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A CHARACTERIZATION OF WEIGHTED POPULAR MATCHINGS UNDER MATROID CONSTRAINTS

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Abstract The popular matching problem introduced by Abraham, Irving, Kavitha, and Mehlhorn is one of bipartite matching problems with one-sided preference lists. In this paper, we first propose a matroid generalization of the weighted variant of popular matchings introduced by Mestre. Then we give a characterization of weighted popular matchings in bipartite graphs with matroid constraints and one-sided preference lists containing no ties. This characterization is based on the characterization of weighted popular matchings proved by Mestre. Lastly we prove that we can decide whether a given matching is a weighted popular matching under matroid constraints in polynomial time by using our characterization.

Keywords: Discrete optimization, popular matching, matroid

1. Introduction

The popular matching problem introduced by Abraham, Irving, Kavitha, and Mehlhorn [1] is one of assignment problems in bipartite graphs with one-sided preference lists. Roughly speaking, a matching M is said to be popular, if there exists no other matching N such that the number of agents that prefer N to M is larger than the number of agents that prefer M to N. The concept of popularity was originally proposed by Gärdenfors [5] in the context of matching problems in bipartite graphs with two-sided preference lists. Abraham, Irving, Kavitha, and Mehlhorn [1] presented polynomial-time algorithms for the problem of deciding whether there exists a popular matching, and finding a popular matching if one exists. Since the seminal paper by Abraham, Irving, Kavitha, and Mehlhorn [1], several variants of the popular matching problem [8, 9, 15, 17, 21] and related problems [8, 10–13, 16, 22, 23] have been extensively investigated. For example, Manlove and Sng [15] proposed polynomial-time algorithms for a many-to-one variant of the popular matching problem. See, e.g, [6] for the application of the popular matching problem to real-world problems.

In this paper, we focus on the weighted variant of the popular matching problem introduced by Mestre [17]. In the (ordinary) popular matching problem, we consider the number of agents that prefer some matching to another matching. In other words, the opinion of every agent is valued equally. On the other hand, in this weighted variant, the opinions of agents are not valued equally. More precisely, in this setting, each agent has a weight. A matching M is said to be popular, if there exists no other matching N such that the sum of the weights of agents that prefer N to M is larger than the sum of the weights of agents that prefer M to N. Mestre [17] gave polynomial-time algorithms for the problem of deciding whether there exists a weighted popular matching, and finding a weighted popular matching if one exists. Furthermore, Sng and Manlove [21] proved that if the preference lists do not contain ties, then this problem in a many-to-one setting can be solved in polynomial time.

In this paper, we first propose a matroid generalization of the weighted variant of popular matchings introduced by Mestre [17]. Then we give a characterization of weighted popular matchings in bipartite graphs with matroid constraints and one-sided preference lists containing no ties. This characterization is based on the characterization of weighted popular matchings proved by Mestre [17]. Lastly we prove that we can decide whether a given matching is a weighted popular matching under matroid constraints in polynomial time by using our characterization.

Matroid approaches to matching problems have been extensively studied (see, e.g., [2–4, 7–9, 19, 24]). For example, Kamiyama [8, 9] proposed polynomial-time algorithms for a matroid generalization of the (ordinary) popular matching problem. A matroid constraint is a generalization of several capacity constraints (see, e.g., [8]).

2. Preliminaries

For each set X and each element u, we define $X + u := X \cup \{u\}$ and $X - u := X \setminus \{u\}$. For each positive integer n, we define $[n] := \{1, 2, \ldots, n\}$. Define $[0] := \emptyset$.

A pair $\mathbf{M} = (U, \mathcal{I})$ of a finite set U and a family $\mathcal{I}$ of subsets of U is called a *matroid*, if it satisfies the following conditions.

(I0) $\emptyset \in \mathcal{I}$.
(I1) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
(I2) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I + u \in \mathcal{I}$ for some element u in $J \setminus I$.

A subset of U belonging to $\mathcal{I}$ is called an *independent set* of $\mathbf{M}$.

2.1. Problem formulation

Throughout this paper, a finite simple (not necessarily complete) bipartite graph $G = (V, E)$ is given. We assume that V is partitioned into subsets A and P, and every edge in E connects a vertex in A and a vertex in P. A vertex in A (resp., P) is called an *applicant* (resp., a *post*). If there exists an edge in E connecting an applicant a in A and a post p in P, then we denote by (a, p) this edge. For each vertex v in V and each subset F of E, we denote by $F(v)$ the set of edges in F incident to v. For each applicant a in A, we are given a strict total order $\succ_a$ on $E(a)$ that represents the preference list of a. For each applicant a in A and each pair of edges e, g in $E(a)$, if $e \succ_a g$, then this means that a prefers e to g. For each applicant a in A and each pair of edges e, g in $E(a)$, we write $e \succeq_a g$, if $e \succ_a g$ or $e = g$. Furthermore, for each applicant a in A, we are given a positive integer $\omega(a)$ that represents of the priority of a. For each post p in P, we are given a matroid $\mathbf{M}_p = (E(p), \mathcal{I}_p)$. We assume that for every edge (a, p) in E, $\{(a, p)\}$ is an independent set of $\mathbf{M}_p$ (see [8] for concrete examples of matroid constraints). As in [1], we assume that for each applicant a in A, there exist *last resort posts* $\ell_1(a), \ell_2(a)$ in P satisfying the following conditions. (Although only one last resort post exists in [17], we prepare two last resort posts for simplicity.)

- $E(\ell_i(a)) = \{(a, \ell_i(a))\}$ for every integer i in $\{1, 2\}$.
- $e \succ_a (a, \ell_1(a))$ and $e \succ_a (a, \ell_2(a))$ for every edge e in $E(a) \setminus \{(a, \ell_1(a)), (a, \ell_2(a))\}$.
- $(a, \ell_1(a)) \succ_a (a, \ell_2(a))$.

A subset M of E is called a *matching* in G, if it satisfies the following conditions.

(M1) For every applicant a in A, $|M(a)| = 1$.
(M2) For every post p in P, $M(p)$ is an independent set of $\mathbf{M}_p$.

For each subset F of E and each applicant a in A such that $|F(a)| = 1$, we denote by $\mu_F(a)$ the unique edge in $F(a)$. For each pair of matchings M, N in G, let $\Phi(M, N)$ be the set of applicants a in A such that $\mu_M(a) \succ_a \mu_N(a)$. Furthermore, for each pair of matchings M, N

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in G, we define $\phi(M,N)$ by

$$\phi(M,N) := \sum_{a\in\Phi(M,N)} \omega(a).$$

A matching M in G is said to be *popular*, if $\phi(M,N) \geq \phi(N,M)$ for every matching N in G. It is known [1] that there exists an instance such that $\omega(a) = 1$ for every applicant a in A and it has no popular matchings.

2.2. Basics of matroids

Assume that we are given a matroid $\mathbf{M} = (U,\mathcal{I})$. A subset C of U is called a *circuit* of $\mathbf{M}$, if C is not an independent set of $\mathbf{M}$, but every proper subset of C is an independent set of $\mathbf{M}$. The following property of circuits is known.

Theorem 2.1 (See, e.g., [20, Lemma 1.1.3])**.** *Assume that we are given a matroid* $\mathbf{M} = (U,\mathcal{I})$ *and distinct circuits* C_1, C_2 *of* $\mathbf{M}$ *such that* $C_1 \cap C_2 \neq \emptyset$*. Then for every element* u *in* $C_1 \cap C_2$*, there exists a circuit* C *of* $\mathbf{M}$ *such that* $C \subseteq (C_1 \cup C_2) - u$*.*

Assume that we are given an independent set I of $\mathbf{M}$ and an element u in $U \setminus I$ such that $I + u$ is not an independent set of $\mathbf{M}$. It is not difficult to see that $I + u$ contains a circuit of $\mathbf{M}$ as a subset, and u belongs to this circuit. Furthermore, Theorem 2.1 implies that such a circuit is uniquely determined. This circuit is called the *fundamental circuit of* u *with respect to* I *and* $\mathbf{M}$. It is known [20, p.20, Exercise 5] that the fundamental circuit of u with respect to I and $\mathbf{M}$ is the set of elements w in $I + u$ such that $I + u - w \in \mathcal{I}$.

