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A STRAIGHTFORWARD APPROACH FOR SOLVING FULLY FUZZY LINEAR PROGRAMMING PROBLEM WITH LR-TYPE FUZZY NUMBERS

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Abstract The fuzzy linear programming problem with triangular fuzzy numbers in its objective functions or constraints has been discussed by many scholars based on using Zadeh's decomposition theorem of fuzzy numbers and transforming it into some crisp linear programming problems. However, the existing methods and the results will be limited when the objective functions (or the constraint functions) of a fuzzy linear programming contain generalized fuzzy numbers. In this paper, we first investigate the approximate representation of the fully fuzzy constraints and the transformation theorem of the fully fuzzy linear programming problem by means of the definition of the extended LR-fuzzy numbers. At the same time, the fully fuzzy linear programming problem is solved by transforming it into a multi-objective linear programming problem under a new ordering of GLR-fuzzy numbers proposed in this paper. Finally, the results obtained are compared with the existing work, and some numerical examples are given.

Keywords: Optimization, LR-fuzzy number, GLR-fuzzy number, fully fuzzy linear programming problem

1. Introduction

Linear programming is an essential mathematical tool in science and technology. Although, it has been investigated and expanded for more than seven decades by many researchers and from the various points of view, and it is still useful to develop new approaches in order to better fit the real world problems within the framework of linear programming. In the conventional approach, the parameters of the linear programming models must be well defined and precise. However, in a real world environment, it is not a realistic assumption. Usually, most of information is not deterministic, and in this situation human has a capability to make a rational decision based on this uncertainty. This is a hard challenge for decision makers to design an intelligent system which can make the same informed decisions as humans. In fact, some of the parameters of the systems may be represented by fuzzy quantity rather than crisp ones in practice, and hence it is necessary to develop the mathematical theory and the numerical schemes to handle the fuzzy linear programming (FLP) problems. So, Bellman and Zadeh [2] proposed the concept of decision making in the fuzzy environments. Since then, a number of researchers have exhibited their interest to various types of the FLP problems and proposed several approaches for solving these problems [4, 7, 9, 15–17, 20, 21, 23]. However, in all of the above mentioned work, those cases of the FLP problems have been studied in which not all parts of the problem were assumed to be fuzzy, e.g., only the right hand side or the objective function coefficients were fuzzy but the variables were not fuzzy. The FLP problems in which all the parameters as well as the variables are represented by fuzzy numbers are known as a fully fuzzy linear programming (FFLP) problems. Many authors [1, 3, 8, 19] have proposed different methods for solving the FFLP problems.

Lotfi [14] proposed a novel method to obtain the approximate solution of the FFLP problems by using the concept of the symmetric triangular fuzzy numbers and introduced an approach to defuzzify a general fuzzy quantity. In Kumar's article [13], an exact optimal solution is achieved using a linear ranking function. In this method, the linear ranking function has been used to convert the fuzzy objective function to a crisp objective function. The shortcoming is that the fuzziness of the objective function has been neglected by the linear ranking function. To the best of our knowledge, till now there is no method in the literature to obtain the exact solution of the FFLP problems in which all the parameters as well as the variables are represented by LR-fuzzy numbers. The LR-fuzzy number and its operations were first introduced by Dubois et al. [6]. We know that triangular fuzzy numbers are just special cases of LR-fuzzy numbers. In 2006, Dehgham et al. [5] discussed a computational method for the fully fuzzy linear systems whose coefficient matrix and the right-hand side vector are denoted by LR fuzzy numbers. In this paper, the approximate representation of the fully fuzzy constraints and the transformation theorem of the FFLP problem (max(min) $\widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}$, s.t.$\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$, $\widetilde{\boldsymbol{x}} \geqslant \boldsymbol{0}$, where $\widetilde{\boldsymbol{C}}^T$, $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{x}}$ are fuzzy number vectors which consist of GLR-fuzzy numbers) are investigated by means of the definition of the extended LR-fuzzy numbers firstly. At the same time, the FFLP problem is solved by transforming it into a MOLP problems under a new ordering of LR-fuzzy numbers. Finally, the results obtained are compared with the existing work, and some numerical examples are given.

The structure of this paper is organized as follows. In Section 2, we review some basic concepts. In Section 3, the approximate representation of the fully fuzzy constraints and the transformation theorem of the FFLP problem are given. In Section 4, the methods for solving the FFLP problem are discussed. In Section 5, some numerical examples are solved and the results obtained are compared. Conclusion is drawn in Section 6.

2. Preliminaries

In this section, some basic definitions and arithmetic operations of LR-fuzzy numbers are presented.

Definition 2.1. A fuzzy number is a fuzzy set like $u : R \to I = [0, 1]$ which satisfies:

(1) u is upper semi-continuous,

(2) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in R$, $\lambda \in [0, 1]$,

(3) u is normal, i.e., there exists $x_0 \in R$ such that $u(x_0) = 1$,

(4) supp$u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp$u$) is compact.

Definition 2.2. A fuzzy number $\widetilde{A}$, defined on universal set of real numbers R, denoted as $\widetilde{A} = (m, \alpha, \beta)_{LR}$, is said to be an LR-fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$u_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & x \leq m, \quad \alpha > 0, \\ R(\frac{x-m}{\beta}), & x \geq m, \quad \beta > 0, \end{cases}$$

where m is the mean value of $\widetilde{A}$, and α and β are left and right spreads, respectively. The function $L(\cdot)$, which is called the left shape function satisfying:

(1) $L(x) = L(-x)$, (2) $L(0) = 1$ and $L(1) = 0$, (3) $L(x)$ is non-increasing on $[0, +\infty)$.

The definition of the right shape function $R(\cdot)$ is usually similar to that of $L(\cdot)$. LR-fuzzy numbers have the following properties [6]:

(1) LR-fuzzy numbers $\widetilde{A_1}=(m_1,\alpha_1,\beta_1)_{LR}$ and $\widetilde{A_2}=(m_2,\alpha_2,\beta_2)_{LR}$ are said to be equal, i.e., $\widetilde{A_1}=\widetilde{A_2}$, if and only if $m_1=m_2,\ \alpha_1=\alpha_2,\ \beta_1=\beta_2$.

(2) An LR fuzzy number $\widetilde{M}=(m,\alpha,\beta)_{LR}$ is said to be a subset of the LR fuzzy number $\widetilde{N}=(n,\gamma,\delta)_{LR}$, if and only if $m-\alpha\geq n-\gamma$ and $m+\beta\leq n+\delta$.

If $L(x)$ and $R(x)$ are linear functions, then the corresponding LR fuzzy number is called a triangular fuzzy number. Note that we use a fixed function $L(\cdot)$ and a fixed function $R(\cdot)$ for all fuzzy numbers in a fixed problem.

Noted that the parameters α and β in an LR fuzzy number defined by Definition 2.2 are restricted to be positive values. In some practical mathematical models, the other situations are often found which restricts the applications of LR fuzzy numbers in some sense. In the following section, we will give a definition of GLR fuzzy numbers and present its arithmetic operations in order to overcome these defects.

