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AN APPLICATION OF A THEOREM OF ALTERNATIVES TO ORIGAMI

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Abstract Theorems of alternatives are useful mathematical tools in optimization. Their objects are pairs of linear systems. Folding a paper with a crease defines a linear inequality in $\mathbb{R}^2$. So one gets convex polygons by folding a paper many times. This paper provides a new perspective of duality to origami mathematics. We show that Gale's theorem of alternatives is useful for the study of twist fold.

Keywords: Optimization, theorem of alternatives, Gale's theorem, origami, twist fold

1. Introduction

One of the most fundamental themes of origami mathematics is flat-foldability of crease patterns, which determines whether the crease pattern can be folded to a flat figure. *Single-vertex fold* is a radical crease pattern incident to a single vertex such as Figure 1.

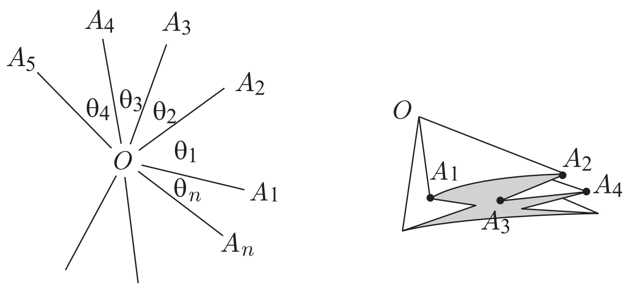

Figure 1: Single-vertex fold

Flat-foldability for single-vertex folds is completely solved by Theorems 1.1 and 1.2, see e.g. [2, 3].

Theorem 1.1. (T. Kawasaki) For any single-vertex fold in Figure 1,

(1) it can be folded flat with a suitable mountain/valley-folds if and only if $n \geq 2$ is even and the alternating sum $\theta_1 - \theta_2 + \cdots - \theta_n$ is zero.

(2) If $\theta_{i-1} > \theta_i < \theta_{i+1}$ for some i, then mountain/valley-folds of OA_i and OA_{i+1} are different.

(3) When $n = 4$, mountain/valley-folds of adjacent half-lines whose angle is a unique maximum are same.

Husimi's theorem presented in 70s is a special case $n = 4$ of Kawasaki's theorem, see [1].

Theorem 1.2. (Maekawa) For any flat-foldable single-vertex fold, the number of mountain folds differs from the number of valley folds by exactly two folds.

Kawasaki [2] also studied flat-foldability of two-vertex fold.

Recently Yamaguchi and Kawasaki [6] gave a criterion for flat-foldability of crease patterns with n vertices of a convex polygon. In this paper we call such a crease pattern an *n-vertex fold*. n-vertex fold is a mathematical model of *twist fold* (Figure 2). In Figure 2 (left), we used a pentagon-shaped paper. Interior solid/dashed lines indicate mountain/valley-folds, respectively. The center figure in Figure 2 is the result, and the right figure is the see-through version of the result.

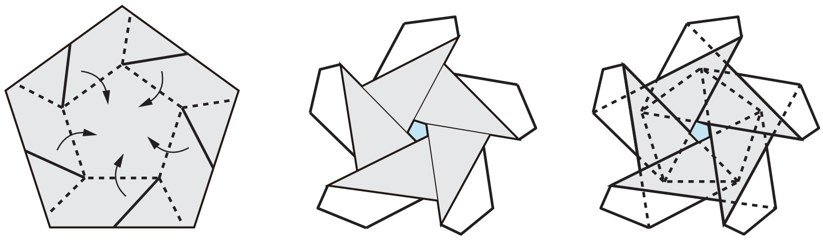

Figure 2: Twist fold

The essential part of the criterion of [6] is whether a convex polygon, called *hole*, defined by the n-vertex fold is empty or not. The small pentagon in Figure 2 (center) is an example of the hole. We will define the hole in Section 3.

A *theorem of alternatives* is a theorem stating that of two linear systems, one or the other has a solution, but not both nor none, refer to Mangasarian [4]. In this paper we show that Gale's theorem of alternatives (Theorem 1.3) is useful to discuss flat-foldability of n-vertex folds.

