see that $V_{INT} = f(\alpha, G^*)$. Thus, by (3.3), $f(\alpha, \xi) \geq F(\alpha, \xi) \geq L = V_{INT}$ for all $\xi \in \Xi$ so the hider can ensure an expected cost of at least V_{INT} by using the strategy α . Hence, when $k = 2$ the value of the game is V_{INT} . \square

5. Discussion of Results

At first sight games on a broken wheel may give the impression that they are fairly straightforward to solve. However we have shown that the expression for the value of a game can depend on whether S has no adjacent nodes or whether S is an interval. Although the supports of our optimal searcher strategies are all contained in the set of pure strategies which, once a node is searched, traverse the cycle searching a node the first time it is visited thereafter, there are qualitative differences between them. In the games when S has no adjacent nodes, the supports of our optimal searcher strategies always contain a strategy which begins by searching a node in S irrespective of whether $c_T > c_S$ or $c_S > c_T$. This is not the case when S is an interval and $c_S > c_T$; in this case the support of our optimal searcher strategies contain only strategies which first search a node of S .

The functions F and H introduced in Sections 3.2 and 3.2.2 meant effectively that, in order to prove that our hider strategies were optimal, it was sufficient to consider a much reduced searcher strategy space. This process is capable of further development and we have used a modification of it to obtain the values of some particular broken wheel games not mentioned in the paper. However to develop a comprehensive theory for broken wheel games seems difficult.

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Appendix

(I) Proof of Lemma 3.1

Suppose $\phi \in \Pi$ and there is an i such that $\phi(i)$ and $\phi(i+1)$ are both in S or both in T . Let j be the least such i and $\phi(j) \in X$ where $X \in \{S, T\}$. Thus, if $i \leq j$, one of $\phi(i-1)$ and $\phi(i)$ is in S and the other is in T . Because $|S| = |T|$, this means that there is an $i > j+1$ such that $\phi(i) \notin X$; let k be the least such i . Define $\xi \in \Xi$ by $\xi(k-1) = \phi(k)$, $\xi(k) = \phi(k-1)$ and $\xi(i) = \phi(i)$ otherwise. It is easily seen that

$$v_i(\xi) = v_i(\phi) \quad \text{for } i \notin \{k-1, k, k+1\},$$

$$\Upsilon_{k+1}(\xi) = \begin{cases} \Upsilon_{k+1}(\phi) & \text{if } \phi(k+1) \in X \\ \Upsilon_{k+1}(\phi) - 2 & \text{if } \phi(k+1) \notin X \end{cases} \quad \text{so } \Upsilon_i(\xi) \leq \Upsilon_i(\phi) \quad \text{for } i \geq k+1,$$

$$c_{\xi(i)} = c_{\phi(i)} \quad \text{and} \quad \alpha_{\xi(i)} = \alpha_{\phi(i)} \quad \text{if } i \notin \{k-1, k\}.$$

Putting $Y_i(\xi) = \alpha_{\xi(i)}(\Upsilon_i(\xi) + C_i(\xi))$, we have

$$Y_{k-1}(\xi) = \alpha_{\xi(k-1)}(\Upsilon_{k-1}(\xi) + C_{k-1}(\xi)) = \alpha_{\phi(k)}(\Upsilon_k(\phi) - 2 + C_k(\phi) - c_{\phi(k-1)}) = Y_k(\phi) - \alpha_{\phi(k)}(2 + c_{\phi(k-1)})$$

$$Y_k(\xi) = \alpha_{\phi(k-1)}(\Upsilon_{k-1}(\phi) + C_{k-1}(\phi) + c_{\phi(k)}) = Y_{k-1}(\phi) + \alpha_{\phi(k-1)}c_{\phi(k)}.$$

Hence $H(\alpha, \phi) - H(\alpha, \xi) \geq \alpha_{\phi(k)}(2 + c_{k-1}(\phi)) - \alpha_{\phi(k-1)}c_{\phi(k)}$.

When $\phi(k) \in S$, we have $H(\alpha, \phi) - H(\alpha, \xi) \geq \alpha_S(2 + c_T) - \alpha_Tc_S = 0$ by (3.11).