A maximal independent set of $\mathbf{M}$ is called a *base* of $\mathbf{M}$. The condition (I2) implies that all bases of $\mathbf{M}$ have the same size. For each subset X of U, we define

$$\mathcal{I}|X := \{I \subseteq X \mid I \in \mathcal{I}\}, \quad \mathbf{M}|X := (X, \mathcal{I}|X).$$

It is known [20, p.20] that for every subset X of U, $\mathbf{M}|X$ is a matroid. For each subset X of U, let $\mathsf{r}_{\mathbf{M}}(X)$ be the size of a base of $\mathbf{M}|X$. For each subset X of U, we define

$$\mathcal{I}/X := \{I \subseteq U \setminus X \mid \mathsf{r}_{\mathbf{M}}(I \cup X) - \mathsf{r}_{\mathbf{M}}(X) = |I|\}, \quad \mathbf{M}/X := (U \setminus X, \mathcal{I}/X).$$

It is known [20, Proposition 3.1.6] that for every subset X of U, $\mathbf{M}/X$ is a matroid. The following facts are known.

Theorem 2.2 (See, e.g., [20, Proposition 3.1.25])**.** *Assume that we are given a matroid* $\mathbf{M} = (U,\mathcal{I})$*. Then for every pair of disjoint subsets* X, Y *of* U*,* $(\mathbf{M}/X)/Y = \mathbf{M}/(X \cup Y)$ *and* $(\mathbf{M}/X)|Y = (\mathbf{M}|(X \cup Y))/X$*.*

Theorem 2.3 (See, e.g., [20, Proposition 3.1.7])**.** *Assume that we are given a matroid* $\mathbf{M} = (U,\mathcal{I})$*, a subset* X *of* U*, and a base* B *of* $\mathbf{M}|X$*. For every subset* I *of* $U \setminus X$*,* I *is an independent set (resp., a base) of* $\mathbf{M}/X$ *if and only if* $B \cup I$ *is an independent set (resp., a base) of* $\mathbf{M}$*.*

Theorem 2.4 (See, e.g., [18, Lemma 2.3.16])**.** *Assume that we are given a matroid* $\mathbf{M}$ *and bases* B, B' *of* $\mathbf{M}$*. Then there exists a bijective mapping* $\pi\colon B \setminus B' \to B' \setminus B$ *such that for every element* u *in* $B \setminus B'$*,* $B - u + \pi(u)$ *is a base of* $\mathbf{M}$*.*

Theorem 2.5 (See, e.g., [14, Lemma 13.27])**.** *Assume that we are given a matroid* $\mathbf{M} = (U,\mathcal{I})$*, an independent set* I *of* $\mathbf{M}$*, and distinct elements* $u_1, u_2, \ldots, u_\delta$ *in* I *and* $v_1, v_2, \ldots, v_\delta$ *in* $U \setminus I$ *satisfying the following conditions.*

(C1) *For any integer* i *in* $[\delta]$*,* $I + v_i$ *is not an independent set of* $\mathbf{M}$*.*

(C2) *For every integer* i *in* $[\delta]$*,* u_i *is contained in the fundamental circuit of* v_i *with respect to* I *and* $\mathbf{M}$*.*

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(C3) *For any pair of integers i, j in $[\delta]$ such that $i < j$, u_i is not contained in the fundamental circuit of v_j with respect to I and $\mathbf{M}$.*

Then $(I \setminus \{u_1, u_2, \ldots, u_\delta\}) \cup \{v_1, v_2, \ldots, v_\delta\}$ is an independent set of $\mathbf{M}$.

Lemma 2.1. *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, an independent set I of $\mathbf{M}$, and distinct elements $u_1, u_2, \ldots, u_\delta$ in I and $v_1, v_2, \ldots, v_\delta$ in $U \setminus I$ satisfying* (C1), (C2), *and* (C3) *in Theorem 2.5. Furthermore, we are given an element v_0 in $U \setminus I$ such that $I + v_0$ is an independent set of $\mathbf{M}$. Then $(I \setminus \{u_1, u_2, \ldots, u_\delta\}) \cup \{v_0, v_1, v_2, \ldots, v_\delta\}$ is an independent set of $\mathbf{M}$.*

Proof. Notice that $v_0 \neq v_i$ for any integer i in $[\delta]$. Define $J := I + v_0$. Then Theorem 2.1 implies that for every integer i in $[\delta]$, the fundamental circuit of v_i with respect to J and $\mathbf{M}$ is equal to the fundamental circuit of v_i with respect to I and $\mathbf{M}$. Thus, this lemma follows from Theorem 2.5. □

3. Well-Formed Matchings

In this section, we introduce the concept of well-formed matchings that plays an important role in our characterization of popular matchings. Let $\omega_1, \omega_2, \ldots, \omega_k$ be positive integers such that $\omega_1 > \omega_2 > \cdots > \omega_k$ and $\{\omega_1, \omega_2, \ldots, \omega_k\} = \{\omega(a) \mid a \in A\}$. For each integer i in $[k]$, we define A_i as the set of applicants a in A such that $\omega(a) = \omega_i$. Define $A_{k+1} := \emptyset$.

For each applicant a in A, we define the *first edge* $f(a)$ and the *second edge* $s(a)$ in $E(a)$ by using Algorithm 1. Notice that for every applicant a in A, since there exist the last resort posts $\ell_1(a), \ell_2(a)$, the edges $f(a), s(a)$ are well-defined.

Algorithm 1

Set $\mathbf{M}_p^0 := \mathbf{M}_p$ for each post p in P.
for each integer i in $[k]$ **do**
  for each applicant a in A_i **do**
    Set $\Pi_f(a) := \{(a,p) \in E(a) \mid \{(a,p)\}$ is an independent set of $\mathbf{M}_p^{i-1}\}$.
    Set $f(a)$ to be the edge e in $\Pi_f(a)$ such that $e \succeq_a g$ for every edge g in $\Pi_f(a)$.
  end for
  Set $F_p^i := \{f(a) \mid a \in A_i, f(a) \in E(p)\}$ for each post p in P.
  Set $\mathbf{M}_p^i := \mathbf{M}_p^{i-1}/F_p^i$ for each post p in P.
  for each applicant a in A_i **do**
    Set $\Pi_s(a) := \{(a,p) \in E(a) - f(a) \mid \{(a,p)\}$ is an independent set of $\mathbf{M}_p^i\}$.
    Set $s(a)$ to be the edge e in $\Pi_s(a)$ such that $e \succeq_a g$ for every edge g in $\Pi_s(a)$.
  end for
end for

Here we give an intuitive explanation of $f(a), s(a)$. Assume that a is an applicant in A_i. Then $f(a)$ is the most preferable edge in $E(a)$ that can be added to a matching even if we add as many first edges as possible for applicants in $A_1 \cup A_2 \cup \cdots \cup A_{i-1}$. Similarly, $s(a)$ is the most preferable edge in $E(a) - f(a)$ that can be added to a matching even if we add as many first edges as possible for applicants in $A_1 \cup A_2 \cup \cdots \cup A_i$.

For each post p in P and each integer i in $[k]$, we define $\Sigma_p^i := F_p^1 \cup F_p^2 \cup \cdots \cup F_p^i$. For each post p in P, we define $\Sigma_p^0 := \emptyset$. A matching M in G is said to be *well formed*, if the following conditions are satisfied.

(F1) For every post p in P and every integer i in $[k]$, $M(p) \cap F_p^i$ is a base of $\mathbf{M}_p^{i-1} | F_p^i$.

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(F2) For every applicant a in A, $\mu_M(a) \in \{f(a), s(a)\}$.