Definition 2.3. A fuzzy number $\widetilde{A}=(m,\alpha,\beta)$ is called a GLR-fuzzy number if it has one of the following forms:

(i) If $\alpha<0,\ \beta>0$, defined $\widetilde{A}=(m,0,\max\{-\alpha,\beta\})_{GLR}$, and its membership functions $\mu_{\widetilde{A}}(x)$ is given by

$$u_{\widetilde{A}}(x)=\begin{cases}0, & x<m,\\ R(\frac{x-m}{\max\{-\alpha,\beta\}}), & x\geqslant m.\end{cases}$$

(ii) If $\alpha>0,\ \beta<0$, defined $\widetilde{A}=(m,\max\{\alpha,-\beta\},0)_{GLR}$, and its membership functions $\mu_{\widetilde{A}}(x)$ is given by

$$u_{\widetilde{A}}(x)=\begin{cases}L(\frac{m-x}{\max\{\alpha,-\beta\}}), & x\leqslant m,\\ 0, & x>m.\end{cases}$$

(iii) If $\alpha<0,\ \beta<0$, defined $\widetilde{A}=(m,-\beta,-\alpha)_{GLR}$, and its membership functions $\mu_{\widetilde{A}}(x)$ is given by

$$u_{\widetilde{A}}(x)=\begin{cases}L(\frac{m-x}{-\beta}), & x<m,\\ R(\frac{x-m}{-\alpha,}), & x\geqslant m.\end{cases}$$

(iv) If $\alpha>0,\ \beta>0$, defined $\tilde{A}=(m,\alpha,\beta)_{GLR}$, and its membership functions $\mu_{\widetilde{A}}(x)$ is given by

$$u_{\widetilde{A}}(x)=\begin{cases}L(\frac{m-x}{\alpha}), & x<m,\\ R(\frac{x-m}{\beta}), & x\geqslant m.\end{cases}$$

In the following Definition 2.4, we propose a new definition to compare two arbitrary GLR-fuzzy numbers [10].

Definition 2.4. Let $\widetilde{A_1}=(m_1,\alpha_1,\beta_1)_{GLR}$ and $\widetilde{A_2}=(m_2,\alpha_2,\beta_2)_{GLR}$ be two GLR-fuzzy numbers. We say that $\widetilde{A_1}$ is relatively less than $\widetilde{A_2}$, which is denoted by $\widetilde{A_1}<\widetilde{A_2}$, if and only if:

(1) $m_1<m_2$ or

(2) $m_1=m_2$ and $\alpha_1+\beta_1>\alpha_2+\beta_2$ or

(3) $m_1=m_2,\ \alpha_1+\beta_1=\alpha_2+\beta_2$ and $2m_1+\beta_1-\alpha_1<2m_2+\beta_2-\alpha_2$.

It is clear that $m_1 = m_2,\ \alpha_1 + \beta_1 = \alpha_2 + \beta_2,\ 2m_1 + \beta_1 - \alpha_1 = 2m_2 + \beta_2 - \alpha_2$ if and only if $\widetilde{A_1} = \widetilde{A_2}$.

$\widetilde{A_1} \leqslant \widetilde{A_2}$ if and only if $\widetilde{A_1} < \widetilde{A_2}$ or $\widetilde{A_1} = \widetilde{A_2}$. $\widetilde{X} = (x, y, z)_{GLR}$ is said to be a non-negative fuzzy number if $x \geqslant 0$ and $x - y \geqslant 0$.

For LR-fuzzy numbers $\widetilde{A}_i = (m_i, \alpha_i, \beta_i)_{LR} (i = 1, 2)$, by means of Zadeh's extension principle, we have $(\widetilde{A}_1 + \widetilde{A}_2)_r = [m_1 + m_2 - (\alpha_1 + \alpha_2)L^{-1}(r), m_1 + m_2 + (\beta_1 + \beta_2)R^{-1}(r)]$. It could be equivalently represented as $\widetilde{A}_1 + \widetilde{A}_2 = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$. However,

$$(\lambda \widetilde{A}_1 + \widetilde{A}_2)_r = [m_2 + \lambda m_1 - (\alpha_2 L^{-1}(r) - \lambda \beta_1 R^{-1}(r)), m_2 + \lambda m_1 + (\beta_2 R^{-1}(r) - \lambda \alpha_1 L^{-1}(r))].$$

for $\lambda < 0$. Therefore, $\lambda \widetilde{A}_1 + \widetilde{A}_2$ is an LR-fuzzy number if and only if $L^{-1}(r) = kR^{-1}(r)$ for $\lambda < 0$. That is $L(x) = R(\frac{x}{k})$, and $k > 0$ since $L(x), R(x)$ with the same monotony. In this paper, the arithmetic operations between two GLR-fuzzy numbers are defined as follows which are different from the results from Zadeh's extension principle.

Let $\widetilde{A_1} = (m_1, \alpha_1, \beta_1)_{GLR}$, $\widetilde{A_2} = (m_2, \alpha_2, \beta_2)_{GLR}$ be two GLR-fuzzy numbers. Then,

(1) Addition

$$\widetilde{A_1} \oplus \widetilde{A_2} = (m_1, \alpha_1, \beta_1)_{GLR} \oplus (m_2, \alpha_2, \beta_2)_{GLR} = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{GLR}.$$

(2) Multiplication

If $\widetilde{A_1} > 0$ and $\widetilde{A_2} > 0$, then

$$\widetilde{A_1} \otimes \widetilde{A_2} = (m_1, \alpha_1, \beta_1)_{GLR} \otimes (m_2, \alpha_2, \beta_2)_{GLR} = (m_1 m_2, m_1\alpha_2 + m_2\alpha_1, m_1\beta_2 + m_2\beta_1)_{GLR}.$$

If $\widetilde{A_1} < 0$ and $\widetilde{A_2} > 0$, then

$$\widetilde{A_1} \otimes \widetilde{A_2} = (m_1, \alpha_1, \beta_1)_{GLR} \otimes (m_2, \alpha_2, \beta_2)_{GLR} = (m_1 m_2, m_2\alpha_1 - m_1\beta_2, m_2\beta_1 - m_1\alpha_2)_{GLR}.$$

(3) Scalar multiplication

$$\lambda \otimes \widetilde{A_1} = \lambda \otimes (m_1, \alpha_1, \beta_1)_{GLR} \cong \begin{cases} (\lambda m_1, \lambda\alpha_1, \lambda\beta_1)_{GLR}, & \lambda > 0, \\ (\lambda m_1, -\lambda\beta_1, -\lambda\alpha_1)_{GLR}, & \lambda < 0. \end{cases}$$

Definition 2.5. Matrix $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ is said to be a fuzzy matrix if each element of $\widetilde{\boldsymbol{A}}$ is a GLR-fuzzy number. If for every element $\widetilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij})_{GLR} \geqslant 0$ (or $\widetilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij})_{GLR} \leqslant 0$), then $\widetilde{\boldsymbol{A}}$ is said to be a non-negative (or non-positive) fuzzy matrix, denoted by $\widetilde{\boldsymbol{A}} \geqslant \boldsymbol{0}$ (or $\widetilde{\boldsymbol{A}} \leqslant \boldsymbol{0}$).

We can represent $m \times n$ fuzzy matrix $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})$, that $\widetilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{GLR}$ with new notation $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{M}, \boldsymbol{N})$, where $\boldsymbol{A} = (a_{ij}), \boldsymbol{M} = (\alpha_{ij})$ and $\boldsymbol{N} = (\beta_{ij})$ are three $m \times n$ matrices [11]. For brevity, a crisp matrix consisting of real numbers is written as a matrix directly throughout this paper, similar to linear systems, linear equations, matrix equations, and so on.

Definition 2.6. Let $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$, $\widetilde{\boldsymbol{B}} = (\widetilde{b}_{ij})_{m\times n}$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$. We say that $\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{B}}$ if and only if $\widetilde{a}_{ij} = \widetilde{b}_{ij}$ for every i, j.