Theorem 1.3. (Gale) Let A be an $n \times m$ matrix and $b \in \mathbb{R}^n$. The system of linear inequalities $Ax \leq b$ has a solution $x \in \mathbb{R}^m$ if and only if (1.1) has no solution $y \in \mathbb{R}^n$.

$$y \geq 0, \quad y^T A = 0, \quad y^T b < 0. \tag{1.1}$$

The rest of the paper is organized as follows. Section 2 reviews n-vertex folds. It introduces the main results of [6] (Theorems 2.1 and 2.2) on flat-foldability in terms of the *hole*. Section 3 treats the hole as the solution set of linear inequalities, and see the hole from the standpoint of duality by applying Gale's theorem to the linear inequality system.

2. n-vertex Fold

It is convenient to regard a crease pattern as a graph. Then the degree of vertex v is the number of edges (including half-lines) incident to v, denoted by $\deg v$.

Let P be a convex polygon with vertices $v_1, \ldots, v_n$. In this paper an *n-vertex fold* means a crease pattern consisting of the edges of P and a finite number of half lines incident to the vertices $v_1, \ldots, v_n$, see Figure 3 (left).

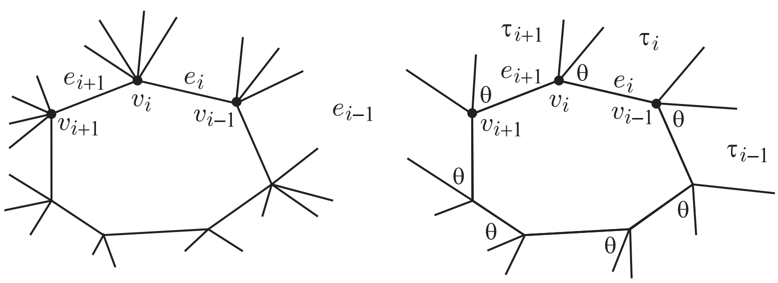

Figure 3: n-vertex fold

An n-vertex fold is said to be *flat-foldable* if it is folded to a flat figure by suitably assigning mountain/valley-folds to each crease. Such an n-vertex fold takes the form of Figure 3 (right). Precisely speaking, it is stated as follows.

Theorem 2.1. (Yamaguchi and Kawasaki [6]) If an n-vertex fold is flat-foldable, then
(1) $\deg v_i = 4$ for any i. Namely, the number of half lines incident to v_i is exactly two.
(2) For any adjacent two vertices v_i and v_{i+1}, (there are totally four half lines incident to them), two half lines inside are parallel and their mountain/valley-folds are different. (We denote by τ_i the area bounded by these parallel half lines and edge $e_i := v_{i-1}v_i$, and call it a *belt*.)
(3) The angle $0 \leq \theta \leq \frac{\pi}{2}$ between τ_i and e_i does not depend on i.

Naturally, any n-vertex fold includes n single-vertex folds. So any flat-foldable n-vertex fold has to satisfy all necessary conditions in Theorems 1.1 and 1.2 around each vertex. Further necessary condition for flat-foldability is stated in terms of the *hole* defined below.

In Figure 4, l_i denotes the left half-line of the belt τ_i. τ_i' and l_i' denote the reflection of τ_i and l_i through edge $v_{i-1}v_i$, respectively. The *hole* denoted by H (gray area in the right figure of Figure 4) is defined as the intersection of the closed half-spaces H_i $(i = 1, 2, \ldots, n)$ (hatched area in the right figure).

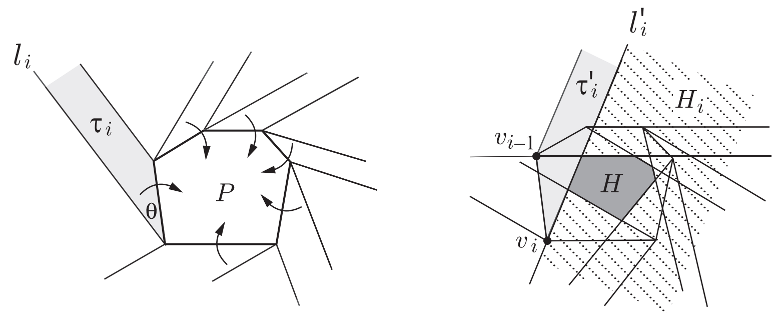

Figure 4: Half-space H_i and the hole H

In the following theorem, the n-vertex fold is assumed to satisfy all necessary conditions in Theorems 1.1 and 1.2 at each vertex and (1)-(3) of Theorem 2.1.