The goal of this section is to prove the following lemma.

Lemma 3.1. *If there exists a popular matching M in G, then M is well formed.*

Proof. This lemma follows from Lemmas 3.7 and 3.9 in the next subsection. □

3.1. Proof of Lemma 3.1

Assume that we are given a post p in P and an integer z in $[k]$. Then an independent set I of $\mathbf{M}_p$ is said to be (p,z)-*tight*, if $I \cap F_p^i$ is a base of $\mathbf{M}_p^{i-1}|F_p^i$ for every integer i in $[z]$.

Lemma 3.2. *Assume that we are given a post p in P, an independent set I of $\mathbf{M}_p$, and an integer z in $[k]$ such that I is (p,z)-tight. Then $I \cap \Sigma_p^z$ is a base of $\mathbf{M}_p|\Sigma_p^z$.*

Proof. Let us fix an integer δ in $[k]$. We prove that this lemma holds in the case of $z = \delta$. If $\delta = 1$, then this lemma follows from $\mathbf{M}_p^0 = \mathbf{M}_p$ and $\Sigma_p^1 = F_p^1$. If $\delta > 1$, then we assume that this lemma holds in the case of $z = \delta - 1$. The induction hypothesis implies that $I \cap \Sigma_p^{\delta-1}$ is a base of $\mathbf{M}_p|\Sigma_p^{\delta-1}$. It is not difficult to see that $\mathbf{M}_p|\Sigma_p^{\delta-1} = (\mathbf{M}_p|\Sigma_p^{\delta})|\Sigma_p^{\delta-1}$. Furthermore, since Theorem 2.2 implies that

$$\mathbf{M}_p^{\delta-1}|F_p^{\delta} = (\mathbf{M}_p/\Sigma_p^{\delta-1})|F_p^{\delta} = (\mathbf{M}_p|(\Sigma_p^{\delta-1} \cup F_p^{\delta}))/\Sigma_p^{\delta-1} = (\mathbf{M}_p|\Sigma_p^{\delta})/\Sigma_p^{\delta-1},$$

$I \cap F_p^{\delta}$ is a base of $(\mathbf{M}_p|\Sigma_p^{\delta})/\Sigma_p^{\delta-1}$. Thus, Theorem 2.3 implies this lemma. □

Lemma 3.3. *Assume that we are given a post p in P, an independent set I of $\mathbf{M}_p$, and an integer z in $[k]$ such that I is (p,z)-tight. Then for every subset J of $E(p) \setminus \Sigma_p^z$, J is an independent set of $\mathbf{M}_p^z$ if and only if $(I \cap \Sigma_p^z) \cup J$ is an independent set of $\mathbf{M}_p$.*

Proof. Lemma 3.2 implies that $I \cap \Sigma_p^z$ is a base of $\mathbf{M}_p|\Sigma_p^z$. Thus, since Theorem 2.2 implies that $\mathbf{M}_p^z = \mathbf{M}_p/\Sigma_p^z$, Theorem 2.3 implies this lemma. □

Lemma 3.4. *Assume that we are given a post p in P, an independent set I of $\mathbf{M}_p$, and an integer z in $[k]$ such that I is (p,z)-tight. Then for every edge e in $F_p^z \setminus I$ and every edge g in the fundamental circuit of e with respect to $I \cap F_p^z$ and $\mathbf{M}_p^{z-1}|F_p^z$, $I + e - g$ is an independent set of $\mathbf{M}_p$.*

Proof. Define $J := I + e - g$. Notice that since Lemma 3.2 implies that $I \cap \Sigma_p^z$ is a base of $\mathbf{M}_p|\Sigma_p^z$, Theorem 2.3 implies that $I \setminus \Sigma_p^z$ is an independent set of $\mathbf{M}_p/\Sigma_p^z$. Since $I \cap F_p^z$ is a base of $\mathbf{M}_p^{z-1}|F_p^z$, (I2) implies that $J \cap F_p^z$ is a base of $\mathbf{M}_p^{z-1}|F_p^z$. Since $J \cap F_p^i = I \cap F_p^i$ for every integer i in $[z-1]$, this and Lemma 3.2 imply that $J \cap \Sigma_p^z$ is a base of $\mathbf{M}_p|\Sigma_p^z$. Thus, since $J = (J \cap \Sigma_p^z) \cup (I \setminus \Sigma_p^z)$, Theorem 2.3 implies that J is an independent set of $\mathbf{M}_p$. □

Lemma 3.5. *Assume that we are given a post p in P, an independent set I of $\mathbf{M}_p$, and an integer z in $[k]$ such that I is (p,z)-tight. Then $(I \cap \Sigma_p^z) + (a,p)$ is not an independent set of $\mathbf{M}_p$ for any edge (a,p) in E such that $a \in A_{z+1}$ and $(a,p) \succ_a f(a)$.*

Proof. Since $a \in A_{z+1}$, the definition of $f(a)$ implies that $\{(a,p)\}$ is not an independent set of $\mathbf{M}_p^z$. Thus, Lemma 3.3 implies this lemma. □

Lemma 3.6. *Assume that we are given a matching M in G and an integer z in $[k]$ such that $M(p)$ is (p,z)-tight for every post p in P. Then $f(a) \succeq_a \mu_M(a)$ for every applicant a in A_{z+1}.*

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Proof. Assume that $\mu_M(a) = (a, p)$. Then since (I1) implies that $(M(p) \cap \Sigma_p^z) + \mu_M(a)$ is an independent set of $\mathbf{M}_p$, Lemma 3.5 implies that $f(a) \succeq_a \mu_M(a)$. □

Lemma 3.7. *Assume that we are given a popular matching M in G. Then for every post p in P and every integer i in $[k]$, $M(p) \cap F_p^i$ is a base of $\mathbf{M}_p^{i-1}|F_p^i$.*

Proof. We prove this lemma by induction on i. Let us fix an integer z in $[k]$. If $z > 1$, then we assume that $M(p')$ is $(p', z-1)$-tight for every post p' in P. If $z = 1$, then we do not need to make any assumption. For proving the case of $i = z$ by contradiction, we assume that there exists a post p in P such that $M(p) \cap F_p^z$ is not a base of $\mathbf{M}_p^{z-1}|F_p^z$. Then (I2) implies that there exists an edge $e = (a, p)$ in $F_p^z \setminus M$ such that $(M(p) \cap F_p^z) + e$ is an independent set of $\mathbf{M}_p^{z-1}|F_p^z$. Define $M_1 := M + e$. Then we divide the proof into the following **Case 1** and **Case 2**.

Case 1: $M(p) + e$ is an independent set of $\mathbf{M}_p$. Define $M' := M_1 - \mu_M(a)$. Then (I1) implies that M' is a matching in G. Furthermore, since Lemma 3.6 implies that $e = f(a) \succ_a \mu_M(a)$, $\phi(M, M') - \phi(M', M) = -\omega(a) < 0$. This contradicts the fact that M is a popular matching in G. This completes the proof.

Case 2: $M(p) + e$ is not an independent set of $\mathbf{M}_p$. Let C_1 be the fundamental circuit of e with respect to $M(p)$ and $\mathbf{M}_p$. Since Lemma 3.3 implies that $(M(p) \cap \Sigma_p^z) + e$ is an independent set of $\mathbf{M}_p$, $C_1 \not\subseteq (M(p) \cap \Sigma_p^z) + e$. Let $g = (b, p)$ be an edge in $C_1 \setminus (\Sigma_p^z + e)$. We divide the rest of the proof of this case into the following **Case 2A** and **Case 2B**.

Case 2A: $b \in A_\delta$ for some integer δ in $[k] \setminus [z]$. Define $M^\dagger := (M_1 \setminus \{g, \mu_M(a)\}) + (b, \ell_1(b))$. Then (I1) implies that $M^\dagger$ is a matching in G. Furthermore,

$$\phi(M, M^\dagger) - \phi(M^\dagger, M) = -\omega(a) + \omega(b) = -\omega_z + \omega_\delta < 0.$$

This contradicts the fact that M is a popular matching in G. This completes the proof.