Definition 2.7. Let $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$ and $\widetilde{\boldsymbol{B}} = (\widetilde{b}_{ij})_{n\times p}$. Then

$$\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{D}} = (\widetilde{d}_{ij})_{m\times p},$$

where

$$\widetilde{d}_{ij} = \sum_{k=1,\ldots,n}^{\oplus} \widetilde{a}_{ik} \otimes \widetilde{b}_{kj}.$$

3. The Fully Fuzzy Linear Programming Problem

Consider the standard form of the FFLP problem with m constraints and n variables as follows:

$$\begin{aligned} &\max(\min)\, \widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}, \\ &\text{s.t.} \quad \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}, \\ &\qquad\ \widetilde{\boldsymbol{x}} \text{ is non-negative fuzzy number vector,} \end{aligned} \tag{3.1}$$

where $\widetilde{\boldsymbol{A}} = (\widetilde{a}_{ij})_{m\times n}$, $\widetilde{\boldsymbol{C}}^T = (\widetilde{c}_j)_{1\times n}$, $\widetilde{\boldsymbol{x}} = (\widetilde{x}_j)_{n\times 1}$, $\widetilde{\boldsymbol{b}} = (\widetilde{b}_i)_{m\times 1}$, and $\widetilde{a}_{ij} \geqslant 0$ (or $\widetilde{a}_{ij} < 0$), $\widetilde{c}_j \geqslant 0$ (or $\widetilde{c}_j < 0$), $\widetilde{b}_i$, $\widetilde{x}_j$ are GLR-fuzzy numbers.

It should be noted that $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \leqslant \widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \geqslant \widetilde{\boldsymbol{b}}$ can be transformed to the standard form by introducing a vector variable $\widetilde{\boldsymbol{T}} = (\widetilde{t_1}, \widetilde{t_2}, \ldots, \widetilde{t_m})$, where $\widetilde{t_j}$ $(j = 1, 2, \ldots, m)$ are GLR-fuzzy numbers, as $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{T}} = \widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} \ominus \widetilde{\boldsymbol{T}} = \widetilde{\boldsymbol{b}}$, respectively.

Definition 3.1. A fuzzy optimal solution of the FFLP problem (3.1) will be a fuzzy number vector $\widetilde{\boldsymbol{x}}^*$ if it satisfies the following characteristics:

(1) $\widetilde{\boldsymbol{x}}^* = (\widetilde{x}_j{}^*)_{n\times 1} \geqslant \boldsymbol{0}$, where $\widetilde{x}_j{}^*(j = 1, 2, \ldots, n)$ are GLR-fuzzy numbers;

(2) $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}}^* = \widetilde{\boldsymbol{b}}$;

(3) for any $\widetilde{\boldsymbol{x}} \in \widetilde{\boldsymbol{S}} = \{\widetilde{\boldsymbol{x}} | \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}.\ \widetilde{\boldsymbol{x}} = (\widetilde{x}_j)_{n\times 1}$, where $\widetilde{x}_j$ are GLR-fuzzy numbers$\}$, we have $\widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}^* \geqslant \widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}$ (in case of the maximization problem), $\widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}^* \leqslant \widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}$ (in case of the minimization problem).

Let $\widetilde{\boldsymbol{x}}^*$ be an exact optimal solution of the FFLP problem (3.1). If there exists an $\widetilde{\boldsymbol{x}}' \in \widetilde{\boldsymbol{S}}$ such that $\widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}^* = \widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}}'$, then $\widetilde{\boldsymbol{x}}'$ is also an exact optimal solution of the FFLP problem (3.1) and is called an alternative exact optimal solution.

Note that the elements ($\widetilde{a}_{ij}$) in coefficient matrix $\widetilde{\boldsymbol{A}}$ of the FFLP problem (3.1) have two forms, i.e., (1) $\widetilde{a}_{ij} \geqslant 0$, (2) $\widetilde{a}_{ij} < 0$. So we define the coefficient matrix $\widetilde{\boldsymbol{A}}$ as follows:

$$(\widetilde{\boldsymbol{A}}_1)_{ij} = \begin{cases} \widetilde{a}_{ij}, & \widetilde{a}_{ij} \geqslant 0, \\ \widetilde{0}, & \widetilde{a}_{ij} < 0; \end{cases} \qquad (\widetilde{\boldsymbol{A}}_2)_{ij} = \begin{cases} \widetilde{a}_{ij}, & \widetilde{a}_{ij} \leqslant 0, \\ \widetilde{0}, & \widetilde{a}_{ij} > 0, \end{cases}$$

where $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. Obviously,

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2, \quad \widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{A}}_1 \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{A}}_2 \otimes \widetilde{\boldsymbol{x}}.$$

Similarly, we define the coefficient matrix ($\widetilde{\boldsymbol{C}}^T$) of objective function of the FFLP problem (3.1) as follows:

$$(\widetilde{\boldsymbol{C}}_1^T)_j = \begin{cases} \widetilde{c}_j, & \widetilde{c}_j \geqslant 0, \\ \widetilde{0}, & \widetilde{c}_j < 0; \end{cases} \qquad (\widetilde{\boldsymbol{C}}_2^T)_j = \begin{cases} \widetilde{c}_j, & \widetilde{c}_j \leqslant 0, \\ \widetilde{0}, & \widetilde{c}_j > 0, \end{cases}$$

where $1 \leqslant j \leqslant n$. Obviously,

$$\widetilde{\boldsymbol{C}}^T = \widetilde{\boldsymbol{C}}_1^T \oplus \widetilde{\boldsymbol{C}}_2^T, \quad \widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{C}}_1^T \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{C}}_2^T \otimes \widetilde{\boldsymbol{x}}.$$

Theorem 3.1. Let $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{M}, \boldsymbol{N})_{GLR} = \widetilde{\boldsymbol{A}_1} \oplus \widetilde{\boldsymbol{A}_2}$, $\widetilde{\boldsymbol{b}} = (\boldsymbol{b}, \boldsymbol{g}, \boldsymbol{h})_{GLR}$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \geqslant \boldsymbol{0}$, where $\widetilde{\boldsymbol{A}_1} = (\boldsymbol{A}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{A}_2} = (\boldsymbol{A}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{GLR} \leqslant \boldsymbol{0}$. Then $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$ can be represented approximately as follows:

$$\begin{cases} \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ \boldsymbol{A}_1\boldsymbol{y} - \boldsymbol{A}_2\boldsymbol{z} + \boldsymbol{M}\boldsymbol{x} = \boldsymbol{g}, \\ \boldsymbol{A}_1\boldsymbol{z} - \boldsymbol{A}_2\boldsymbol{y} + \boldsymbol{N}\boldsymbol{x} = \boldsymbol{h}. \end{cases} \tag{3.2}$$

Proof. Since $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{M}, \boldsymbol{N})_{GLR} = \widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2$, $\widetilde{\boldsymbol{A}}_1 = (\boldsymbol{A}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{A}}_2 = (\boldsymbol{A}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{GLR} \leqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \geqslant \boldsymbol{0}$, we have

$$\begin{aligned}
\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} &= (\widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2) \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{A}}_1 \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{A}}_2 \otimes \widetilde{\boldsymbol{x}} \\
&= (\boldsymbol{A}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{GLR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \oplus (\boldsymbol{A}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{GLR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \\
&\cong (\boldsymbol{A}_1\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{M}_1\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{z} + \boldsymbol{N}_1\boldsymbol{x})_{GLR} \oplus (\boldsymbol{A}_2\boldsymbol{x}, \boldsymbol{M}_2\boldsymbol{x} - \boldsymbol{A}_2\boldsymbol{z}, \boldsymbol{N}_2\boldsymbol{x} - \boldsymbol{A}_2\boldsymbol{y})_{GLR} \\
&= (\boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{M}_1\boldsymbol{x} + \boldsymbol{M}_2\boldsymbol{x} - \boldsymbol{A}_2\boldsymbol{z}, \boldsymbol{A}_1\boldsymbol{z} + \boldsymbol{N}_1\boldsymbol{x} + \boldsymbol{N}_2\boldsymbol{x} - \boldsymbol{A}_2\boldsymbol{y})_{GLR} \\
&= (\boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{y} - \boldsymbol{A}_2\boldsymbol{z} + \boldsymbol{M}\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{z} - \boldsymbol{A}_2\boldsymbol{y} + \boldsymbol{N}\boldsymbol{x})_{GLR} \\
&= \widetilde{\boldsymbol{b}} = (\boldsymbol{b}, \boldsymbol{g}, \boldsymbol{h})_{GLR}.
\end{aligned}$$