When $\phi(k) \in T$, we have $H(\alpha, \phi) - H(\alpha, \xi) \geq \alpha_T(2 + c_S) - \alpha_Sc_T = ((2 + c_T)(2 + c_S) - c_Sc_T)/W > 0$.

Thus $H(\alpha, \phi) \geq H(\alpha, \xi)$ so $\xi \in \Pi$.

The argument can be repeated to give a $\theta \in \Pi$ such that either $\theta(i)$ and $\theta(i + 1)$ are in different sets for all i or the first i such that $\theta(i)$ and $\theta(i + 1)$ are in the same set satisfies $i > j$. We write this i as $i = j(\theta)$. But now the same reasoning gives the existence of a $\psi \in \Pi$ such that either $\psi(i)$ and $\psi(i + 1)$ are in different sets for all i or the first i such that $\psi(i)$ and $\psi(i + 1)$ are in the same set satisfies $i = j(\psi) > j(\theta)$. Continuing this process it must terminate giving the existence of an $\omega \in \Pi$ for which $\omega(i)$ and $\omega(i + 1)$ are in different sets for all i .

Because $|S| = |T|$, there are only two possibilities for ω ; one arising from $\vec{1}$ and the other from 1^\rightarrow .

(II) Proof of Theorem 4.2

We first show that the searcher can make the expected cost equal to (4.3). Let G_1 denote the searcher strategy which uses each of \vec{k} and $\overleftarrow{k-1}$ with probability 1/2.

If the hider hides at node i , the expected cost for G_1 is

$$\frac{n + 1 + C + c_i}{2} = \frac{W + 1 + c_i}{2}.$$

Let G_2 denote the searcher strategy which uses each of $\overleftarrow{1}$ and k^\rightarrow with probability 1/2.

If the hider hides at a node of S , the expected cost for G_2 is

$$\frac{2n + 2 - (k - 1) + 2(n - k)c_T + (k + 1)c_S}{2} = \frac{2W + 2 - (k - 1)(1 + c_S)}{2}.$$

If the hider hides at a node of T , the expected cost for these nodes is

$$\frac{n - k + 3 + (n - k + 1)c_T}{2} = \frac{W + 3 + c_T - k(1 + c_S)}{2}.$$

Let G^* denote the searcher strategy which employs G_1 with probability $1 + 2\epsilon$ and G_2 with probability -2ϵ .

If the hider hides at a node of S , the expected cost for G^* is

$$\begin{aligned} & \frac{W + 1 + c_S}{2} - 2\epsilon \frac{2W + 2 - (k - 1)(1 + c_S) - W - (1 + c_S)}{2} \\ &= \frac{W + 1 + c_S}{2} + \frac{c_T - c_S}{2} + \frac{c_T - c_S}{W} \frac{2 - k(1 + c_S)}{2} \\ &= \frac{W + 1 + c_T}{2} - [2 - k(1 + c_S)]\epsilon \\ &= V_{INT}. \end{aligned}$$

If the hider hides at a node of T , the expected cost for G^* is

$$\begin{aligned} & \frac{W + 1 + c_T}{2} + -2\epsilon \frac{W + 3 + c_T - k(1 + c_S) - W - 1 - c_T}{2} \\ &= \frac{W + 1 + c_T}{2} - [2 - k(1 + c_S)]\epsilon \\ &= V_{INT}. \end{aligned}$$

Thus G^* restricts the hider to V_{INT} given by (4.3).

We now prove that the value of the game is given by V_{INT} when $k = 2$.

Put

$$\alpha_i = \begin{cases} \alpha_S = c_S/W & \text{if } i \in S \\ \alpha_T = (n/(n-2) + c_T)/W & \text{if } i \in T \end{cases} \tag{A.1}$$

We will show that $L = \min_{\xi \in \Xi} F(\alpha, \xi)$ equals V_{INT} where F and V_{INT} are given by (3.3) and (4.3) respectively. Let $\Pi = \{\xi \in \Xi : F(\alpha, \xi) = L\}$.