Case 2B: $b \in A_\delta$ for some integer δ in $[z]$. Since $g = \mu_M(b)$, Lemma 3.6 implies that $f(b) \succeq_b g$. Thus, since $g \notin \Sigma_p^z$ (i.e., $g \neq f(b)$), $f(b) \succ_b g$ holds. Define $M_2 := M_1 - g + f(b)$. Assume that $f(b) = (b, q)$. We divide the rest of the proof of this case into the following **Case 2B-I** and **Case 2B-II**.

Case 2B-I: $\delta \leq z - 1$. Since the induction hypothesis implies that $M(q) \cap F_q^\delta$ is a base of $\mathbf{M}_q^{\delta-1}|F_q^\delta$, $(M(q) \cap F_q^\delta) + f(b)$ is not an independent set of $\mathbf{M}_q^{\delta-1}|F_q^\delta$. We denote by C_2 the fundamental circuit of $f(b)$ with respect to $M(q) \cap F_q^\delta$ and $\mathbf{M}_q^{\delta-1}|F_q^\delta$. Since $\{f(b)\}$ is an independent set of $\mathbf{M}_q^{\delta-1}$, there exists an edge $h = (c, q)$ in $C_2 - f(b)$. Since $c \in A_\delta$ and $a \notin A_\delta$, we have $c \neq a$. Define $M^\bullet := M_2 - h + (c, \ell_1(c))$. Then Lemma 3.4 implies that $M^\bullet(q)$ is an independent set of $\mathbf{M}_q$. Define $M^\star := M^\bullet - \mu_M(a)$. Then (I1) implies that $M^\star$ is a matching in G. Furthermore,

$$\phi(M, M^\star) - \phi(M^\star, M) = -\omega(a) - \omega(b) + \omega(c) = -\omega_z - \omega_\delta + \omega_\delta < 0.$$

This contradicts the fact that M is a popular matching in G. This completes the proof.

Case 2B-II: $\delta = z$. We divide the rest of the proof into the following **Case 2B-II-a** and **Case 2B-II-b**.

Case 2B-II-a: $M(q) + f(b)$ is an independent set of $\mathbf{M}_q$. Define $M^\circ := M_2 - \mu_M(a)$. Then (I1) implies that M° is a matching in G. Furthermore,

$$\phi(M, M^\circ) - \phi(M^\circ, M) = -\omega(a) - \omega(b) = -\omega_z - \omega_z < 0.$$

This contradicts the fact that M is a popular matching in G. This completes the proof.

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Case 2B-II-b: $M(q) + f(b)$ is not an independent set of $\mathbf{M}_q$. Let C_3 be the fundamental circuit of $f(b)$ with respect to $M(q)$ and $\mathbf{M}_q$. Then since $\{f(b)\}$ is an independent set of $\mathbf{M}_q^{z-1}$, Lemma 3.3 implies that $(M(q) \cap \Sigma_q^{z-1}) + f(b)$ is an independent set of $\mathbf{M}_q$. Thus, $C_3 \not\subseteq (M(q) \cap \Sigma_q^{z-1}) + f(b)$. Let $h = (c, q)$ be an edge in $C_3 \setminus (\Sigma_q^{z-1} + f(b))$. We divide the rest of the proof into the following **Case 2B-II-b-i** and **Case 2B-II-b-ii**.

Case 2B-II-b-i: $c \in A_\xi$ for some integer ξ in $[k] \setminus [z-1]$. If $c = a$, then $h = \mu_M(a)$. Thus, by defining $M^\circ := M_2 - \mu_M(a)$, we can prove this case in the same way as **Case 2B-II-a**. Assume that $c \neq a$. Define $M^* := (M_2 \setminus \{h, \mu_M(a)\}) + (c, \ell_1(c))$. Then (I1) implies that M^* is a matching in G. Furthermore,

$$\phi(M, M^*) - \phi(M^*, M) = -\omega(a) - \omega(b) + \omega(c) = -\omega_z - \omega_z + \omega_\xi < 0.$$

This contradicts the fact that M is a popular matching in G. This completes the proof.

Case 2B-II-b-ii: $c \in A_\xi$ for some integer ξ in $[z-1]$. Since $a \in A_z$, we have $c \neq a$. Since $h = \mu_M(c)$, Lemma 3.6 implies that $f(c) \succeq_c h$. Thus, since $h \notin \Sigma_q^{z-1}$ (i.e., $f(c) \neq h$), we have $f(c) \succ_c h$. Define $M_3 := M_2 - h + f(c)$. Assume that $f(c) = (c, r)$. Notice that r may be equal to p. Since $a, b \in A_z$, we have $M_2(r) \cap F_r^\xi = M(r) \cap F_r^\xi$. Thus, the induction hypothesis implies that $M_2(r) \cap F_r^\xi$ is a base of $\mathbf{M}_r^{\xi-1}|F_r^\xi$. Thus, $(M_2(r) \cap F_r^\xi) + f(c)$ is not an independent set of $\mathbf{M}_r^{\xi-1}|F_r^\xi$. Let C_4 be the fundamental circuit of $f(c)$ with respect to $M_2(r) \cap F_r^\xi$ and $\mathbf{M}_r^{\xi-1}|F_r^\xi$. Since $\{f(c)\}$ is an independent set of $\mathbf{M}_r^{\xi-1}$, there exists an edge $o = (d, r)$ in $C_4 - f(c)$. Notice that since $o \in F_r^\xi$, we have $d \in A_\xi$. This implies that $d \neq a, b$. Define $M_4 := M_3 - o + (d, \ell_1(d))$. Since $M_2(r)$ is (r, ξ)-tight, Lemma 3.4 implies that $M_4(r)$ is an independent set of $\mathbf{M}_r$. Define $M_5 := M_4 - \mu_M(a)$. Then (I1) implies that M_5 is a matching in G. Furthermore,

$$\begin{aligned}\phi(M, M_5) - \phi(M_5, M) &= -\omega(a) - \omega(b) - \omega(c) + \omega(d) \\ &= -\omega_z - \omega_z - \omega_\xi + \omega_\xi < 0.\end{aligned}$$

This contradicts the fact that M is a popular matching in G. This completes the proof. □

Lemma 3.8. *Assume that we are given a popular matching M in G. Then there does not exist an applicant a in A such that $f(a) \succ_a \mu_M(a) \succ_a s(a)$.*

Proof. We prove this lemma by contradiction. Assume that there exists an applicant a in A such that $f(a) \succ_a \mu_M(a) \succ_a s(a)$, and $\mu_M(a) = (a, p)$. Let z be the integer in $[k]$ such that $a \in A_z$. Since M is a popular matching in G, Lemma 3.7 implies that $M(p)$ is (p, z)-tight. Since (M2) and (I1) imply that $(M(p) \cap \Sigma_p^z) + \mu_M(a)$ is an independent set of $\mathbf{M}_p$, Lemma 3.3 implies that $\{\mu_M(a)\}$ is an independent set of $\mathbf{M}_p^z$. Thus, since $\mu_M(a) \neq f(a)$, $\mu_M(a) \in \Pi_s(a)$. This contradicts the fact that $\mu_M(a) \succ_a s(a)$. This completes the proof. □

Lemma 3.9. *Assume that we are given a popular matching M in G. For every applicant a in A, $\mu_M(a) \in \{f(a), s(a)\}$.*

Proof. We prove this lemma by contradiction. Assume that there exist an integer z in $[k]$ and an applicant a in A_z such that $\mu_M(a) \notin \{f(a), s(a)\}$. Lemmas 3.6, 3.7, and 3.8 imply that $s(a) \succ_a \mu_M(a)$. Define $M_1 := M + s(a)$. Assume that $s(a) = (a, p)$. Then we divide the proof into the following **Case 1** and **Case 2**.