That is

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \ \boldsymbol{A}_1\boldsymbol{y} - \boldsymbol{A}_2\boldsymbol{z} + \boldsymbol{M}\boldsymbol{x} = \boldsymbol{g},\ \ \boldsymbol{A}_1\boldsymbol{z} - \boldsymbol{A}_2\boldsymbol{y} + \boldsymbol{N}\boldsymbol{x} = \boldsymbol{h}.$$

Hence, $\widetilde{\boldsymbol{A}} \otimes \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$ can be represented approximately as (3.2). □

Theorem 3.2 (Transformation theorem of the FFLP problem)**.** Let $\widetilde{\boldsymbol{C}} = (\boldsymbol{c}, \boldsymbol{p}, \boldsymbol{q})_{GLR} = \widetilde{\boldsymbol{C}}_1 \oplus \widetilde{\boldsymbol{C}}_2$ ($\widetilde{\boldsymbol{C}}_1 = (\boldsymbol{c}_1, \boldsymbol{p}_1, \boldsymbol{q}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{C}}_2 = (\boldsymbol{c}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)_{GLR} \leqslant \boldsymbol{0}$), $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{M}, \boldsymbol{N})_{GLR} = \widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2$ ($\widetilde{\boldsymbol{A}}_1 = (\boldsymbol{A}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{A}}_2 = (\boldsymbol{A}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{GLR} \leqslant \boldsymbol{0}$), $\widetilde{\boldsymbol{b}} = (\boldsymbol{b}, \boldsymbol{g}, \boldsymbol{h})_{GLR}$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \geqslant \boldsymbol{0}$. Then the FFLP problem (3.1) can be converted into an instance of the MOLD problem with three crisp objective functions as follows:

$$\begin{aligned}
&\max(\min)\ (\boldsymbol{c}^T\boldsymbol{x}), \\
&\min(\max)\ [(\boldsymbol{p}^T + \boldsymbol{q}^T)\boldsymbol{x} + (\boldsymbol{c}_1^T - \boldsymbol{c}_2^T)(\boldsymbol{y} + \boldsymbol{z})], \\
&\max(\min)\ [(2\boldsymbol{c}^T + \boldsymbol{q}^T - \boldsymbol{p}^T)\boldsymbol{x} + \boldsymbol{c}^T(\boldsymbol{z} - \boldsymbol{y})], \\
&\quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{A}_1\boldsymbol{y} - \boldsymbol{A}_2\boldsymbol{z} + \boldsymbol{M}\boldsymbol{x} = \boldsymbol{g},\ \boldsymbol{A}_1\boldsymbol{z} - \boldsymbol{A}_2\boldsymbol{y} + \boldsymbol{N}\boldsymbol{x} = \boldsymbol{h}, \\
&\qquad\qquad \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x} - \boldsymbol{y} \geqslant \boldsymbol{0}.
\end{aligned} \tag{3.3}$$

Proof. Since $\widetilde{\boldsymbol{C}} = (\boldsymbol{c}, \boldsymbol{p}, \boldsymbol{q})_{GLR} = \widetilde{\boldsymbol{C}}_1 \oplus \widetilde{\boldsymbol{C}}_2(\widetilde{\boldsymbol{C}}_1 = (\boldsymbol{c}_1, \boldsymbol{p}_1, \boldsymbol{q}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{C}}_2 = (\boldsymbol{c}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)_{GLR} \leqslant \boldsymbol{0})$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \geqslant \boldsymbol{0}$, we have

$$\begin{aligned}
\widetilde{\boldsymbol{C}}^T \otimes \widetilde{\boldsymbol{x}} &= \widetilde{\boldsymbol{C}}_1^T \otimes \widetilde{\boldsymbol{x}} \oplus \widetilde{\boldsymbol{C}}_2^T \otimes \widetilde{\boldsymbol{x}} \\
&= (\boldsymbol{c}_1^T, \boldsymbol{p}_1^T, \boldsymbol{q}_1^T)_{GLR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \oplus (\boldsymbol{c}_2^T, \boldsymbol{p}_2^T, \boldsymbol{q}_2^T)_{GLR} \otimes (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \\
&\cong (\boldsymbol{c}^T\boldsymbol{x},\ \boldsymbol{c}_1^T\boldsymbol{y} - \boldsymbol{c}_2^T\boldsymbol{z} + \boldsymbol{p}^T\boldsymbol{x},\ \boldsymbol{c}_1^T\boldsymbol{z} - \boldsymbol{c}_2^T\boldsymbol{y} + \boldsymbol{q}^T\boldsymbol{x})_{GLR}.
\end{aligned}$$

By Definition 2.4 and Theorem 3.1, we can get that the FFLP problem (3.1) can be converted to the MOLP problem (3.3) with three crisp objective functions. □

4. A Method to Find the Fuzzy Optimal Solution of the FFLP Problem

In this section, we are going to introduce a method based on the ordering and the arithmetic operations of GLR-fuzzy numbers, and to find an exact fuzzy optimal solution of the FFLP problem (3.1). The steps of the proposed method are given as follows:

Step 1: Let $\widetilde{\boldsymbol{C}} = (\boldsymbol{c}, \boldsymbol{p}, \boldsymbol{q})_{GLR} = \widetilde{\boldsymbol{C}}_1 \oplus \widetilde{\boldsymbol{C}}_2$ ($\widetilde{\boldsymbol{C}}_1 = (\boldsymbol{c}_1, \boldsymbol{p}_1, \boldsymbol{q}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{C}}_2 = (\boldsymbol{c}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)_{GLR} \leqslant \boldsymbol{0}$), $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{M}, \boldsymbol{N})_{GLR} = \widetilde{\boldsymbol{A}}_1 \oplus \widetilde{\boldsymbol{A}}_2$ ($\widetilde{\boldsymbol{A}}_1 = (\boldsymbol{A}_1, \boldsymbol{M}_1, \boldsymbol{N}_1)_{GLR} \geqslant \boldsymbol{0}$, $\widetilde{\boldsymbol{A}}_2 = (\boldsymbol{A}_2, \boldsymbol{M}_2, \boldsymbol{N}_2)_{GLR} \leqslant \boldsymbol{0}$), $\widetilde{\boldsymbol{b}} = (\boldsymbol{b}, \boldsymbol{g}, \boldsymbol{h})_{GLR}$, $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR} \geqslant \boldsymbol{0}$. Then according to Theorem 3.2, the FFLP problem (3.1) can be converted to the MOLP problem (3.3) with three crisp objective functions.

Step 2: In terms of the preference of objective functions, the method will be used to obtain an optimal solution of the problem (3.3). So, we have:

$$\begin{aligned}&\max(\min)(\boldsymbol{c}^T\boldsymbol{x}),\\ &\quad \text{s.t. } \boldsymbol{Ax}=\boldsymbol{b},\ \boldsymbol{A}_1\boldsymbol{y}-\boldsymbol{A}_2\boldsymbol{z}+\boldsymbol{Mx}=\boldsymbol{g},\ \boldsymbol{A}_1\boldsymbol{z}-\boldsymbol{A}_2\boldsymbol{y}+\boldsymbol{Nx}=\boldsymbol{h},\\ &\qquad \boldsymbol{x}\geqslant \boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{y}\geqslant \boldsymbol{0}.\end{aligned} \tag{4.1}$$

If the problem (4.1) has a unique optimal solution, namely $\widetilde{\boldsymbol{x}}^*=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})_{GLR}$, then it is an optimal solution of the problem (3.3) and stop. Otherwise go to Step 3.