Theorem 2.2. ([6]) Assume that all interior angles of P are obtuse. If an n-vertex fold is flat-foldable by assigning valley-folds to all edges $e_1, \ldots, e_n$, then the hole is non-empty. Conversely, if the hole is non-empty, then the n-vertex fold is flat-foldable.

3. Hole as the Solution Set of Linear System

In this section we treat the hole as the solution set of linear inequalities. We don't impose any assumption on the angles of P as in Theorem 2.2. So Theorems 3.1, 3.3, and 3.4 are applicable to any convex polygon P. Let

$$a_i := v_i - v_{i-1}, \quad n_i := R(\tfrac{\pi}{2} - \theta)a_i \quad (i = 1, \ldots, n) \tag{3.1}$$

where $R(\alpha)$ denotes the rotation matrix through an angle α, see Figure 4 (left). Then n_i is a normal vector of l_i', and

$$H_i = \{x \in \mathbb{R}^2 \mid (R(\tfrac{\pi}{2} - \theta)a_i)^T(x - v_i) \geq 0\} = \{x \in \mathbb{R}^2 \mid a_i^T R(\theta - \tfrac{\pi}{2})(x - v_i) \geq 0\}, \tag{3.2}$$

where we used $R(\alpha)^T = R(-\alpha)$. Therefore the hole is expressed as

$$H = \{x \in \mathbb{R}^2 \mid a_i^T R(\theta - \tfrac{\pi}{2})x \geq a_i^T R(\theta - \tfrac{\pi}{2})v_i \quad (i = 1, \ldots, n)\}. \tag{3.3}$$

Similarly, the convex polygon P is expressed as

$$P = \{x \in \mathbb{R}^2 \mid a_i^T R(-\tfrac{\pi}{2})x \geq a_i^T R(-\tfrac{\pi}{2})v_i \quad (i = 1, \ldots, n)\}. \tag{3.4}$$

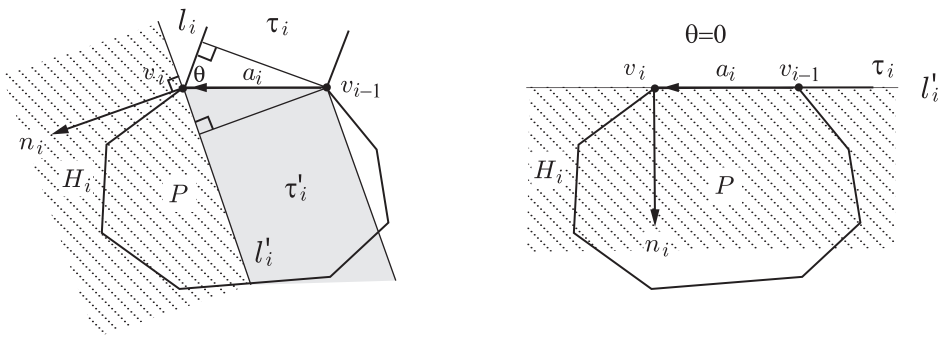

Figure 5: Formulation of the half-space H_i

When $\theta = 0$, the belt τ_i degenerates to the half line parallel to the edge a_i. Even in this case, the hole H is well-defined, and H coincides with P, see Figure 4 (right).

Theorem 3.1. (1) The hole is non-empty if and only if there exists no $y = (y_1, \dots, y_n)$ satisfying (3.5).

$$y \geq 0, \quad \sum_{i=1}^{n} y_i a_i = 0, \quad \sum_{i=1}^{n} y_i a_i^T R(\theta - \tfrac{\pi}{2}) v_i > 0. \tag{3.5}$$

(2) When $\theta = \frac{\pi}{2}$ (each belt τ_i is perpendicular to the edge a_i), the hole is empty.
(3) When $\theta = 0$, the hole coincides with P.
(4) The hole is nonincreasing and continuous w.r.t θ.
(5) There exists some $0 < \bar{\theta} < \frac{\pi}{2}$ such that the hole is non-empty for any $0 \leq \theta \leq \bar{\theta}$, and empty for any $\bar{\theta} < \theta \leq \frac{\pi}{2}$.