Case 1: $M(p) + s(a)$ is an independent set of $\mathbf{M}_p$. Define $M' := M_1 - \mu_M(a)$. Then (I1) implies that M' is a matching in G. Furthermore, $\phi(M, M') - \phi(M', M) = -\omega(a) < 0$. This contradicts the fact that M is a popular matching in G. This completes the proof.

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Case 2: $M(p) + s(a)$ is not an independent set of $\mathbf{M}_p$. Let C_1 be the fundamental circuit of $s(a)$ with respect to $M(p)$ and $\mathbf{M}_p$. The definition of $s(a)$ implies that $\{s(a)\}$ is an independent set of $\mathbf{M}_p^z$. Thus, Lemmas 3.3 and 3.7 imply that $(M(p) \cap \Sigma_p^z) + s(a)$ is an independent set of $\mathbf{M}_p$. This implies that $C_1 \not\subseteq (M(p) \cap \Sigma_p^z) + s(a)$. Let $g = (b, p)$ be an edge in $C_1 \setminus (\Sigma_p^z + s(a))$. Then the rest of this proof is the same as Lemma 3.7. □

4. Promotion Paths

Here we give the definition and properties of a promotion path that play an important role in our characterization. Assume that we are given a well-formed matching M in G. Then a sequence $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ of applicants $a_1, a_2, \ldots, a_\delta$ in A (we will prove that $a_1, a_2, \ldots, a_\delta$ are distinct later) and (not necessarily distinct) posts $p_1, p_2, \ldots, p_\delta$ in P is called a *promotion path with respect to M*, if the following conditions are satisfied.

(P1) $\mu_M(a_i) = (a_i, p_i)$ for every integer i in $[\delta]$.

(P2) $(a_i, p_{i+1}) \in E \setminus M$ and $(a_i, p_{i+1}) \succ_{a_i} (a_i, p_i)$ for every integer i in $[\delta - 1]$.

(P3) For every integer i in $[\delta - 1]$, the following conditions are satisfied.

- $M(p_{i+1}) + (a_i, p_{i+1})$ is not an independent set of $\mathbf{M}_{p_{i+1}}$.
- (a_{i+1}, p_{i+1}) belongs to the fundamental circuit of (a_i, p_{i+1}) with respect to $M(p_{i+1})$ and $\mathbf{M}_{p_{i+1}}$.

If there exists an edge (a_δ, p) in $E(a_\delta)$ such that

(P4) $(a_\delta, p) \succ_{a_\delta} (a_\delta, p_\delta)$ and

(P5) $M(p) + (a_\delta, p)$ is an independent set of $\mathbf{M}_p$,

then L is called a *type-1 promotion path with respect to M*. Otherwise, L is called a *type-2 promotion path with respect to M*. Furthermore, we define

$$\mathsf{c}(L) := \begin{cases} -\sum_{i=1}^{\delta} \omega(a_i) & \text{if } L \text{ is a type-1 promotion path with respect to } M \\ \omega(a_\delta) - \sum_{i=1}^{\delta-1} \omega(a_i) & \text{if } L \text{ is a type-2 promotion path with respect to } M. \end{cases}$$

The goal of this section, we prove the following lemma.

Lemma 4.1. *Assume that we are given a well-formed matching M in G. Then M is a popular matching in G if and only if $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M.*

Proof. This lemma follows from Lemmas 4.6 and 4.8 in the subsequent subsections. □

4.1. Properties of promotion paths

In this subsection, we prove several properties of promotion paths.

Lemma 4.2. *Assume that we are given a post p in P, an independent set I of $\mathbf{M}_p$, an integer z in $[k]$, and an edge (a, p) in $E \setminus I$ such that $(I \cap \Sigma_p^z) + (a, p)$ is not an independent set of $\mathbf{M}_p$. Then for every edge (a', p) in the fundamental circuit of (a, p) with respect to I and $\mathbf{M}_p$, if $(a, p) \neq (a', p)$, then we have $(a', p) = f(a')$ and $\omega(a') \geq \omega_z$.*

Proof. Let C be the fundamental circuit of (a, p) with respect to $I \cap \Sigma_p^z$ and $\mathbf{M}_p$. Then Theorem 2.1 implies that the fundamental circuit of (a, p) with respect to I and $\mathbf{M}_p$ is equal to C. Since $(a', p) \in C - (a, p)$, $(a', p) \in \Sigma_p^z$. Thus, $(a', p) = f(a')$ and $\omega(a') \geq \omega_z$. □

Lemma 4.3. *Assume that we are given a well-formed matching M in G and a promotion path $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ with respect to M such that $\delta \geq 2$. Then the following statements hold.*

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- $(a_i, p_i) = f(a_i)$ *for every integer* i *in* $[\delta] \setminus \{1\}$.
- $\omega(a_1) \leq \omega(a_2) < \cdots < \omega(a_\delta)$. *If* $\omega(a_1) = \omega(a_2)$, *then* $\mu_M(a_1) = s(a_1)$.

Proof. We first prove that $(a_2, p_2) = f(a_2)$ and $\omega(a_1) \leq \omega(a_2)$. We denote by z the integer in $[k]$ such that $a_1 \in A_z$. Since M is a well-formed matching in G, (F2) implies that exactly one of the following **Case 1** and **Case 2** holds.

Case 1: $\mu_M(a_1) = f(a_1)$**.** The condition (P2) implies that $(a_1, p_2) \succ_{a_1} f(a_1)$. If $z = 1$, then $\{(a_1, p_2)\}$ is not an independent set of $\mathbf{M}_{p_2}$. However, this contradicts the assumption that $\{(a_1, p_2)\}$ is an independent set of $\mathbf{M}_{p_2}$. Thus, we can assume that $z > 1$. Then (F1) and Lemma 3.5 imply that $(M(p_2) \cap \Sigma_{p_2}^{z-1}) + (a_1, p_2)$ is not an independent set of $\mathbf{M}_{p_2}$. Thus, it follows from (P3) and Lemma 4.2 that $(a_2, p_2) = f(a_2)$ and $\omega(a_2) \geq \omega_{z-1}$. This implies that $\omega(a_1) = \omega_z < \omega(a_2)$.

Case 2: $\mu_M(a_1) = s(a_1)$**.** We divide the proof of this case into the following **Case 2A** and **Case 2B**.

Case 2A: $(a_1, p_2) = f(a_1)$**.** Notice that (F1) and Lemma 3.2 imply that $M(p_2) \cap \Sigma_{p_2}^z$ is a base of $\mathbf{M}_{p_2} | \Sigma_{p_2}^z$. Since $(a_1, p_2) \in \Sigma_{p_2}^z \setminus M$, this implies that $(M(p_2) \cap \Sigma_{p_2}^z) + (a_1, p_2)$ is not an independent set of $\mathbf{M}_{p_2}$. Thus, (P3) and Lemma 4.2 imply that $(a_2, p_2) = f(a_2)$ and $\omega(a_2) \geq \omega_z$. This implies that $\omega(a_1) = \omega_z \leq \omega(a_2)$.

Case 2B: $(a_1, p_2) \neq f(a_1)$**.** If $(a_1, p_2) \succ_{a_1} f(a_1)$, then we can prove this case in the same way as **Case 1**. Assume that $f(a_1) \succeq_{a_1} (a_1, p_2)$. Then (P2) implies that $(a_1, p_2) \succ_{a_1} s(a_1)$. Since $(a_1, p_2) \neq f(a_1)$, $\{(a_1, p_2)\}$ is not an independent set of $\mathbf{M}_{p_2}^z$. Thus, since (F1) and Lemma 3.3 imply that $(M(p_2) \cap \Sigma_{p_2}^z) + (a_1, p_2)$ is not an independent set of $\mathbf{M}_{p_2}$, it follows from (P3) and Lemma 4.2 that $(a_2, p_2) = f(a_2)$ and $\omega(a_2) \geq \omega_z$. This implies that $\omega(a_1) = \omega_z \leq \omega(a_2)$.