Step 3: Solve the following problem over the optimal solutions that are achieved in Step 2 as follows:

$$\begin{aligned}&\min(\max)[(\boldsymbol{p}^T+\boldsymbol{q}^T)\boldsymbol{x}+(\boldsymbol{c}_1^T-\boldsymbol{c}_2^T)(\boldsymbol{y}+\boldsymbol{z})],\\ &\quad \text{s.t. } \boldsymbol{c}^T\boldsymbol{x}=m^*,\\ &\qquad \boldsymbol{Ax}=\boldsymbol{b},\ \boldsymbol{A}_1\boldsymbol{y}-\boldsymbol{A}_2\boldsymbol{z}+\boldsymbol{Mx}=\boldsymbol{g},\ \boldsymbol{A}_1\boldsymbol{z}-\boldsymbol{A}_2\boldsymbol{y}+\boldsymbol{Nx}=\boldsymbol{h},\\ &\qquad \boldsymbol{x}\geqslant \boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{y}\geqslant \boldsymbol{0},\end{aligned} \tag{4.2}$$

where m^* is the optimal value of the objective function of (4.1). If the problem (4.2) has a unique optimal solution, namely $\widetilde{\boldsymbol{x}}^*=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})_{GLR}$ then it is also an optimal solution of the problem (3.3) and stop. Otherwise go to Step 4.

Step 4: Solve the following problem over the optimal solutions that are achieved in Step 3 as follows:

$$\begin{aligned}&\max(\min)[(2\boldsymbol{c}^T+\boldsymbol{q}^T-\boldsymbol{p}^T)\boldsymbol{x}+\boldsymbol{c}^T(\boldsymbol{z}-\boldsymbol{y})],\\ &\quad \text{s.t. } (\boldsymbol{p}^T+\boldsymbol{q}^T)\boldsymbol{x}+(\boldsymbol{c}_1^T-\boldsymbol{c}_2^T)(\boldsymbol{y}+\boldsymbol{z})=n^*,\\ &\qquad \boldsymbol{c}^T\boldsymbol{x}=m^*,\\ &\qquad \boldsymbol{Ax}=\boldsymbol{b},\ \boldsymbol{A}_1\boldsymbol{y}-\boldsymbol{A}_2\boldsymbol{z}+\boldsymbol{Mx}=\boldsymbol{g},\ \boldsymbol{A}_1\boldsymbol{z}-\boldsymbol{A}_2\boldsymbol{y}+\boldsymbol{Nx}=\boldsymbol{h},\\ &\qquad \boldsymbol{x}\geqslant \boldsymbol{0},\ \boldsymbol{x}-\boldsymbol{y}\geqslant \boldsymbol{0},\end{aligned} \tag{4.3}$$

where n^* is the optimal value of the objective function of (4.2). So, the optimal solution of the problem (3.3), namely $\widetilde{\boldsymbol{x}}^*=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})_{GLR}$ is obtained by solving the problem (4.3).

Now by the following Theorem 4.1, it shows that the obtained the optimal solution of the problem (3.3) can be considered as an exact optimal solution of the problem (3.1).

Theorem 4.1. If $\widetilde{\boldsymbol{x}}^*=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})_{GLR}$ be an optimal solution of the problems (4.1)-(4.3) (naturally, it is an optimal solution of the problem (3.3)), then it is also an exact optimal solution of the problem (3.1).

Proof. Assume $\widetilde{\boldsymbol{x}}^*=(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{z}^*)_{GLR}$ is an optimal solution of the problems (4.1)-(4.3), but it is not the exact optimal solution of the problem (3.1). Therefore, there exists a feasible solution of the problem (3.1), namely $\widetilde{\boldsymbol{x}}^0=(\boldsymbol{x}^0,\boldsymbol{y}^0,\boldsymbol{z}^0)_{GLR}\neq\widetilde{\boldsymbol{x}}^*$ such that

$$\widetilde{\boldsymbol{C}}^T\otimes\widetilde{\boldsymbol{x}}^*=(\boldsymbol{c}^T\boldsymbol{x}^*,\ \boldsymbol{c}_1^T\boldsymbol{y}^*-\boldsymbol{c}_2^T\boldsymbol{z}^*+\boldsymbol{p}^T\boldsymbol{x}^*,\ \boldsymbol{c}_1^T\boldsymbol{z}-\boldsymbol{c}_2^T\boldsymbol{y}^*+\boldsymbol{q}^T\boldsymbol{x}^*)_{GLR}$$

$$<\widetilde{\boldsymbol{C}}^T\otimes\widetilde{\boldsymbol{x}}^0=(\boldsymbol{c}^T\boldsymbol{x}^0,\ \boldsymbol{c}_1^T\boldsymbol{y}^0-\boldsymbol{c}_2^T\boldsymbol{z}^0+\boldsymbol{p}^T\boldsymbol{x}^0,\ \boldsymbol{c}_1^T\boldsymbol{z}^0-\boldsymbol{c}_2^T\boldsymbol{y}^0+\boldsymbol{q}^T\boldsymbol{x}^0)_{GLR}$$

(in case of the maximization problem). Hence, according to Definition 2.4, we have three conditions as follows:

(1) If $\boldsymbol{c}^T\boldsymbol{x}^*<\boldsymbol{c}^T\boldsymbol{x}^0$, with respect to the assumption we have

$\boldsymbol{A x}^0 = \boldsymbol{b}$,
$\boldsymbol{A}_1 \boldsymbol{y}^0 - \boldsymbol{A}_2 \boldsymbol{z}^0 + \boldsymbol{M x}^0 = \boldsymbol{g}$,
$\boldsymbol{A}_1 \boldsymbol{z}^0 - \boldsymbol{A}_2 \boldsymbol{y}^0 + \boldsymbol{N x}^0 = \boldsymbol{h}$,
$\boldsymbol{x}^0 \geqslant \boldsymbol{0},\ \boldsymbol{x}^0 - \boldsymbol{y}^0 \geqslant \boldsymbol{0}$.
Therefore, $\widetilde{\boldsymbol{x}}^0 = (\boldsymbol{x}^0, \boldsymbol{y}^0, \boldsymbol{z}^0)_{GLR}$ is a feasible solution of the problem (4.1) in which the objective value in $\widetilde{\boldsymbol{x}}^0$ is greater than the objective value in $\widetilde{\boldsymbol{x}}^*$. This is a contradiction.

(2) If $\boldsymbol{c}^T \boldsymbol{x}^* = \boldsymbol{c}^T \boldsymbol{x}^0$ and

$$(\boldsymbol{p}^T + \boldsymbol{q}^T)\boldsymbol{x}^0 + (\boldsymbol{c}_1^T - \boldsymbol{c}_2^T)(\boldsymbol{y}^0 + \boldsymbol{z}^0) < (\boldsymbol{p}^T + \boldsymbol{q}^T)\boldsymbol{x}^* + (\boldsymbol{c}_1^T - \boldsymbol{c}_2^T)(\boldsymbol{y}^* + \boldsymbol{z}^*),$$

then $\widetilde{\boldsymbol{x}}^0 = (\boldsymbol{x}^0, \boldsymbol{y}^0, \boldsymbol{z}^0)_{GLR}$ is a feasible solution of the problem (4.2) in which the objective value in $\widetilde{\boldsymbol{x}}^0$ is less than the objective value in $\widetilde{\boldsymbol{x}}^*$. This is a contradiction.