Proof. (3) has already been proved. (1): Applying Gale's theorem to the linear system in (3.3), its dual linear system (1.1) reduces to (3.6).

$$y = (y_1, \dots, y_n) \geq 0, \quad \sum_{i=1}^{n} y_i a_i^T R(\theta - \tfrac{\pi}{2}) = 0, \quad \sum_{i=1}^{n} y_i a_i^T R(\theta - \tfrac{\pi}{2}) v_i > 0. \tag{3.6}$$

Since $R(\theta - \frac{\pi}{2})$ is nonsingular, we get (3.5) from (3.6). (2): When $\theta = \frac{\pi}{2}$, we show that the dual linear system (3.5) has a solution y. Since $\{a_i\}_{i=1}^n$ are directed edges of P, by taking $y := (1, \dots, 1)$ we have

$$\sum_{i=1}^{n} y_i a_i = \sum_{i=1}^{n} a_i = 0.$$

Since $R(\theta - \frac{\pi}{2}) = R(0)$ is identity, the last term in (3.5) is

$$\sum_{i=1}^{n} y_i a_i^T R(\theta - \tfrac{\pi}{2}) v_i = \sum_{i=1}^{n} a_i^T v_i = \sum_{i=1}^{n} (v_i - v_{i-1})^T v_i = \sum_{i=1}^{n} \frac{||v_i - v_{i-1}||^2}{2} > 0,$$

where $v_0 := v_n$. Therefore (3.5) has a solution y, so that the hole is empty. (4): It is clear from Figure 4 that $H_i \cap P$ is nonincreasing and continuous w.r.t. θ, so is

$$H = H_1 \cap \cdots \cap H_n = H_1 \cap \cdots \cap H_n \cap P = (H_1 \cap P) \cap \cdots \cap (H_n \cap P).$$

(5) follows from (2), (3), and (4). □

Theorem 3.2. (Helly's theorem, [5, Theorem 21.6]) Let $C_1, \dots, C_n$ be a finite collection of convex sets in $\mathbb{R}^d$. If their intersection is empty, then there exists a $d+1$ subcollection of $C_1, \dots, C_n$ whose intersection is empty.

Combining Theorem 3.1 (1) and Helly's theorem, we directly obtain the following.

Theorem 3.3. The following three conditions are equivalent to each other.

(1) The hole is empty.

(2) There exists $p, q, r \in \{1, \dots, n\}$ such that $H_p \cap H_q \cap H_r$ is empty.

(3) There exists $p, q, r \in \{1, \dots, n\}$ and $y_p, y_q, y_r \geq 0$ such that

$$y_p a_p + y_q a_q + y_r a_r = 0, \quad y_p a_p^T R v_p + y_q a_q^T R v_q + y_r a_r^T R v_r > 0, \tag{3.7}$$

where $R := R(\theta - \frac{\pi}{2})$.

Let H_i^c denote the complement of H_i. Then the hole is surrounded by H_i^c $(i = 1, \dots, n)$, in other words, surrounded by the reflected belts τ_i' $(i = 1, \dots, n)$, see Figure 6. The following theorem can be regarded as a sufficient condition for the hole be empty.

Theorem 3.4. Let P_i denote the closed half-space including P determined by the edge a_i, and $\mathring{\tau}_i'$ denote the interior of the reflected belt τ_i'. Assume that the hole is non-empty and $\cap_{i \in I} P_i$ is bounded for some $I \subset \{1, \dots, n\}$. Then both $\cap_{i \in I} H_i^c$ and $\cap_{i \in I} \mathring{\tau}_i'$ are empty. In particular, if three lines extending edges a_p, a_q, and a_r form a triangle as in Figure 6 (right), then $H_p^c \cap H_q^c \cap H_r^c$ and $\mathring{\tau}_p' \cap \mathring{\tau}_q' \cap \mathring{\tau}_r'$ are empty.

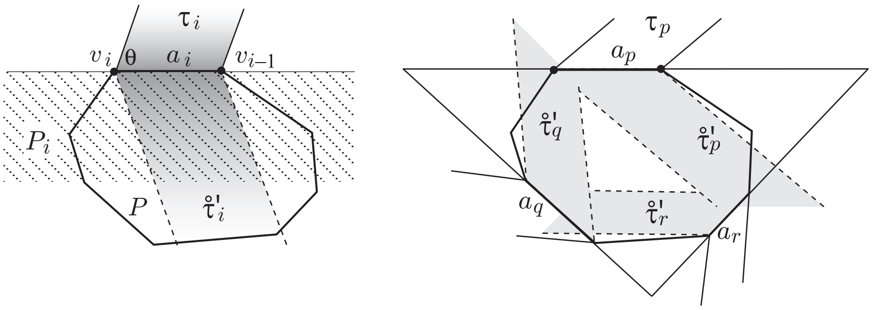

Figure 6: The closed half-space P_i and the triangle formed by edges a_p, a_q, and a_r