We can prove that $\omega(a_i) < \omega(a_{i+1})$ for every integer i in $[\delta - 1] \setminus \{1\}$ in the same way as **Case 1**. This completes the proof. □

Corollary 4.1. *Assume that we are given a well-formed matching* M *in* G *and a promotion path* $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ *with respect to* M. *If* $\delta \geq 2$, *then* $a_1, a_2, \ldots, a_\delta$ *are distinct.*

Proof. Since G is a simple graph, we have $a_1 \neq a_2$. Thus, this corollary immediately follows from Lemma 4.3. This completes the proof. □

Lemma 4.4. *Assume that we are given a well-formed matching* M *in* G *and a promotion path* $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ *with respect to* M. *Furthermore, we assume that* $p_i = p_j$ *for some integers* i, j *in* $[\delta]$ *such that* $i < j$. *Then* (a_i, p_i) *does not belong to the fundamental circuit of* (a_{j-1}, p_j) *with respect to* $M(p_j)$ *and* $\mathbf{M}_{p_j}$.

Proof. Since (P2) implies that $(a_i, p_{i+1}) \neq (a_i, p_i)$ (i.e., $p_{i+1} \neq p_i$), $j - i \geq 2$. This implies that $j - 1 \geq 2$. Thus, Lemma 4.3 implies that $(a_{j-1}, p_{j-1}) = f(a_{j-1})$. Let z be the integer in $[\delta]$ such that $a_{j-1} \in A_z$. Since Lemma 4.3 implies that $\omega(a_{j-1}) < \omega(a_j)$, we have $z > 1$. Since (P2) implies that $(a_{j-1}, p_j) \succ_{a_{j-1}} (a_{j-1}, p_{j-1}) = f(a_{j-1})$, (F1) and Lemma 3.5 imply that $(M(p_j) \cap \Sigma_{p_j}^{z-1}) + (a_{j-1}, p_j)$ is not an independent set of $\mathbf{M}_{p_j}$. Let C be the fundamental circuit of (a_{j-1}, p_j) with respect to $M(p_j) \cap \Sigma_{p_j}^{z-1}$ and $\mathbf{M}_{p_j}$. Theorem 2.1 implies that the fundamental circuit of (a_{j-1}, p_j) with respect to $M(p_j)$ and $\mathbf{M}_{p_j}$ is equal to C. Thus, since Lemma 4.3 implies that $\omega(a_i) \leq \omega(a_{j-1}) = \omega_z$ (i.e., $(a_i, p_i) \notin \Sigma_{p_j}^{z-1}$), we have $(a_i, p_i) \notin C$. This completes the proof. □

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4.2. Proof of the "only if" part

We first prove the "only if" part of Lemma 4.1, i.e., Lemma 4.6. Assume that we are given a well-formed matching M in G and a promotion path $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ with respect to M. We define $M \oplus L$ as follows. We first consider the case where L is a type-1 promotion with respect to M. We arbitrarily choose an edge (a_δ, p) in $E(a_\delta)$ satisfying (P4) and (P5). Then we define $M \oplus L$ as

$$((M \setminus \{\mu_M(a_i) \mid i \in [\delta]\}) \cup \{(a_i, p_{i+1}) \mid i \in [\delta - 1]\}) + (a_\delta, p).$$

If L is a type-2 promotion with respect to M, then we define $M \oplus L$ as

$$((M \setminus \{\mu_M(a_i) \mid i \in [\delta]\}) \cup \{(a_i, p_{i+1}) \mid i \in [\delta - 1]\}) + (a_\delta, \ell_1(a_\delta)).$$

Lemma 4.5. *Assume that we are given a well-formed matching M in G and a promotion path L with respect to M. Then $M \oplus L$ is a matching in G.*

Proof. Define $N := M \oplus L$. Then Corollary 4.1 implies that $|N(a)| = 1$ for every applicant a in A. It follows from (I1), Theorem 2.5, and Lemmas 2.1 and 4.4 that $N(p) \in \mathcal{I}_p$ for every post p in P. This completes the proof. □

Lemma 4.6. *Assume that we are given a well-formed matching M in G. If M is a popular matching in G, then $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M.*

Proof. Assume that M is a popular matching in G and there exists a promotion path L with respect to M such that $\mathsf{c}(L) < 0$. Lemma 4.5 implies that $M \oplus L$ is a matching in G. Furthermore, Corollary 4.1 implies that $\phi(M, M \oplus L) - \phi(M \oplus L, M) = \mathsf{c}(L) < 0$. This contradicts the fact that M is a popular matching in G. □

4.3. Proof of the "if" part

In this subsection, we prove the "if" part of Lemma 4.1, i.e., Lemma 4.8. The following lemma plays an important role in the proof of Lemma 4.8.

Lemma 4.7. *Assume that we are given a well-formed matching M in G and a matching N in G such that $\Phi(N, M) \neq \emptyset$. Furthermore, we assume that there does not exist a type-1 promotion path with respect to M. Then there exists a set of promotion paths*

$$\begin{aligned} L_1 &= (p_{1,1}, a_{1,1}, p_{1,2}, a_{1,2}, \ldots, p_{1,\delta_1}, a_{1,\delta_1}), \\ L_2 &= (p_{2,1}, a_{2,1}, p_{2,2}, a_{2,2}, \ldots, p_{2,\delta_2}, a_{2,\delta_2}), \\ &\vdots \\ L_d &= (p_{d,1}, a_{d,1}, p_{d,2}, a_{d,2}, \ldots, p_{d,\delta_d}, a_{d,\delta_d}) \end{aligned}$$

with respect to M satisfying the following conditions.

1. *$a_{i,j} \in \Phi(N, M)$ for every pair of integers i in $[d]$ and j in $[\delta_i - 1]$.*
2. *$a_{i,\delta_i} \in \Phi(M, N)$ for every integer i in $[d]$.*
3. *For every applicant a in $\Phi(N, M)$, there exists a pair of integers i in $[d]$ and j in $[\delta_i - 1]$ such that $a = a_{i,j}$.*
4. *$a_{i,\delta_i} \neq a_{j,\delta_j}$ for any pair of distinct integers i, j in $[d]$.*

Before proving Lemma 4.7, we prove the main result of this subsection by Lemma 4.7.

Lemma 4.8. *Assume that we are given a well-formed matching M in G. If $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M, then M is a popular matching in G.*

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Proof. Let N be a matching in G such that $\Phi(N, M) \neq \emptyset$. Since $\mathsf{c}(L) < 0$ for every type-1 promotion path L with respect to M, there does not exist a type-1 promotion path L with respect to M. Thus, there exist promotion paths $L_1, L_2, \ldots, L_d$ satisfying the conditions in Lemma 4.7. Since $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M, we have

$$\phi(M, N) - \phi(N, M) \geq \mathsf{c}(L_1) + \mathsf{c}(L_2) + \cdots + \mathsf{c}(L_d) \geq 0.$$

This implies that M is a popular matching in G. □

4.3.1. Proof of Lemma 4.7

Here we prove Lemma 4.7. Assume that we are given a well-formed matching M in G and a matching N in G such that $\Phi(N, M) \neq \emptyset$. Furthermore, we assume that there does not exist a type-1 promotion path with respect to M. For each post p in P, let D_p be the set of edges e in $N(p) \setminus M(p)$ such that $M(p) + e$ is not an independent set of $\mathbf{M}_p$.

Lemma 4.9. *For every post p in P, there exists an injective mapping $\sigma_p \colon \mathsf{D}_p \to M(p) \setminus N(p)$ such that $M(p) - \sigma_p(e) + e$ is an independent set of $\mathbf{M}_p$ for every edge e in D_p.*

Proof. Let M' (resp., N') be an arbitrary base of $\mathbf{M}_p|(M(p) \cup N(p))$ such that $M(p) \subseteq M'$ (resp., $N(p) \subseteq N'$). Notice that (I2) guarantees the existence of M', N'. Then Theorem 2.4 implies that there exists a bijective mapping $\pi \colon M' \setminus N' \to N' \setminus M'$ such that for every edge e in $M' \setminus N'$, $M' - e + \pi(e)$ is an independent set of $\mathbf{M}_p$.