(3) If $\boldsymbol{c}^T \boldsymbol{x}^* = \boldsymbol{c}^T \boldsymbol{x}^0$,

$$(\boldsymbol{p}^T + \boldsymbol{q}^T)\boldsymbol{x}^0 + (\boldsymbol{c}_1^T - \boldsymbol{c}_2^T)(\boldsymbol{y}^0 + \boldsymbol{z}^0) = (\boldsymbol{p}^T + \boldsymbol{q}^T)\boldsymbol{x}^* + (\boldsymbol{c}_1^T - \boldsymbol{c}_2^T)(\boldsymbol{y}^* + \boldsymbol{z}^*),$$

and

$$(2\boldsymbol{c}^T + \boldsymbol{q}^T - \boldsymbol{p}^T)\boldsymbol{x}^* + \boldsymbol{c}^T(\boldsymbol{z}^* - \boldsymbol{y}^*) < (2\boldsymbol{c}^T + \boldsymbol{q}^T - \boldsymbol{p}^T)\boldsymbol{x}^0 + \boldsymbol{c}^T(\boldsymbol{z}^0 - \boldsymbol{y}^0),$$

then $\widetilde{\boldsymbol{x}}^0 = (\boldsymbol{x}^0, \boldsymbol{y}^0, \boldsymbol{z}^0)_{GLR}$ is a feasible solution of the problem (4.3) in which the objective value in $\widetilde{\boldsymbol{x}}^0$ is greater than the objective value in $\widetilde{\boldsymbol{x}}^*$. This is a contradiction.

Therefore, $\widetilde{\boldsymbol{x}}^* = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})_{GLR}$ is an exact optimal solution of the problem (3.1). For the case of the minimization, the proof is similar. □

5. Examples

In this section, we will demonstrate the efficiency and superiority of the proposed method using numerical examples. At the same time, the shortcomings of the existing methods [13, 14] for solving the FFLP problems with equality constraints are pointed out.

Example 5.1. Let $L(x) = R(x) = \max\{0, 1 - |x|\}$. Consider the following FFLP:

$$\begin{aligned} \min\ & (2,1,3) \otimes \widetilde{x}_1 \oplus (-3,2,1) \otimes \widetilde{x}_2, \\ \text{s.t.}\ & (2,1,0) \otimes \widetilde{x}_1 \oplus (-1,2,1) \otimes \widetilde{x}_2 = (2,1,2), \\ & (-3,1,2) \otimes \widetilde{x}_1 \oplus (2,1,1) \otimes \widetilde{x}_2 = (1,0,1), \\ & \widetilde{x}_1 \geqslant 0,\ \widetilde{x}_2 \geqslant 0. \end{aligned} \tag{5.1}$$

Let $\widetilde{x}_1 = (x_1, y_1, z_1)_{GLR},\ \widetilde{x}_2 = (x_2, y_2, z_2)_{GLR}$. Then
$\widetilde{\boldsymbol{A}} = \begin{pmatrix} (2,1,0) & (-1,2,1) \\ (-3,1,2) & (2,1,1) \end{pmatrix}$ $\widetilde{\boldsymbol{C}}^T = ((2,1,3),(-3,2,1))$, $\widetilde{\boldsymbol{b}}^T = ((2,1,2),(1,0,1))$ $\boldsymbol{A} = \begin{pmatrix} 2,-1 \\ -3,2 \end{pmatrix}$, $\boldsymbol{A}_1 = \begin{pmatrix} 2,0 \\ 0,2 \end{pmatrix}$, $\boldsymbol{A}_2 = \begin{pmatrix} 0,-1 \\ -3,0 \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} 1,2 \\ 1,1 \end{pmatrix}$, $\boldsymbol{N} = \begin{pmatrix} 0,1 \\ 2,1 \end{pmatrix}$.
$\boldsymbol{c}^T = (2,-3), \boldsymbol{c}_1^T = (2,0)$, $\boldsymbol{c}_2^T = (0,-3)$, $\boldsymbol{p}^T = (1,2)$, $\boldsymbol{q}^T = (3,1)$, $\boldsymbol{b}^T = (2,1)$, $\boldsymbol{g}^T = (1,0)$, $\boldsymbol{h}^T = (2,1)$.
Using step 1, above FFLP problem (5.1) is converted to a MOLP problem as follows:

$$\begin{aligned} & \min(2x_1 - 3x_2), \\ & \max(4x_1 + 3x_2 + 2y_1 + 3y_2 + 2z_1 + 3z_2), \\ & \min(6x_1 - 7x_2 - 2y_1 + 3y_2 + 2z_1 - 3z_2), \\ & \text{s.t. } 2x_1 - x_2 = 2, \\ & \qquad -3x_1 + 2x_2 = 1, \\ & \qquad 2y_1 + z_2 + x_1 + 2x_2 = 1, \end{aligned} \tag{5.2}$$

$$\begin{aligned}&2y_2+3z_1+x_1+x_2=0,\\&2z_1+y_2+x_2=2,\\&2z_2+3y_1+2x_1+x_2=1,\\&x_1\geqslant 0,\ x_2\geqslant 0,\\&x_1-y_1\geqslant 0,\ x_2-y_2\geqslant 0.\end{aligned}$$

Using step 2, 3 and 4, the optimal solution of the MOLP problem (5.2) is

$$x_1^*=5,\ x_2^*=8,\ y_1^*=-23,\ y_2^*=-8,\ z_1^*=1,\ z_2^*=26.$$

Note that the left and the right spread of $\widetilde{x}_1,\widetilde{x}_2$ are not all non-negative, Therefore, using the definition of the GLR-fuzzy numbers, we can get: $\widetilde{x}_1^*=(5,0,23)_{GLR}$, $\widetilde{x}_2^*=(8,0,26)_{GLR}$, and their membership functions are given by

$$u_{\widetilde{x}_1}(t)=\begin{cases}0, & t<5,\\ R(\frac{t-5}{23}), & t\geqslant 5;\end{cases}\qquad u_{\widetilde{x}_2}(t)=\begin{cases}0, & t<8,\\ R(\frac{t-8}{26}), & t\geqslant 8.\end{cases}$$

That is,

$$u_{\widetilde{x}_1}(t)=\begin{cases}\frac{28-t}{23}, & 5<t<28,\\ 0, & \text{others};\end{cases}\qquad u_{\widetilde{x}_2}(t)=\begin{cases}\frac{34-t}{26}, & 8<t<34,\\ 0, & \text{others}.\end{cases}$$

Hence, the fuzzy optimal solution of the FFLP problem (5.1) is

$$\begin{cases}\widetilde{x}_1^*=(5,0,23)_{GLR},\\ \widetilde{x}_2^*=(8,0,26)_{GLR}.\end{cases}$$

The optimal value of objective function is obtained by putting $\widetilde{\boldsymbol{x}}^*$ in $\widetilde{\boldsymbol{C}}^T\otimes\widetilde{\boldsymbol{x}}$. Therefore, the fuzzy optimal value of the problem (5.1) may be written as follows:

$$\widetilde{\boldsymbol{C}}^T\otimes\widetilde{\boldsymbol{x}}^*=(2,1,3)\otimes(5,0,23)\oplus(-3,2,1)\otimes(8,0,26)=(-14,99,69)_{GLR},$$

and its membership function is given by

$$u_{\widetilde{z}}(t)=\begin{cases}L(\frac{-14-t}{99}), & t<-14,\\ R(\frac{t+14}{69}), & t\geqslant -14.\end{cases}$$

That is,

$$u_{\widetilde{z}}(t)=\begin{cases}\frac{t+113}{99}, & -113<t\leqslant -14,\\ \frac{55-t}{69}, & -14<t<55,\\ 0, & \text{others}.\end{cases}$$

Example 5.2. Using the method proposed by Lotfi [14] to solve the FFLP problem (5.1).