Proof. As we have seen in (3.4), P_i is expressed as

$$P_i = \{x \in \mathbb{R}^2 \mid a_i^T R(-\tfrac{\pi}{2}) x \geq a_i^T R(-\tfrac{\pi}{2}) v_i\} \quad (i \in I). \tag{3.8}$$

Choose any $j \in I$ and a point $u \in \mathbb{R}^2$, and put

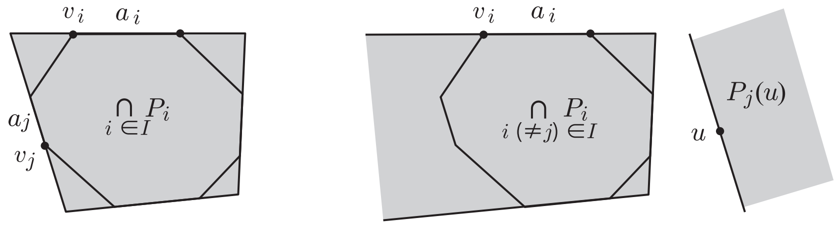

Figure 7: The reason for (3.9)

$$P_j(u) := \{x \in \mathbb{R}^2 \mid a_j^T R(-\tfrac{\pi}{2}) x \leq a_j^T R(-\tfrac{\pi}{2}) u\}.$$

Since $\cap_{i\in I} P_i$ is bounded, we can take a point $u \in \mathbb{R}^2$ such that

$$P_j(u) \cap \left(\bigcap_{i(\neq j)\in I} P_i \right) = \emptyset, \tag{3.9}$$

see Figure 7. Namely, the following linear system has no solution $x \in \mathbb{R}^2$.

$$a_i^T R(-\tfrac{\pi}{2})x \leq a_i^T R(-\tfrac{\pi}{2})v_i \quad (i(\neq j) \in I), \quad a_j^T R(-\tfrac{\pi}{2})x \leq a_j^T R(-\tfrac{\pi}{2})u. \tag{3.10}$$

By Gale's theorem, there exists $z_i \geq 0$ $(i \in I)$ such that

$$\sum_{i\in I} z_i a_i^T R(-\tfrac{\pi}{2}) = 0, \qquad \sum_{i(\neq j)\in I} z_i a_i^T R(-\tfrac{\pi}{2})v_i + z_j a_i^T R(-\tfrac{\pi}{2})u > 0. \tag{3.11}$$

Here we remark that $z_i > 0$ for some $i \in I$. Define $y \in \mathbb{R}^n$ by

$$y_i := \begin{cases} z_i & (i \in I) \\ 0 & (i \notin I). \end{cases}$$

Then y satisfies the first two conditions of (3.5). Since the hole is non-empty, y does not satisfy the last condition of (3.5), so that,

$$0 \leq \sum_{i=1}^{n} y_i a_i^T R(\theta - \tfrac{\pi}{2})v_i = \sum_{i\in I} z_i a_i^T R(\theta - \tfrac{\pi}{2})v_i. \tag{3.12}$$

Suppose that there exists some $x \in \bigcap_{i\in I} H_i^c$. Then we see from (3.2) that

$$a_i^T R(\theta - \tfrac{\pi}{2})(x - v_i) < 0 \quad (i \in I).$$

Hence

$$\sum_{i\in I} z_i a_i^T R(\theta - \tfrac{\pi}{2})x < \sum_{i\in I} z_i a_i^T R(\theta - \tfrac{\pi}{2})v_i \leq 0.$$

However, since $\sum_{i\in I} z_i a_i = 0$ by (3.11), we have $\sum_{i\in I} z_i a_i^T R(\theta - \frac{\pi}{2})x = 0$. Therefore $\bigcap_{i\in I} H_i^c$ is empty. Further, since $\overset{\circ}{\tau}{}_i' \subset H_i^c$ for any i, $\bigcap_{i\in I} \overset{\circ}{\tau}{}_i'$ is also empty. □

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