Define the mapping $\sigma_p \colon \mathsf{D}_p \to M(p) \setminus N(p)$ as follows. It is not difficult to see that (I1) implies that for every edge e in D_p, e does not belong to M', which implies that e belongs to $N' \setminus M'$. For each edge e in D_p, we define $\sigma_p(e) := g$, where g is the unique edge in $M' \setminus N'$ such that $\pi(g) = e$.

We prove that σ_p satisfies the condition in this lemma. Let us fix an edge e in D_p. Since $M(p)$ and M' are independent sets of $\mathbf{M}_p$, Theorem 2.1 implies that the fundamental circuit of e with respect to $M(p)$ and $\mathbf{M}_p$ and the fundamental circuit of e with respect to M' and $\mathbf{M}_p$ are the same. Thus, $M(p) - \sigma_p(e) + e$ is an independent set of $\mathbf{M}_p$. □

For each post p in P, let σ_p be an injective mapping from D_p to $M(p) \setminus N(p)$ such that $M(p) - \sigma_p(e) + e$ is an independent set of $\mathbf{M}_p$ for every edge e in D_p. Then we consider the following Algorithm 2. The input of Algorithm 2 is some applicant a in $\Phi(N, M)$.

Algorithm 2

1: Set $i := 1$ and $a_1 := a$. Set p_1 to be the post q in P such that $\mu_M(a_1) = (a_1, q)$.
2: **while** $a_i \in \Phi(N, M)$ **do**
3: Set p_{i+1} to be the post q in P such that $\mu_N(a_i) = (a_i, q)$.
4: Set a_{i+1} to be the applicant b in A such that $\sigma_{p_{i+1}}((a_i, p_{i+1})) = (b, p_{i+1})$.
5: Set $i := i + 1$.
6: **end while**
7: Output $(p_1, a_1, p_2, a_2, \ldots, p_i, a_i)$, and halt.

For proving that Algorithm 2 is well-defined, it suffices to prove that $(a_i, p_{i+1}) \in \mathsf{D}_{p_{i+1}}$ in Step 4. It is not difficult to see that during Algorithm 2, the definition of the functions σ_p implies that $(p_1, a_1, p_2, a_2, \ldots, p_i, a_i)$ is a promotion path with respect to M. Thus, if $(a_i, p_{i+1}) \notin \mathsf{D}_{p_{i+1}}$, then since $a_i \in \Phi(N, M)$, $(p_1, a_1, p_2, a_2, \ldots, p_i, a_i)$ is a type-1 promotion path with respect to M. This contradicts the assumption that there does not exist a type-1

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promotion path with respect to M. Furthermore, Corollary 4.1 implies that the number of iterations of Steps 2 to 6 is at most $|A|$.

Let $\mathsf{L}(a)$ be the promotion path with respect to M obtained by applying Algorithm 2 for an input applicant a in $\Phi(N, M)$. For each applicant a in $\Phi(N, M)$, let $\mathsf{AL}(a)$ be the set of applicants contained in $\mathsf{L}(a)$. Precisely speaking, for each applicant a in $\Phi(N, M)$, if $\mathsf{L}(a) = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$, then we define $\mathsf{AL}(a) := \{a_1, a_2, \ldots, a_\delta\}$. For each subset T of $\Phi(N, M)$, let $\mathsf{min}(T)$ be the set of applicants a in T such that $\omega(a) = \min\{\omega(a') \mid a' \in T\}$.

Lemma 4.10. *Assume that we are given an applicant a in $\Phi(N, M)$. In addition, we assume that $\mathsf{L}(a) = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$. Then $a_\delta \in \Phi(M, N)$.*

Proof. The definition of Algorithm 2 implies that $a_\delta \notin \Phi(N, M)$ and $(a_\delta, p_\delta) \notin N(p_\delta)$, which implies that $a_\delta \in \Phi(M, N)$. This completes the proof. □

Lemma 4.11. *Assume that we are given a subset T of $\Phi(N, M)$ and an applicant a in T such that $\mu_M(a) = s(a)$. Then for every applicant b in $T \setminus \mathsf{AL}(a)$, $\mathsf{AL}(a) \cap \mathsf{AL}(b) = \emptyset$.*

Proof. Let b be an applicant in $T \setminus \mathsf{AL}(a)$. Assume that $\mathsf{L}(a) = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ and $\mathsf{L}(b) = (q_1, b_1, q_2, b_2, \ldots, q_{\delta'}, b_{\delta'})$. In addition, we assume that there exist integers i in $[\delta]$ and j in $[\delta']$ such that $a_i = b_j$. Since $b \notin \mathsf{AL}(a)$, $j \geq 2$. If $i = 1$, then Lemma 4.3 implies that $\mu_M(a) = f(a)$. This contradicts the fact that $\mu_M(a) = s(a)$. Assume that $i \geq 2$. Then the definition of Algorithm 2 implies that $p_i = q_j$ and $a_{i-1} = b_{j-1}$. By repeating this, we can see that there exists an integer j' in $[j] \setminus \{1\}$ such that $a = b_{j'}$. This and Lemma 4.3 imply that $\mu_M(a) = f(a)$. This contradicts the fact that $\mu_M(a) = s(a)$. This completes the proof. □

Lemma 4.12. *Assume that we are given a subset T of $\Phi(N, M)$ such that $\mu_M(a) = f(a)$ for every applicant a in T. Then for every applicant a in $\mathsf{min}(T)$ and every applicant b in $T \setminus \mathsf{AL}(a)$, $\mathsf{AL}(a) \cap \mathsf{AL}(b) = \emptyset$.*

Proof. Let a (resp., b) be an applicant in $\mathsf{min}(T)$ (resp., $T \setminus \mathsf{AL}(a)$). Assume that $\mathsf{L}(a) = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ and $\mathsf{L}(b) = (q_1, b_1, q_2, b_2, \ldots, q_{\delta'}, b_{\delta'})$. In addition, we assume that there exist integers i in $[\delta]$ and j in $[\delta']$ such that $a_i = b_j$. Since $b \notin \mathsf{AL}(a)$, we can prove that there exists an integer j' in $[j] \setminus \{1\}$ such that $a = b_{j'}$ in the same way as the proof of Lemma 4.11. Thus, Lemma 4.3 implies that $\omega(b) < \omega(a)$. This contradicts the fact that $a \in \mathsf{min}(T)$. This completes the proof. □

By using Algorithm 2, we give an algorithm for finding promotion paths satisfying the conditions in this lemma. Our algorithm is described in Algorithm 3.

Algorithm 3

```
Set i := 1 and T := Φ(N, M).
while T ≠ ∅ do
    if there exists an applicant a in T such that μ_M(a) = s(a) then
        Set a_i := a.
    else
        Set a_i to be an arbitrary applicant in min(T).
    end if
    Set L_i := L(a_i), T := T \ AL(a_i), and i := i + 1.
end while
Set d := i − 1. Then output L_1, L_2, ..., L_d, and halt.
```

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What remains is to prove that $L_1, L_2, \ldots, L_d$ satisfy the conditions in this lemma. The definition of Algorithm 3 implies the conditions 1 and 3. Furthermore, Lemma 4.10 implies the condition 2. Lastly Lemmas 4.11 and 4.12 imply the condition 4. This completes the proof of Lemma 4.7.

5. Characterization

We are now ready to give the main result of this paper.

Theorem 5.1. *Assume that we are given a matching M in G. Then M is a popular matching in G if and only if M is well formed and $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M.*

Proof. This theorem immediately follows from Lemmas 3.1 and 4.1. □

In the rest of this section, we prove that we can solve the following problem by using Theorem 5.1.

Problem 1

Input: A matching M in G.

Task: Decide whether M is a popular matching in G.