Let $\widetilde{x}_1=(x_1-y_1,x_1,x_1+z_1)$, $\widetilde{x}_2=(x_2-y_2,x_2,x_2+z_2)$ be triangular fuzzy numbers. Using the method of Lotfi [14], the FFLP problem (5.1) can be converted to the first problem which is related to the core of the solution as follows:

$$\begin{aligned}&\min(2.5x_1-3.25x_1-0.33y_1+1.08y_2+z_1-0.58z_2),\\ &\text{s.t.}\ \ 1.75x_1-1.25x_1-0.33y_1+0.58y_2+0.5z_1-0.08z_2=2.25,\\ &\quad -2.75x_1+2x_1+0.917y_1-0.33y_2-0.417z_1+0.67z_2=1.25,\\ &\quad 0.5x_1+1.5x_1+0.625y_1-1.25y_2+z_1-0.125z_2=1.5,\\ &\quad 1.5x_1+x_1-1.875y_1+0.625y_2-0.75z_1+1.375z_2=0.5,\\ &\quad x_1\geqslant 0,\ x_2\geqslant 0,\ y_1\geqslant 0,\ y_2\geqslant 0,\ z_1\geqslant 0,\ z_2\geqslant 0,\\ &\quad x_1-y_1\geqslant 0,\ x_2-y_2\geqslant 0.\end{aligned}\tag{5.3}$$

The crisp linear programming problem (5.3) has no feasible solution. Hence, there is no fuzzy optimal solution for the FFLP problem (5.1).

Example 5.3. Using the method proposed by Kumar [13] to solve the FFLP problem (5.1). Let $\widetilde{x}_1 = (x_1, y_1, z_1)$, $\widetilde{x}_2 = (x_2, y_2, z_2)$ be triangular fuzzy numbers. Using the method of Kumar [12], the FFLP problem (5.1) can be converted to a crisp linear programming problem as follows:

$$\begin{aligned} &\min(0.25x_2 + 0.5y_1 + y_2 + 0.5z_1 + z_2), \\ &\text{s.t. } \; x_1 - 3z_2 = 1, \\ &\qquad 2y_1 - y_2 = 2, \\ &\qquad 2z_1 = 4, \\ &\qquad -4z_1 + x_2 = 1, \\ &\qquad 3y_1 + 2y_2 = 1, \\ &\qquad -x_1 + 3z_2 = 2, \\ &\qquad y_1 - x_1 \geqslant 0,\ z_1 - y_1 \geqslant 0,\ y_2 - x_2 \geqslant 0,\ z_2 - y_2 \geqslant 0. \end{aligned} \tag{5.4}$$

The crisp linear programming problem (5.4) does not have any feasible solution. Therefore, there is no fuzzy optimal solution for the FFLP problem (5.1).

In order to compare with the existed methods, if we still use the data of Example 5.1 and solve it by using the methods of Lotfi [14] and Kumar [13], the results show that the linear programming problems which are converted by the FFLP problem (5.1) does not have any feasible solution. Therefore, there is no fuzzy optimal solution for the FFLP problem (5.1).

Note that for a fully fuzzy linear programming problem, the fuzzy optimal solution $\widetilde{\boldsymbol{x}}^*$ is more general than the result that is calculated in Lotfi [14] when the left spreads shape functions $L(\cdot)$ and right spreads shape functions $R(\cdot)$ are linear functions. Meanwhile all the parameters as well as the variables are represented by the symmetric fuzzy numbers. Furthermore, even the uncertain elements in a fuzzy linear programming problem were extended fuzzy numbers, we can convert it into a MOLP problem and use the method to find its fuzzy optimal solution. However, the transformed crisp linear programming problem through the method of [14] always has too many constraints which makes their computation especially numerical implementation is inconvenient.

Comparing with the method and the results in Kumar's article [13], the fuzzy optimal solution of the FFLP problem can be obtained in [13] by using the arithmetic operations of triangular fuzzy numbers and linear ranking function which is used to convert the fuzzy objective function to the real objective function. Although the ranking function is convenient for a specific numerical computation, the fuzziness of the objective function is neglected. However, the method proposed in this paper based on a new ordering and arithmetic operations of GLR-fuzzy numbers without ranking function, the FFLP problems could be transformed to a MOLP problem with three crisp objective functions.

Besides, for the coefficients $\widetilde{\boldsymbol{C}}$, $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{b}}$ are represented by GLR-fuzzy numbers, we could define the predetermined left spreads shape functions $L(\cdot)$ and right spreads shape functions $R(\cdot)$ as follows according to the need of mathematical modeling:

(1) $L(x) = \max\{0, 1 - |x|^p\}(p > 0)$;

(2) $L(x) = \max\{0, \frac{1}{e-1}(e^{1-|x|^p} - 1)\}(p > 0)$;

(3) $L(x) = \max\{0, \frac{2}{1+|x|^p} - 1(p > 0)$,

and it is similar to $R(\cdot)$.

Example 5.4. Let $L(x) = \max\{0, 1 - |x|\}, R(x) = \max\{0, \frac{1}{e-1}(e^{1-x^2} - 1)\}$. Consider a FFLP problem with non-negative variables as follows:

$$\begin{aligned} \min\ & (12,5,2)\otimes\widetilde{x}_1 \oplus (6,6,4)\otimes\widetilde{x}_2 \oplus (14,4,3)\otimes\widetilde{x}_3 \oplus (8,2,2)\otimes\widetilde{x}_4, \\ \text{s.t.}\ & (10,2,3)\otimes\widetilde{x}_1 \oplus (11,1,2)\otimes\widetilde{x}_2 \oplus (12,3,1)\otimes\widetilde{x}_3 \oplus (15,4,2)\otimes\widetilde{x}_4 = (421,80,115), \\ & (14,2,2)\otimes\widetilde{x}_1 \oplus (18,4,1)\otimes\widetilde{x}_2 \oplus (17,7,3)\otimes\widetilde{x}_3 \oplus (14,1,4)\otimes\widetilde{x}_4 = (637,134,138), \\ & \widetilde{x}_1 \geqslant 0,\ \widetilde{x}_2 \geqslant 0,\ \widetilde{x}_3 \geqslant 0,\ \widetilde{x}_4 \geqslant 0. \end{aligned} \tag{5.5}$$

Let $\widetilde{x}_1 = (x_1, y_1, z_1)_{GLR},\ \widetilde{x}_2 = (x_2, y_2, z_2)_{GLR},\ \widetilde{x}_3 = (x_3, y_3, z_3)_{GLR},\ \widetilde{x}_3 = (x_4, y_4, z_4)_{GLR}$. Then,

$$\widetilde{\boldsymbol{C}} = \begin{bmatrix} (12,5,2) \\ (6,6,4) \\ (14,4,3) \\ (8,2,2) \end{bmatrix}, \qquad \widetilde{\boldsymbol{A}} = \begin{bmatrix} (10,2,3) & (11,1,2) & (12,3,1) & (15,4,2) \\ (14,2,2) & (18,4,1) & (17,3,3) & (14,1,4) \end{bmatrix},$$

$$\widetilde{\boldsymbol{b}} = \begin{bmatrix} (421,80,115) \\ (637,134,138) \end{bmatrix}, \boldsymbol{A} = \boldsymbol{A}_1 = \begin{pmatrix} 10,11,12,15 \\ 14,18,17,14 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 2,1,3,4 \\ 2,4,3,1 \end{pmatrix},$$

$$\boldsymbol{N} = \begin{pmatrix} 3,2,1,2 \\ 2,1,3,4 \end{pmatrix}, \boldsymbol{A}_2 = \boldsymbol{0}, \quad \boldsymbol{c}^T = \boldsymbol{c}_1^T = (12,6,14,8), \quad \boldsymbol{p}^T = (5,6,4,2),$$

$$\boldsymbol{q}^T = (2,4,3,2),\ \boldsymbol{c}_2^T = \boldsymbol{0}, \quad \boldsymbol{b}^T = (421,637), \quad \boldsymbol{g}^T = (80,134), \quad \boldsymbol{h}^T = (115,138).$$