Suppose $\phi \in \Pi$ and there is an j with $1 < j < n$ satisfying $\phi(j) \in S$ and $\phi(j+1) \in T$. Define ζ by $\zeta(j) = \phi(j+1)$, $\zeta(j+1) = \phi(j)$ and $\zeta(i) = \phi(i)$ otherwise. Because $\zeta(1) = \phi(1)$, the corresponding terms in the sums for $F(\alpha, \phi)$ and $F(\alpha, \zeta)$ are equal except possibly when $i = j$ or $j + 1$. Hence, putting $v = v_1(\phi) = v_1(\zeta)$,

$$\begin{aligned} F(\alpha, \phi) - F(\alpha, \zeta) &= \alpha_S(j-1 + v + C_j(\phi)) + \alpha_T(j + v + C_{j+1}(\phi)) \\ &\quad - \alpha_T(j-1 + v + C_j(\zeta)) - \alpha_S(j + v + C_{j+1}(\zeta)) \\ &= \alpha_T(1 + c_S) - \alpha_S(1 + c_T) = \frac{n/(n-2) + c_T}{W}(1 + c_S) - \frac{c_S}{W}(1 + c_T) > 0. \end{aligned}$$

Because $F(\alpha, \phi) > F(\alpha, \zeta)$, it follows that, if $\xi \in \Pi$, satisfies $\xi(j) \in T$ for $j > 1$, then either $\xi(i) \in T$ for $i = 1, \dots, n-2$ or $\xi(1) \in S$ and $\xi(i) \in T$ for $i = 2, 3, \dots, n-1$. Thus $L = F(\alpha, 2^\rightarrow) = f(\alpha, 2^\rightarrow)$ or $L = F(\alpha, \overleftarrow{1}) = f(\alpha, \overleftarrow{1})$

Now

$$F(\alpha, 2^\rightarrow) = \alpha_T \sum_{i=1}^{n-2} (1 + i + ic_T) + \alpha_S(2n + 1 + 3c_S + 2(n-2)c_T)$$

and

$$F(\alpha, \overleftarrow{1}) = \alpha_T \sum_{i=1}^{n-2} (1 + c_S + i + ic_T) + \alpha_S(n + 1 + 3c_S + (n-2)c_T).$$

so

$$F(\alpha, 2^\rightarrow) - F(\alpha, \overleftarrow{1}) = -\alpha_T \sum_{i=1}^{n-2} c_S + \alpha_S(n + (n-2)c_T) = -(n-2)\alpha_T c_S + \alpha_S(n + (n-2)c_T) = 0$$

by (A.1). Thus $L = F(\alpha, 2^\rightarrow) = F(\alpha, \overleftarrow{1})$

By symmetry $f(\alpha, \overleftarrow{1}) = f(\alpha, 2^\rightarrow)$ and $f(\alpha, \overrightarrow{2}) = f(\alpha, \overleftarrow{1})$. In the first part of the proof, we showed that

$$V_{INT} = \left(\frac{1}{2} + \epsilon\right)f(\alpha, \overrightarrow{2}) + \left(\frac{1}{2} + \epsilon\right)f(\alpha, \overleftarrow{1}) - \epsilon f(\alpha, \overleftarrow{1}) - \epsilon f(\alpha, 2^\rightarrow)$$

so

$$V_{INT} = (1 + 2\epsilon)f(\alpha, \overleftarrow{1}) - 2\epsilon f(\alpha, 2^\rightarrow) = (1 - 2\epsilon)L + 2\epsilon L = L = \min_{\xi \in \Xi} F(\alpha, \xi).$$

Thus, by (3.3), $f(\alpha, \xi) \geq V_{INT}$ for all $\xi \in \Xi$ so the hider can ensure an expected cost of at least V_{INT} by using the strategy α . Hence, when $k = 2$ the value of the game is V_{INT} .

Kensaku Kikuta
 School of Business Administration
 University of Hyogo
 Gakuen-nishi, 8-2-1
 Nishi-Ku, Kobe 651-2197, JAPAN
 E-mail: kikuta@biz.u-hyogo.ac.jp