In what follows, we assume that we can check in time bounded by a polynomial in the size of G whether I is an independent set of $\mathbf{M}_p$ for every post p in P and every subset I of $E(p)$ (i.e., we assume the oracle model). Furthermore, we assume that we can check in $O(1)$ time whether $e \succ_a g$ for every applicant a in A and every pair of edges e, g in $E(a)$.

The framework of our algorithm for **Problem 1** is described as follows.

Step 1: We first check whether a given matching M is well formed.

Step 2: We check whether $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M.

We first consider the time complexity of **Step 1**. For checking whether M is well formed, we have to compute the edges $f(a), s(a)$ for all applicants a in A. For this, we consider the time required to check whether $\{(a,p)\}$ is an independent set of $\mathbf{M}_p^{i-1}$ for every integer i in $[k]$, every applicant a in A_i, and every edge (a,p) in $E(a)$. In the case of $i = 1$, this is equivalent to checking wether $\{(a,p)\}$ is an independent set of $\mathbf{M}_p$. Assume that $i \geq 2$. Since Theorem 2.2 implies that $\mathbf{M}_p^{i-1} = \mathbf{M}_p / \Sigma_p^{i-1}$, we can check in polynomial time whether $\{(a,p)\}$ is an independent set of $\mathbf{M}_p^{i-1}$ by computing a base of $\mathbf{M}_p | \Sigma_p^{i-1}$. Similarly, we can check in polynomial time whether $\{(a,p)\}$ is an independent set of $\mathbf{M}_p^{i}$ for every integer i in $[k]$, every applicant a in A_i, and every edge (a,p) in $E(a) - f(a)$. Furthermore, for every integer i in $[k]$, since Theorem 2.2 implies that $\mathbf{M}_p^{i-1} | F_p^i = (\mathbf{M}_p | \Sigma_p^i)/\Sigma_p^{i-1}$, we can check in polynomial time whether $M(p) \cap F_p^i$ is a base $\mathbf{M}_p^{i-1} | F_p^i$ by computing a base of $\mathbf{M}_p | \Sigma_p^{i-1}$. This completes the proof that **Step 1** can be done in polynomial time.

In what follows, we assume that M is well formed. Then we consider the time complexity of **Step 2**. We define the auxiliary directed graph $D_M = (V_M, A_M)$ as follows. The vertex set V_M is $E \cup \{s, t\}$, where s, t are new vertices. The arc set A_M is defined as follows.

- For each pair of edges $e = (a,p)$ in M and $g = (a,q)$ in $E \setminus M$ such that $g \succ_a e$, there exists an arc from e to g whose cost is equal to $-\omega(a)$.
- For each pair of edges $e = (a,p)$ in M and $g = (b,p)$ in $E \setminus M$ such that $M(p) + g$ is not an independent set of $\mathbf{M}_p$ and $M(p) - e + g$ an independent set of $\mathbf{M}_p$, there exists an arc from g to e whose cost is equal to 0.
- For each edge e in M, there exists an arc from s to e whose cost is equal to 0.

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- For each edge $e = (a, p)$ in M, if $M(p) + g$ is not an independent set of $\mathbf{M}_p$ for any edge $g = (a, q)$ in $E \setminus M$ such that $g \succ_a e$, there exists an arc from e to t whose cost is equal to $\omega(a)$.
- For each edge $g = (a, p)$ in $E \setminus M$ such that $M(p) + g$ is an independent set of $\mathbf{M}_p$, there exists an arc from g to t whose cost is equal to 0.

We first prove that D_M is acyclic.

Lemma 5.1. *There does not exist a directed cycle in D_M.*

Proof. Assume that there exists a directed cycle P in D_M. It is not difficult to see that P goes through only vertices corresponding to edges in E. Assume that P goes through the vertices in D_M corresponding to edges $e_1, g_1, e_2, g_2, \ldots, e_\delta, g_\delta, e_1$ in E in this order, where $e_1, e_2, \ldots, e_\delta$ are edges in M and $g_1, g_2, \ldots, g_\delta$ are edges in $E \setminus M$. Notice that since G is a simple graph, $\delta \geq 2$. Furthermore, for each integer i in $[\delta]$, we assume that $e_i = (a_i, p_i)$. Notice that for every integer i in $[\delta]$, we have $g_i = (a_i, p_{i+1})$, where $p_{\delta+1} = p_1$. Let P_1 (resp., P_2) be the directed path in D_M going through the vertices in D_M corresponding to $e_1, g_1, e_2, g_2, \ldots, e_\delta$ (resp., e_δ, g_δ, e_1) in this order. Then the definition of D_M implies that $L_1 = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ (resp., $L_2 = (p_\delta, a_\delta, p_1, a_1)$) is a promotion path with respect to M. We first consider the case where $\delta = 2$. In this case, the existence of L_1 and Lemma 4.3 imply that (i) $(a_1, p_1) = s(a_1)$, $(a_2, p_2) = f(a_2)$, and $\omega(a_1) \leq \omega(a_2)$, or (ii) $\omega(a_1) < \omega(a_2)$. On the other hand, the existence of L_2 and Lemma 4.3 imply that (i) $(a_1, p_1) = f(a_1)$, $(a_2, p_2) = s(a_2)$, and $\omega(a_1) \geq \omega(a_2)$, or (ii) $\omega(a_1) > \omega(a_2)$. These statements contradict. Next, we consider the case where $\delta > 2$. In this case, the existence of L_1 and Lemma 4.3 imply that $\omega(a_1) < \omega(a_\delta)$. On the other hand, the existence of L_2 and Lemma 4.3 imply that $\omega(a_1) \geq \omega(a_\delta)$. These statements contradict. This completes the proof. □

We define the *cost* of a directed path P in D_M from s to t as the sum of the costs of the arcs that P goes through.

Assume that we are given a directed path P in D_M from s to t, and P goes through the vertices corresponding to edges $e_1, e_2, \ldots, e_\delta$ in M in this order. For each integer i in $[\delta]$, we assume that $e_i = (a_i, p_i)$. The definition of D_M implies that $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ is a promotion path with respect to M, and $\mathsf{c}(L)$ is equal to the cost of P. Conversely, we assume that we are given a promotion path $L = (p_1, a_1, p_2, a_2, \ldots, p_\delta, a_\delta)$ with respect to M. We first consider the case where L is a type-1 promotion path. For each integer i in $[\delta]$, we define $e_i := (a_i, p_i)$ and $g_i := (a_i, p_{i+1})$, where p_{i+1} is a post in P such that $(a_\delta, p_{\delta+1})$ satisfies (P4) and (P5). Then there exists a directed path P in D_M going through the vertices in D_M corresponding to $s, e_1, g_1, e_2, g_2, \ldots, e_\delta, g_\delta, t$ in this order. Furthermore, the cost of P is equal to $\mathsf{c}(L)$. Next, we consider the case where L is a type-2 promotion path. For each integer i in $[\delta]$ (resp., i in $[\delta - 1]$), we define $e_i := (a_i, p_i)$ (resp., $g_i := (a_i, p_{i+1})$). Then there exists a directed path P in D_M going through the vertices in D_M corresponding to $s, e_1, g_1, e_2, g_2, \ldots, e_{\delta-1}, g_{\delta-1}, e_\delta, t$ in this order. Furthermore, the cost of P is equal to $\mathsf{c}(L)$. These observations imply that $\mathsf{c}(L) \geq 0$ for every promotion path L with respect to M if and only if there does not exist a directed path in D_M from s to t whose cost is less than 0.

Since Lemma 5.1 implies that the minimum cost of a directed path from s to t is finite, we can check whether M is a popular matching by computing the minimum cost of a directed path from s to t. Furthermore, Lemma 5.1 implies that this can be done in polynomial time (see, e.g., [14, Chapter 7]). This completes the proof.

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6. Conclusion

In this paper, we consider a characterization of weighted popular matchings under matroid constraints. An apparent next step is to clarify whether we can solve the problem of checking the existence of a weighted popular matching under matroid constraints by using the characterization of this paper.

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