Using step 1, above FFLP problem (5.5) is converted to a MOLP problem as follows:

$$\begin{aligned} & \min(12x_1 + 6x_2 + 14x_3 + 8x_4), \\ & \max(7x_1 + 10x_2 + 7x_3 + 4x_4 + 12y_1 + 6y_2 + 14y_3 + 8y_4 + 12z_1 + 6z_2 + \\ & \quad 14z_3 + 8z_4), \\ & \min(21x_1 + 10x_2 + 27x_3 + 16x_4 - 12y_1 - 6y_2 - 14y_3 - 8y_4 + 12z_1 + \\ & \quad 6z_2 + 14z_3 + 8z_4), \\ & \text{s.t. } 10x_1 + 11x_2 + 12x_3 + 15x_4 = 421, \\ & \quad 14x_1 + 18x_2 + 17x_3 + 14x_4 = 637, \\ & \quad 10y_1 + 11y_2 + 12y_3 + 15y_4 + 2x_1 + x_2 + 3x_3 + 4x_4 = 80, \\ & \quad 14y_1 + 18y_2 + 17y_3 + 14y_4 + 2x_1 + 4x_2 + 3x_3 + x_4 = 134, \\ & \quad 10z_1 + 11z_2 + 12z_3 + 15z_4 + 3x_1 + 2x_2 + x_3 + 2x_4 = 115, \\ & \quad 14z_1 + 18z_2 + 17z_3 + 14z_4 + 2x_1 + x_2 + 3x_3 + 4x_4 = 138, \\ & \quad x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ x_4 \geqslant 0, \\ & \quad x_1 - y_1 \geqslant 0,\ x_2 - y_2 \geqslant 0,\ x_3 - y_3 \geqslant 0,\ x_4 - y_4 \geqslant 0. \end{aligned} \tag{5.6}$$

Using step 2, 3 and 4, the optimal solution of the MOLP problem (12) is:

$$x_1^* = 0,\ x_2^* = 31.56,\ x_3^* = 0,\ x_4^* = 4.92,\ y_1^* = -0.39,\ y_2^* = -2.42,$$

$$y_3^* = -0.6,\ y_4^* = 4.43,\ z_1^* = 1.53,\ z_2^* = 3.06,\ z_3^* = 2.91,\ z_4^* = -2.79.$$

Note that the left and the right spread of $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4$ are not all non-negative. Therefore, using the definition of GLR-fuzzy numbers, we can get a fuzzy optimal solution of FFLP problem (5.5) is:

$$\widetilde{x}^* = \begin{pmatrix} (0.00, 0.00, 1.53)_{GLR} \\ (31.56, 0.00, 3.06)_{GLR} \\ (0.00, 0.00, 2.91)_{GLR} \\ (4.92, 4.43, 0.00)_{GLR} \end{pmatrix},$$

and their membership functions are given by

$$u_{\tilde{x}_1^*}(t)=\begin{cases}0, & t<0,\\ R(\frac{t}{1.53}), & t\geqslant 0;\end{cases}\qquad u_{\tilde{x}_2^*}(t)=\begin{cases}0, & t<31.56,\\ R(\frac{t-31.56}{3.06}), & t\geqslant 31.56;\end{cases}$$

$$u_{\tilde{x}_3^*}(t)=\begin{cases}0, & t<0,\\ R(\frac{t}{2.91}), & t\geqslant 0;\end{cases}\qquad u_{\tilde{x}_4^*}(t)=\begin{cases}L(\frac{4.92-t}{4.43}), & t<4.92,\\ 0, & t\geqslant 4.92.\end{cases}$$

That is,

$$u_{\tilde{x}_1^*}(t)=\begin{cases}\frac{1}{e-1}(e^{1-\frac{t^2}{1.53^2}}-1), & 0\leqslant t<1.53,\\ 0, & \text{others};\end{cases}$$

$$u_{\tilde{x}_2^*}(t)=\begin{cases}\frac{1}{e-1}(e^{1-\frac{(t-31.56)^2}{3.06^2}}-1), & 31.56\leqslant t<34.62,\\ 0, & \text{others};\end{cases}$$

$$u_{\tilde{x}_3^*}(t)=\begin{cases}\frac{1}{e-1}(e^{1-\frac{t^2}{2.91^2}}-1), & 0\leqslant t<2.91,\\ 0, & \text{others};\end{cases}\qquad u_{\tilde{x}_4^*}(t)=\begin{cases}\frac{t-0.49}{4.43}, & 0.49\leqslant t<4.92,\\ 0, & \text{others}.\end{cases}$$

The fuzzy optimal value of objective function is obtained by putting $\tilde{\boldsymbol{x}}^*$ in $\tilde{\boldsymbol{C}}^T\otimes\tilde{\boldsymbol{x}}$, therefore, the optimal value of the problem (5.5) may be written as follows:

$$\begin{aligned}\tilde{\boldsymbol{C}}^T\otimes\tilde{\boldsymbol{x}}^* &= (12,5,2)\otimes(0,0,1.53)\oplus(6,6,4)\otimes(31.56,0,3.06)\oplus(14,4,3)\otimes(0,0,2.91)\\ &\oplus(8,2,2)\otimes(4.92,4.43,0)\ =(228.74,234.64,213.54)_{GLR},\end{aligned}$$

and its membership function is given by

$$u_{\tilde{z}}(t)=\begin{cases}L(\frac{228.74-t}{234.64}), & t<228.74,\\ R(\frac{t-228.74}{213.54}), & t\geqslant 228.74.\end{cases}$$

That is,

$$u_{\tilde{z}}(t)=\begin{cases}\frac{t+5.90}{234.64}, & -5.90<t<228.74,\\ \frac{1}{e-1}(e^{1-\frac{(t-228.74)^2}{213.54^2}}-1), & 228.74\leqslant t<442.28,\\ 0, & \text{others}.\end{cases}$$

We note that the fully fuzzy linear programming (FFLP) problem is also considered in the papers [12, 18, 19, 21, 22]. Meanwhile, the different methods are proposed to find the fuzzy optimal solution of the fully fuzzy linear programming (FFLP) problems with LR fuzzy numbers. In the articles [18] and [19], the authors first defined a ranking function which could be used to convert the fuzzy objective function to a real objective function. Although the ranking function is convenient for the specific numerical computation, the fuzziness of the objective function is neglected. However, the method proposed in this paper based on a new ordering and arithmetic operations of GLR-fuzzy numbers without ranking function, the FFLP problems could be transformed to a MOLP problem with three crisp objective functions directly. In the paper [12, 21], new kinds of the FFLP problems were introduced respectively. The FFLP also could be converted into a multi-objective linear programming (MOLP) according to a order relation for comparing the LR flat fuzzy numbers. Compared with [12, 18, 19, 21], the method proposed in this paper is more simple and practical for solving a full fuzzy linear programming problem. The method proposed can successfully solve the FFLP problem in which all the parameters as well as the variables are represented by extended fuzzy numbers. The FFLP problem could be converted to its equivalent MOLP problem based on a new ordering and arithmetic operations of extended LR-fuzzy numbers, and the fuzzy optimal solution of the FFLP problem, occurring in real life situation, not only can be easily obtained, but also the scope of it can be extended by using this method.

6. Conclusion

In this paper, we propose a simple and practical method to solve a fully fuzzy linear programming problem. The proposed method can successfully solve the FFLP problem which all the parameters as well as the variables are represented by the extended fuzzy numbers. The FFLP problem could be converted to its equivalent MOLP problem based on a new ordering and the arithmetic operations of extended LR-fuzzy numbers. The fuzzy optimal solution of the FFLP problem, occurring in real life situation, not only can be easily obtained, but also the scope of it can be extended by using this method. By some numerical examples, the results obtained in this paper have been compared with that of Lotfi [14] and Kumar [13] and shown the reliability and applicability of our methods.

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